Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2017) 41: 267 – 281
© TÜBİTAK
doi:10.3906/mat-1510-66

On the volume of the indicatrix of a complex Finsler space

Elena POPOVICI*
Department of Mathematics and Informatics, Faculty of Mathematics and Informatics,
Transilvania University of Braşov, Romania

Received: 17.10.2015 • **Accepted/Published Online:** 20.04.2016 • **Final Version:** 03.04.2017

Abstract: Following the study on volume of indicatrices in a real Finsler space, in this paper we are investigating some volume properties of the indicatrix considered in an arbitrary fixed point of a complex Finsler manifold. Since for each point of a complex Finsler space the indicatrix is an embedded CR-hypersurface of the punctured holomorphic tangent bundle, by means of its normal vector, the volume element of the indicatrix is determined. Thus, the volume function is pointed out and its variation is studied. Conditions under which the volume is constant are also determined and some classes of complex Finsler spaces with constant indicatrix volume are given. Moreover, the length of the complex indicatrix of Riemann surfaces is found to be constant. In addition, considering submersions from the complex indicatrix onto almost Hermitian surfaces, we obtain that the volume of the submersed manifold is constant.

Key words: Complex Finsler space, complex indicatrix, volume element, volume variation, indicatrix length, submersion volume

1. Introduction

The study of the unit tangent sphere, or indicatrix, in real Finsler spaces is one of interest ([15, 19, 21, 22], etc.), mainly because it is a compact and strictly convex set surrounding the origin. For example, the indicatrix plays a special role in the volume definition of a Finsler space. However, in the present paper, based on some ideas from the real case, the volume element and volume function of the indicatrix in a complex Finsler manifold (M, F) are introduced and several of their properties are obtained.

First we recall some basic notions about complex Finsler geometry in Section 1. By taking $z \in M$ as an arbitrary point, the punctured holomorphic tangent bundle $T'_z M$ can be locally viewed as a Kähler manifold and the complex indicatrix, considered in the fixed point z, can be approached as a CR-hypersurface of $T'_z M$. Thus, regarding the volume element of a complex Finsler space, in Section 2 we establish the volume element of complex indicatrices. In Section 3 we will then be able to express the volume function of the complex indicatrix and obtain its variation by the z variable. Thus, some conditions under which the volume of complex indicatrices is constant (i.e. it does not depend on the choice of $z \in M$) are given. Even so, these conditions do not offer much information about the type of the complex Finsler spaces, but for certain classes of complex Finsler manifolds, as complex Berwald manifolds, the constant volume property is verified. Furthermore, by taking the particular case of Riemann surfaces, we study the length of the indicatrix and we obtain that it has constant value, equal to the unit circle value. This contrasts sharply with the situation from two-real-dimensional Finsler

*Correspondence: elena.c.popovici@gmail.com
2010 *AMS Mathematics Subject Classification:* 53B40, 53C60, 53C40, 53B25.

geometry, where the indicatrix length is in general a function of x. In the last section, using an example of a submersion from the indicatrix I_zM onto the complex projective bundle P_zM and the result presented in [23], we state that the volume of any Hermitian manifold, on which exists a submersion from the complex indicatrix, has constant value.

Now we provide a short overview of the concepts and terminology used in complex Finsler geometry (for more see [1, 20]). Let M be an n-dimensional complex manifold, with $z := (z^k),\ k = 1,..,n$, the complex coordinates on a local chart (U, φ). The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is in its turn a $2n$-dimensional complex manifold and the local coordinates in a local chart in $u \in T'M$ are $u := (z^k, \eta^k),\ k = 1,..,n$.

Definition 1 *A* complex Finsler space *is a pair* (M, F)*, with* $F : T'M \to \mathbb{R}^+,\ F = F(z, \eta)$ *a continuous function that satisfies the following conditions:*

i. F *is a smooth function on* $\widetilde{T'M} := T'M \setminus \{0\}$*;*

ii. $F(z, \eta) \geq 0$*, the equality holding if and only if* $\eta = 0$*;*

iii. $F(z, \lambda\eta) = |\lambda| F(z, \eta), \forall \lambda \in \mathbb{C}$*;*

iv. The Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ *is positive definite, where* $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ *is the fundamental metric tensor, with* $L := F^2$ *the complex Lagrangian associated to the complex Finsler function* F*.*

The positivity from the fourth condition is equivalent to the convexity of L and to the strongly pseudo-convex property of the complex indicatrix in a fixed point $I_zM = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$, for any $z \in M$. It also ensures the existence of the inverse $(g^{\bar{j}i})$, with $g^{\bar{j}i} g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$.

Moreover, condition iii represents the fact that L is homogeneous with respect to the complex norm, $L(z, \lambda\eta) = \lambda\bar{\lambda} L(z, \eta),\ \forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1}$$

The geometry of complex Finsler spaces consists of the study of geometric objects on the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. A first step represents the analysis of the sections on the complexified tangent bundle $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$. Let $V(T'M) = span\{\frac{\partial}{\partial \eta^k}\} \subset T'(T'M)$ be the vertical bundle and the complex nonlinear connection, briefly (c.n.c.), is the supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}$, where $N^j_k(z, \eta)$ are the coefficients of a (c.n.c.). Then we will call the adapted frame of the (c.n.c.) the pair $\{\delta_k := \frac{\delta}{\delta z^k},\ \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$, with the dual adapted base $\{\mathrm{d}z^k,\ \delta\eta^k := \mathrm{d}\eta^k + N^k_j \mathrm{d}z^j\}$.

One basic (c.n.c.) of a complex Finsler space is the Chern–Finsler (c.n.c.) ([1, 20]), with $N^k_j = g^{\bar{m}k} \frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l$. By means of the adapted base determined by the Chern–Finsler (c.n.c) it can be determined a Hermitian connection D of $(1,0)$-type, named in [1] the *Chern–Finsler linear connection*. In notations from [20] it is $D\Gamma N = (L^i_{jk}, L^{\bar{i}}_{\bar{j}k}, C^i_{jk}, C^{\bar{i}}_{\bar{j}k})$, locally given by the next set of coefficients $L^i_{jk} = g^{\bar{l}i}\delta_k(g_{j\bar{l}})$, $C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k(g_{j\bar{l}})$, $L^{\bar{i}}_{\bar{j}k} = 0$, $C^{\bar{i}}_{\bar{j}k} = 0$, with $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, $D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i$, etc., and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$ from (1). Further we will use the notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to denote a conjugate object.

In the terminology of [1], the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$, and *weakly Kähler* iff $g_{i\bar{l}}T^i_{jk}\eta^j\bar{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. However, the notions of strongly Kähler and Kähler actually coincide (cf. [14]). A complex Finsler space that is Kähler and $L^i_{jk} = L^i_{jk}(z)$ is named a *complex Berwald space* [3]. An extension of this is the *generalized Berwald space*, which has the coefficients of a complex spray $G^i = \frac{1}{2}N^i_j\eta^j$ holomorphic in η, i.e. $\dot{\partial}_{\bar{h}}G^i = 0$ [7].

Cauchy–Riemann (CR) submanifolds of almost Hermitian manifolds, introduced by Bejancu[10–12], were extended to the Finsler geometry by Dragomir in [16, 17]. A real submanifold $\tilde{M}$ of an almost Hermitian Finsler space (M, g), is a *CR-submanifold* if it carries a pair of complementary Finslerian distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ of $T\tilde{M}$, such that $\mathcal{D}$ is invariant, $J(\mathcal{D}_u) = \mathcal{D}_u$, and $\mathcal{D}^\perp$ is antiinvariant, $J(\mathcal{D}^\perp_u) \subset (T_u\tilde{M})^\perp$, for each $u \in \tilde{M}$, where J is an almost complex structure on $\tilde{M}$. Any real hypersurface $\tilde{M}$ of M is a CR-submanifold, with $\mathcal{D}^\perp_u = J(T_u\tilde{M})^\perp$ and $\mathcal{D}$ the complementary orthogonal distribution of $\mathcal{D}^\perp$.

2. The volume element of the complex indicatrix

Consider T'_zM the corresponding holomorphic tangent space of a complex Finsler manifold (M, F) and F_z the Finsler metric in an arbitrary fixed point $z \in M$. Then (T'_zM, F_z) can be regarded as a locally complex n-dimensional Minkowski space, with (η^i) its complex coordinate system, $\eta = (\eta^i) = \eta^i\frac{\partial}{\partial z^i}|_z$. Let g be the Hermitian structure on $T'(\widetilde{T'M})$ associated to F_z, which is usually extended in Hermitian geometry to a complex bilinear form $\mathcal{G}$ by $\mathcal{G}(Z, \bar{W}) = g(Z, W)$, $\mathcal{G}(Z, W) = \mathcal{G}(\bar{Z}, \bar{W}) = 0$, $\mathcal{G}(\bar{Z}, W) = \overline{\mathcal{G}(Z, \bar{W})}$, $\forall Z, W \in T'(T'_zM)$. Thus, $\mathcal{G}$ defines a Hermitian metric on $\widetilde{T'_zM}$, which is smooth at $\eta = 0$ if and only if F_z is a Hermitian norm and has the explicit form:

$$\mathcal{G} := \frac{\partial^2 F_z^2}{\partial\eta^i\partial\bar{\eta}^j}\mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k = g_{j\bar{k}}(z, \eta)\mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k. \tag{2}$$

For a fixed point $z \in M$, the unit sphere in (T'_zM, F_z), also called the *complex indicatrix* in z, is:

$$I_zM = \{\eta \in T'_zM \mid F(z, \eta) = 1\}.$$

I_zM is a strictly pseudoconvex submanifold and since it has only one defining equation, which involves the real valued Finsler function F, it is a real hypersurface of the holomorphic tangent bundle T'_zM, and thus a CR-hypersurface, for any $z \in M$.

Let $(u^1, ..., u^{2n-1})$ be local coordinates on I_zM and $\eta^j = \eta^j(u^1, .., u^{2n-1})$, $\forall j \in \{1, .., n\}$ the equations of inclusion $I_zM \overset{i}{\hookrightarrow} \widetilde{T'_zM}$ [17]. Then $F(z, \eta(u)) = 1$ yields by derivation after u variables to

$$l_j\frac{\partial\eta^j}{\partial u^\alpha} + l_{\bar{j}}\frac{\partial\eta^{\bar{j}}}{\partial u^\alpha} = 0, \quad \alpha \in \{1, \ldots, 2n-1\},\ j \in \{1, \ldots, n\}, \tag{3}$$

where $l_j = 2\frac{\partial F}{\partial \eta^j} = g_{j\bar{k}} l^{\bar{k}}$, with $l^j = \frac{1}{F}\eta^j$. Since the inclusion tangent map $i_* : T_R(I_zM) \to T_C(\widetilde{T'_zM})$ acts on tangent vectors of I_zM as

$$i_*\left(\frac{\partial}{\partial u^\alpha}\right) = X_\alpha := \frac{\partial \eta^k}{\partial u^\alpha}\frac{\partial}{\partial \eta^k} + \frac{\partial \bar{\eta}^k}{\partial u^\alpha}\frac{\partial}{\partial \bar{\eta}^k},$$

where X_α is a tangent vector of the complex indicatrix expressed in terms of tangent vectors of $T_C(T'M)$, from (3), we set

$$N = l^j \dot{\partial}_j + l^{\bar{j}} \dot{\partial}_{\bar{j}} \tag{4}$$

and thus we obtain $G_R(X_\alpha, N) = 0$, where G_R is the Riemannian metric applied to real vector fields as

$$G_R(X, Y) = \operatorname{Re} \mathcal{G}(X', \overline{Y'}). \tag{5}$$

Here X', $\overline{Y'}$ are the holomorphic, respectively, the antiholomorphic part of tangent vectors X and Y of the complexified tangent bundle of $T'M$, obtained by $X' = \frac{1}{2}(X - \mathrm{i}JX)$, $\overline{Y'} = \frac{1}{2}(Y + \mathrm{i}JY)$, $\mathrm{i} = \sqrt{-1}$. Consequently, $N \in T_R(I_zM)^\perp$ and $G_R(N, N) = 1$, so that N is the unit normal vector of the indicatrix bundle.

Remark. A similar result of the complex indicatrix normal vector was obtained in [24] by considering T'_zM as a $2n$-dimensional vector space $V_\mathbb{R}$, via the diffeomorphism $\mathcal{I} : V_\mathbb{R} \to T'_zM$, $(x^k, y^k) \mapsto (\eta^k) = (x^k + \sqrt{-1}y^k)$. Then g induces a Riemannian metric on $V_\mathbb{R}\backslash\{0\}$, defined by

$$\hat{g}_v(X, Y) := \operatorname{Re} g_{\mathcal{I}(v)}(\mathcal{I}_*(X), \mathcal{I}_*(Y)),\ \forall X, Y \in T_vV_\mathbb{R},\ v \neq 0, \tag{6}$$

where $\operatorname{Re}(\tau) = \frac{1}{2}(\tau + \bar{\tau})$, for any given form $\tau \in \mathcal{A}^{p,q}(M)$. Set the indicatrix $I_\mathbb{R} := \mathcal{I}^{-1}(I_zM)$, $\mathcal{I}^{-1}(\eta) = (v^k)$, with (v^k) taken as $v^{2k-1} := x^k$ and $v^{2k} := y^k$ the coordinates of $V_\mathbb{R}$ and let $X(t)$, $t \in (-\epsilon, \epsilon)$ be a smooth curve on $I_\mathbb{R}$ with $X(0) = v^k \frac{\partial}{\partial v^k}$. Then, by taking $\dot{X}(0) = \frac{dv^k}{dt}\frac{\partial}{\partial v^k}|_{t=0}$, a tangent vector of the curve, and using $\frac{\partial L}{\partial \eta^k} = g_{k\bar{j}}\eta^{\bar{j}}$, it is obtained that $\hat{g}_{X(0)}(X(0), \dot{X}(0)) = 0$. Hence, $\mathbf{n}|_{(v^k)} = X(0) = v^k\frac{\partial}{\partial v^k}$ is the normal vector to the indicatrix, which expressed in terms of the I_zM tangent vector field is exactly $N = \eta^j\frac{\partial}{\partial \eta^j} + \eta^{\bar{j}}\frac{\partial}{\partial \bar{\eta}^j}$ if $\eta \in I_zM$, or, equivalently, $N = l^j\frac{\partial}{\partial \eta^j} + l^{\bar{j}}\frac{\partial}{\partial \bar{\eta}^j}$ for an arbitrary $\eta \in \widetilde{T'_zM}$.

By considering the volume form of (T'_zM, F_z) induced by the Hermitian metric $\mathcal{G}$ as

$$d\mu = \frac{1}{n!}\left(\frac{\sqrt{-1}}{2}g_{i\bar{j}}d\eta^i \wedge d\bar{\eta}^j\right)^n = \frac{(-1)^{\frac{n^2}{2}}\mathbf{g}}{2^n}d\eta^1 \wedge d\eta^2 \wedge .. \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge d\bar{\eta}^2 \wedge .. \wedge d\bar{\eta}^n,$$

with $\mathbf{g} = \det(g_{i\bar{j}})$, we get the volume element of the induced metric on I_zM by a contraction of $d\mu$ with the outward-pointing normal $N = \eta^j\dot{\partial}_j + \eta^{\bar{j}}\dot{\partial}_{\bar{j}}$ (restricted to I_zM) in the first slot [9], i.e. the interior product with respect to N. Thus, we have

$$\begin{aligned} dV_z|_\eta = \ & d\mu\left(\eta^j\dot{\partial}_j + \eta^{\bar{j}}\dot{\partial}_{\bar{j}}, ...\right) = \iota_N(d\mu) = \\ = \ & \tfrac{(-1)^{\frac{n^2}{2}}\mathbf{g}}{2^n}\Big(\textstyle\sum_{j=1}^n(-1)^{j-1}\eta^j d\eta^1 \wedge .. \wedge \widehat{d\eta^j} \wedge .. \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge .. \wedge d\bar{\eta}^n + \\ & \quad + \textstyle\sum_{j=1}^n(-1)^{n+j-1}\eta^{\bar{j}}d\eta^1 \wedge .. \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge .. \wedge \widehat{d\bar{\eta}^j} \wedge .. \wedge d\bar{\eta}^n\Big), \end{aligned}$$

where ι is the interior product operator and by $\widehat{d\eta^j}$ we denote absence of the $d\eta^j$ term in the exterior product above. Using that $\tau + \bar{\tau} = 2\mathrm{Re}(\tau)$ for any given form τ, we obtain:

Proposition 1 *For each $\eta \in I_zM$, the volume element of the complex indicatrix is*

$$dV_z|_\eta = \frac{\mathbf{g}}{2^{n-1}}\mathrm{Re}[(-1)^{\frac{n^2}{2}}\sum_{i=1}^{n}(-1)^{i-1}\eta^i d\eta^1 \wedge ... \wedge \widehat{d\eta^i} \wedge ... \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge ... \wedge d\bar{\eta}^n].$$

Considering that I_zM is the unit sphere in (T'_zM, F_z), we take the map $r : \widetilde{T'_zM} \to I_zM$, $r(\eta) = \frac{\eta}{F_z(\eta)}$ and define a form Θ_z on $\widetilde{T'_zM}$ by $\Theta_z := r^*dV_z$. Then, without any constraint on the η^js, the normal vector is $N = l^j\dot{\partial}_j + l^{\bar{j}}\dot{\partial}_{\bar{j}}$, and for any $\eta \in \widetilde{T'_zM}$ the $2n-1$ form Θ_z has the expression

$$\Theta_z|_\eta = \frac{\mathbf{g}}{2^{n-1}F^{2n}}\mathrm{Re}[(-1)^{\frac{n^2}{2}}\sum_{i=1}^{n}(-1)^{i-1}\eta^i d\eta^1 \wedge .. \wedge \widehat{d\eta^i} \wedge .. \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge .. \wedge d\bar{\eta}^n], \tag{7}$$

for any $\eta \in \widetilde{T'_zM}$. The Θ_z form has an important role in calculating the local degree of a holomorphic vector field at a nondegenerate zero.

We also notice that the same form of volume element dV_z and the Θ_z form were obtained in [24], Proposition 3.1 and Proposition 3.3, respectively, by taking $dV_z := \mathcal{I}^{-1*}[\iota(\mathbf{n})d\mu_{\hat{g}}]$, where $d\mu_{\hat{g}}$ is the Riemannian volume form induced by the Riemannian metric $\hat{g}$ from (6).

3. Some properties of the indicatrices volume

In this section we study the volume of unit tangent spheres I_zM in complex Finsler manifolds (M, F), using the volume form obtained in the previous section in Proposition 1. Thus, we obtain conditions for a space to have indicatrix of constant volume, i.e. independent of the position variables $z \in M$, and we study this property for several cases of complex Finsler spaces.

The volume function of the indicatrix I_zM is considered to be

$$Vol(z) := Vol(I_zM) = \int_{I_zM} dV_z. \tag{8}$$

This definition is natural since the indicatrices volume of Hermitian manifolds is always equal to the volume of unit Euclidean sphere $\mathbb{S}^{2n-1}$ in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, as we can see from the following example.

Example 1. *The Euclidean norm*, $F|_z = \|\cdot\|$, i.e. $\|\eta\|^2 := |\eta|^2 = \sum \eta^i\bar{\eta}^i$, on the complex Minkowski space $(T'_z\mathbb{C}^n, F|_z)$ considered in an arbitrary point $z \in \mathbb{C}^n$. Since the volume element dV is independent of the choices of (η^i) coordinates and the Hermitian structure satisfies $g(\dot{\partial}_i, \dot{\partial}_j) = \delta_{ij}$, so that $\mathbf{g} = 1$, and by applying (8) and Proposition 1, we have

$$Vol(z) = \int_{\|\eta\|=1} \mathrm{Re}(\frac{\sqrt{-1}^{n^2}}{2^{n-1}}\sum_{i=1}^{n}(-1)^{i-1}\eta^i d\eta^1 \wedge .. \wedge \widehat{d\eta^i} \wedge .. \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge .. \wedge d\bar{\eta}^n).$$

By taking each $\eta^k = v^{2k-1} + \sqrt{-1}v^{2k}$, we obtain $\|\eta\|^2 = \sum_{k=1}^{2n}(v^k)^2$ and the integrand volume form $\omega = \sum_{i=1}^{2n}(-1)^{i-1}v^i dv^1 \wedge ... \wedge \widehat{dv^i} \wedge ... \wedge dv^{2n}$, which is exactly the volume element of a $(2n-1)$-sphere with radius $R = 1$. Thus, we obtain that $Vol(z) = Vol(\mathbb{S}^{2n-1})$.

As we could expect, indicatrices of complex Finsler manifolds have different properties from those considered in real Finsler manifolds. For instance, if for real Finsler manifolds the volume of the indicatrix can never exceed $Vol(\mathbb{S}^{n-1})$ if the real Finsler metric is positively homogeneous of degree 1, with equality for the case of the Euclidean norm (cf. [8], Proposition 14.9.1), in the complex Finsler case this result cannot be generalized, as we can see from the next example.

Example 2. Take the *complex version of the Antonelli–Shimada metric*,

$$F_{AS}^2 = L_{AS}(z, w; p, q) := e^{2\sigma}(|p|^4 + |q|^4)^{\frac{1}{2}}, \quad \text{with } p, q \neq 0,$$

on a domain D from $\widetilde{T'M}$, $\dim_C M = 2$, such that its metric tensor is nondegenerated. Here, z, w, p, q represent a relabel of the local coordinates z^1, z^2, η^1, η^2, respectively; the exponent $\sigma(z, w)$ is a real valued function; and $|\eta^k|^2 := \eta^k\bar{\eta}^k$, $\eta^k \in \{p, q\}$. A direct computation leads to (cf. [7])

$$\mathbf{g} = \det(g_{j\bar{k}}) = \frac{2e^{4\sigma(z,w)}|p|^2|q|^2}{|p|^4 + |q|^4}$$

and by taking $n = 2$ in Proposition 1, the volume element of the complex indicatrix becomes

$$dV_z = \frac{1}{4}\mathbf{g}(pdq \wedge d\bar{p} \wedge d\bar{q} - qdp \wedge d\bar{p} \wedge d\bar{q} + \bar{p}d\bar{q} \wedge dp \wedge dq - \bar{q}d\bar{p} \wedge dp \wedge dq).$$

If we consider the algebraic form of the complex coordinates $p = x + \mathrm{i}y$, $q = u + \mathrm{i}v$, $\mathrm{i} = \sqrt{-1}$, we obtain the volume function

$$Vol(z) = \int_\Sigma \mathbf{g}(xdy \wedge du \wedge dv - ydx \wedge du \wedge dv + udx \wedge dy \wedge dv - vdx \wedge dy \wedge dv), \tag{9}$$

where $\mathbf{g} = 2e^{8\sigma(z,w)}(x^2 + y^2)(u^2 + v^2)$ and the indicatrix condition $F(z, \eta) = 1$ becomes the equation of the surface $\Sigma : (x^2 + y^2)^2 + (u^2 + v^2)^2 = e^{-4\sigma(z,w)}$. If we denote $r = e^{-\sigma(z,w)}$, the coordinates of Σ can be changed using a type of Hopf coordinates: $x = r\cos\alpha\sqrt{\sin\theta}$, $y = r\sin\alpha\sqrt{\sin\theta}$, $u = r\cos\beta\sqrt{\cos\theta}$, $v = r\sin\beta\sqrt{\cos\theta}$, where $\theta \in [0, \frac{\pi}{2}]$, $\alpha, \beta \in [0, 2\pi]$. Thus, we easily obtain that $Vol(z) \equiv Vol(\mathbb{S}^3)$.

However, in the general case $Vol(z)$ does vary with z, as can be verified by the following example.

Example 3. Consider the case $n = 2$, $M = \mathbb{C}^2$, of local coordinates $z = (z^1, z^2)$ and $\eta = (p, q)$, and we define

$$F(z, \eta) := \sqrt{\lambda(z)(|p|^2 + |q|^2) + \sqrt{|p|^4 + |q|^4}},$$

where λ is an arbitrary smooth nonnegative function of the z variables only and $|\eta^k|^2 := \eta^k\bar{\eta}^k$, $\eta^k \in \{p, q\}$. When $\lambda \equiv$ *constant* the Minkowskian structure is obtained. It can be checked that $L = F^2$ is actually a smooth function on $\widetilde{T'M}$ and F is a complex Finsler function. Calculations give

$$(g_{i\bar{j}}) = \begin{pmatrix} \lambda + \frac{|p|^2(|p|^4+2|q|^4)}{(|p|^4+|q|^4)^{3/2}} & -\frac{\bar{p}q|p|^2|q|^2}{(|p|^4+|q|^4)^{3/2}} \\ -\frac{p\bar{q}|p|^2|q|^2}{(|p|^4+|q|^4)^{3/2}} & \lambda + \frac{|q|^2(2|p|^4+|q|^4)}{(|p|^4+|q|^4)^{3/2}} \end{pmatrix}.$$

Hence,

$$\mathbf{g} = \det(g_{i\bar{j}}) = \lambda^2 + \lambda\frac{(|p|^2+|q|^2)^3 - |p|^2|q|^2(|p|^2+|q|^2)}{(|p|^4+|q|^4)^{3/2}} + \frac{2|p|^2|q|^2}{|p|^4+|q|^4}.$$

If we consider the algebraic form $p = x + \mathrm{i}y$, $q = u + \mathrm{i}v$, $\mathrm{i} = \sqrt{-1}$, we obtain volume function (9), with $\Sigma : \lambda(x^2+y^2+u^2+v^2) + \sqrt{(x^2+y^2)^2+(u^2+v^2)^2} = 1$. By changing the coordinates of Σ as

$$x = \sqrt{\tfrac{\cos\theta}{1+\lambda(\cos\theta+\sin\theta)}}\sin\alpha, \qquad u = \sqrt{\tfrac{\sin\theta}{1+\lambda(\cos\theta+\sin\theta)}}\cos\beta,$$
$$y = \sqrt{\tfrac{\cos\theta}{1+\lambda(\cos\theta+\sin\theta)}}\cos\alpha, \qquad v = \sqrt{\tfrac{\sin\theta}{1+\lambda(\cos\theta+\sin\theta)}}\sin\beta,$$

with $\theta \in [0, \frac{\pi}{2}]$, $\alpha, \beta \in [0, 2\pi]$, the volume function of the complex indicatrix will have the form

$$Vol(z) = 2\pi^2 \int_0^{\pi/2} [\lambda^2 + \lambda(\cos\theta+\sin\theta)(1+\frac{\sin 2\theta}{2}) + \sin 2\theta]\frac{d\theta}{[1+\lambda(\cos\theta+\sin\theta)]^2}.$$

It can be seen that the result depends on $\lambda(z)$ and thus $Vol(z)$ is nonconstant.

In the following, using the ideas from [9], we study the condition under which a complex Finsler space has indicatrix with constant volume. Thus, we take $\eta \mapsto \sigma(\eta)$ a Minkowski norm on $\mathbb{C}^n$. Since the restriction of F to T'_zM gives a Minkowski norm F_z, any Finsler manifold may be viewed as a smoothly varying family of Minkowski spaces $\{(T'_zM, F_z)\}$. Thus, if we assign the same σ to each tangent space of $\mathbb{C}^n$, identifiable with $\mathbb{C}^n$ itself, every Minkowski space $(\mathbb{C}^n, \sigma)$ is a Finsler manifold and the resulting Finsler manifold is an example of locally Minkowski space.

In each $z \in M$ the indicatrix of F is strongly convex and $\sigma : \mathbb{C}^n \simeq T'_zM \to \mathbb{R}$, $\sigma(\eta) = F(z,\eta)$, is a strictly convex function, cf. [1], p. 167. The embedding $\varphi_z : \mathbb{I} \to I_zM, \varphi_z(\eta^1,...,\eta^n) := \eta^i \frac{\partial}{\partial z^i}\big|_z$, with $\mathbb{I} = \{\eta \in \mathbb{C}^n : \sigma(\eta) = 1\}$ the indicatrix of the Minkowski space $(\mathbb{C}^n, \sigma)$ and image just I_zM, allow us to switch the integrating domain in the volume function (8) over to $\mathbb{I}$. Thus,

$$Vol(z) = \int_{I_zM} dV_z = \int_{\mathbb{I}} \varphi_z^* dV_z.$$

Using Proposition 1 and the tangential map φ_z^*, we can express $Vol(z)$ as

$$Vol(z) = \frac{1}{2^{n-1}} \int_{\mathbb{I}} \mathbf{g}\mathrm{Re}[(-1)^{\frac{n^2}{2}} \sum_{i=1}^{n} (-1)^{i-1}\eta^i d\eta^1 \wedge .. \wedge \widehat{d\eta^i} \wedge .. \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge .. \wedge d\bar{\eta}^n]. \tag{10}$$

Therefore, the volume of the indicatrix is independent of z or $\bar{z}$ variables if the variational formula for $Vol(z)$ vanishes:

$$b^r(z,\bar{z})\frac{\partial Vol(z)}{\partial z^r} = 0 \quad \text{and} \quad b^r(z,\bar{z})\frac{\partial Vol(z)}{\partial \bar{z}^r} = 0 \tag{11}$$

for any vector fields $b^r\frac{\partial}{\partial z^r}$ and $b^r\frac{\partial}{\partial \bar{z}^r}$ on M, where the $b^r = b^r(z,\bar{z})$s are functions of z and $\bar{z}$ only on M. Since we work in local coordinates (z^i) and their induced global coordinates (η^i) on each tangent space T'_zM and the domain of integration is thus $\mathbb{I}$, which does not vary with z or $\bar{z}$, using (10) the conditions from (11) become

$$b^r(z,\bar{z})\frac{\partial \mathbf{g}}{\partial z^r} = 0 \quad \text{and} \quad b^r(z,\bar{z})\frac{\partial \mathbf{g}}{\partial \bar{z}^r} = 0.$$

By applying the Jacobi formula for the determinant of the positive definite Hermitian matrix $\left(g_{i\bar{j}}(z,\eta)\right)$, the above equalities are rewritten as

$$b^r(z,\bar{z})\mathbf{g}\Gamma^i_{ir} = 0 \quad \text{and} \quad b^r(z,\bar{z})\mathbf{g}\Gamma^i_{i\bar{r}} = 0,$$

with $\Gamma^i_{ir} = g^{\bar{m}i}\frac{\partial g_{i\bar{m}}}{\partial z^r}$ and $\Gamma^i_{i\bar{r}} = g^{\bar{m}i}\frac{\partial g_{i\bar{m}}}{\partial \bar{z}^r}$. Since $\mathbf{g} \neq 0$, $Vol(z) = const.$ if and only if $b^r(z,\bar{z})\Gamma^i_{ir} = b^r(z,\bar{z})\Gamma^i_{i\bar{r}} = 0$. Moreover, we can state:

Theorem 1 *Let (M,F) be a complex Finsler manifold, $z \in M$ an arbitrary point, and $Vol(z)$ the volume function of the complex indicatrix I_zM. Then, for any vector fields $b^r\frac{\partial}{\partial z^r}$ and $b^r\frac{\partial}{\partial \bar{z}^r}$ on M, with $b^r = b^r(z,\bar{z})$, we have*

$$b^r(z,\bar{z})\frac{\partial Vol(z)}{\partial z^r} = \frac{1}{2^{n-1}}\int_{\mathbb{I}} \mathbf{g}b^r\Gamma^i_{ir}\mathrm{Re}[(-1)^{\frac{n^2}{2}}\sum_{i=1}^n (-1)^{i-1}\eta^i d\eta^1\wedge ... \wedge \widehat{d\eta^i} \wedge ... \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge ... \wedge d\bar{\eta}^n]$$

and, respectively,

$$b^r(z,\bar{z})\frac{\partial Vol(z)}{\partial \bar{z}^r} = \frac{1}{2^{n-1}}\int_{\mathbb{I}} \mathbf{g}b^r\Gamma^i_{i\bar{r}}\mathrm{Re}[(-1)^{\frac{n^2}{2}}\sum_{i=1}^n (-1)^{i-1}\eta^i d\eta^1\wedge ... \wedge \widehat{d\eta^i} \wedge ... \wedge d\eta^n \wedge d\bar{\eta}^1 \wedge ... \wedge d\bar{\eta}^n]$$

where $\Gamma^i_{ir} = g^{\bar{m}i}\frac{\partial g_{i\bar{m}}}{\partial z^r}$ and $\Gamma^i_{i\bar{r}} = g^{\bar{m}i}\frac{\partial g_{i\bar{m}}}{\partial \bar{z}^r}$.

Corollary 1 *Let (M,F) be a complex Finsler manifold. If $\Gamma^i_{ir} = \Gamma^i_{i\bar{r}} = 0$, the volume function $Vol(z)$ of the complex indicatrix is constant.*

Remark. Any local Minkowski metric verifies the corollary's conditions; thus, $Vol(z)$ is constant (see Example 1 and Example 2 with σ a constant function). However, in Example 2, $Vol(z)$ is constant for any function $\sigma(z,w)$ and thus there are other types of complex Finsler metrics with constant $Vol(z)$. For instance, further we will provide an example of a pure Hermitian metric (i.e. $g = g(z)$) for which we obtain $Vol(z) \equiv Vol(S^{2n-1})$.

Example 4. *A pure Hermitian metric.* Take $M = \mathbb{C}^2$ endowed with a purely Hermitian metric

$$F^2(z,\eta) = a(z,w)|p|^2 + b(z,w)|q|^2,$$

with a, b real valued functions and local coordinates $z = (z,w)$, $\eta = (p,q)$. It can be easily verified that $g_{1\bar{1}} = a(z,w)$, $g_{1\bar{2}} = g_{2\bar{1}} = 0$, $g_{2\bar{2}} = b(z,w)$, such that F represents a pure Hermitian metric, and thus $\mathbf{g} = ab$. Considering the algebraic form $p = x + \mathrm{i}y$, $q = u + \mathrm{i}v$, we obtain the volume (9) with the surface $\Sigma : a(x^2+y^2) + b(u^2+v^2) = 1$. Taking $x = a^{-1/2}\cos\alpha\sin\theta$, $y = a^{-1/2}\sin\alpha\sin\theta$, $u = b^{-1/2}\cos\beta\cos\theta$, $v =$

$b^{-1/2}\sin\beta\cos\theta$, where $\theta\in[0,\frac{\pi}{2}]$, $\alpha,\beta\in[0,2\pi]$, we easily obtain $Vol(z)\equiv Vol(\mathbb{S}^3)$. The same result is obtained if we choose to use the spherical coordinates $x=a^{-1/2}\cos\alpha$, $y=a^{-1/2}\sin\alpha\cos\beta$, $u=b^{-1/2}\sin\alpha\sin\beta\cos\theta$, $v=b^{-1/2}\sin\alpha\sin\beta\sin\theta$, with $\theta\in[0,\pi]$, $\alpha,\beta\in[0,2\pi]$.

By generalizing to an n-dimensional complex manifold M, take the purely Hermitian metric $F^2(z,\eta)=\sum_{i=1}^{n}a_i(z)|\eta^i|^2$, with $a_i(z)$s real valued functions and $\mathbf{g}=\prod_{i=1}^{n}a_i(z)$. Using $\eta^j=x^j+\mathrm{i}y^j$, condition $F=1$ is equivalent to $\sum_{i=1}^{n}a_i(x^i+y^i)=1$. If we apply an adapted form of the spherical coordinates as $x^1=a_1^{-1/2}\cos\alpha_1$, $y^1=a_1^{-1/2}\sin\alpha_1\cos\alpha_2$, $x^2=a_2^{-1/2}\sin\alpha_1\sin\alpha_2\cos\alpha_3$, $y^2=a_2^{-1/2}\sin\alpha_1\sin\alpha_2\sin\alpha_3\cos\alpha_4$, .., $x^n=a_n^{-1/2}\sin\alpha_1..\sin\alpha_{2n-2}\cos\alpha_{2n-1}$ and $y^n=a_n^{-1/2}\sin\alpha_1..\sin\alpha_{2n-2}\sin\alpha_{2n-1}$, with $\alpha_{2n-1}\in[0,\pi]$, $\alpha_1,..,\alpha_{2n-2}\in[0,2\pi]$, we obtain that $Vol(z)=Vol(S^{2n-1})$.

However, the variational formula of $Vol(z)$ does not offer much information about classes of complex Finsler manifolds that may verify the constant volume conditions. The following result slightly improves the one from [24].

Proposition 2 *Let (M,F) be a weakly Kähler manifold, with $L^i_{jk}=L^i_{jk}(z)$. Then $Vol(z)$ is constant.*

Proof Since (M,F) is a weakly Kähler manifold, according to [1], p. 101, Corollary 2.4.2, for any $p\in M$ and $X_p\in\widetilde{T'_pM}$, there exists a unique geodesic γ through (p,X_p), such that $\gamma(0)=p$ and $\dot\gamma(0)=X_p\in I_pM$. Thus, given p and q in M, we can take $\gamma(t)$, $t\in[0,1]$ as a unique geodesic such that $\gamma(0)=p$ and $\gamma(1)=q$, which for the weakly Kähler metric and its Chern–Finsler connection D satisfies ([1], p. 101, [20], p. 79)

$$D_{\dot\gamma^h+\overline{\dot\gamma^h}}\dot\gamma^h=0,$$

with $\dot\gamma$ the tangent map of γ and $\dot\gamma^h=l^h(\dot\gamma(t))$ the horizontal lift of the vector $\dot\gamma(t)$. In local coordinates it becomes

$$[\ddot\gamma^i+\dot\gamma^jL^i_{jk}(\gamma,\dot\gamma)\dot\gamma^k]\delta_i=0. \tag{12}$$

Let $P_t:T'_pM\to T'_{\gamma(t)}M$, $\dot\gamma(0)\mapsto\dot\gamma(t)$, be the parallel translation induced by (12). By taking the Chern–Finsler horizontal coefficients L^i_{jk} to be dependent only on the position z, the parallel translation P_t is a complex linear isomorphism and an isometry, so we have $||\dot\gamma(t)||=F(\gamma(t),\dot\gamma(t))=F(p,\dot\gamma(0))=||\dot\gamma(0)||$.

Set (z^i,η^i), (ξ^i,ζ^i) local coordinates on T'_pM, $T'_{\gamma(t)}M$, respectively. In a local chart consider $\dot\gamma(0)=\frac{d\gamma^i}{dt}(0)\frac{\partial}{\partial z^i}|_{\gamma(0)}$ and denote $X^i_0:=\frac{d\gamma^i}{dt}(0)$ and by $\{e_i\}_{i=1,n}$ the natural basis $\{\frac{\partial}{\partial z^i}\}_{i=1,n}$ in the fixed point p. If we take $E_i(t)=P_t(e_i)$ to be the translated basis for $T'_{\gamma(t)}M$, there exists a nonsingular matrix $(A^j_i(t))$ such that $E_i(t)=A^j_i(t)\frac{\partial}{\partial\xi^j}$. If we consider $\dot\gamma(t)=X^i(t)\frac{\partial}{\partial z^i}|_{\gamma(t)}=X^i_0E_i(t)$ along γ, where $X^i(t)=\frac{d\gamma^i}{dt}(t)$, the condition from (12) becomes $\frac{dX^i}{dt}+X^jL^i_{jk}(\xi,X^i_0A^j_i(t)\frac{\partial}{\partial\xi^j})X^k=0$. We observe that, in general, the coefficients L^i_{jk} are dependent on the translated frame, but in the considered case, L^i_{jk} depending only on position, that does not happen. Thus, by applying it to $E_m(t)$, we obtain

$$\frac{dA^i_m}{dt}=-A^j_mA^k_mL^i_{jk}(\xi).$$

Now we follow the same steps as in [24]. Since $\zeta^i = A^i_j \eta^j$, the tangent and cotangent application of P_t satisfies

$$P_{t*}\frac{\partial}{\partial \eta^i} = A^j_i \frac{\partial}{\partial \zeta^j}, \quad P^*_t d\zeta^i = A^i_j d\eta^j.$$

Relative to the Hermitian metric $\mathcal{G}$ from (2), using the last three relations, it can be easily checked that

$$\frac{d}{dt}\left[\mathcal{G}_{P_t(\eta)}\left(P_{t*}\frac{\partial}{\partial \eta^l}, P_{t*}\frac{\partial}{\partial \eta^{\bar{m}}}\right)\right] = \frac{d}{dt}\left[g_{i\bar{j}}(\gamma(t), \dot{\gamma}(t))A^i_l A^{\bar{j}}_{\bar{m}}\right] = 0.$$

This implies $\mathcal{G}_{P_t(\eta)}(P_{t*}(\dot{\partial}_i), P_{t*}(\dot{\partial}_{\bar{j}})) = \mathcal{G}_\eta(\dot{\partial}_i, \dot{\partial}_{\bar{j}})$, $\det[\mathcal{G}_{\dot{\gamma}(t)}(P_{t*}(\dot{\partial}_i), P_{t*}(\dot{\partial}_{\bar{j}}))] = \det[\mathcal{G}_\eta(\dot{\partial}_i, \dot{\partial}_{\bar{j}})]$.

Take $B_{\gamma(t)}(r) := \{\zeta \in T'_{\gamma(t)}M : F(\gamma(t), \zeta) < r\}$, $B_{\gamma(t)}(r) = B_{\gamma(0)}(r)$, and using Stokes' formula, the above results, and the fact that $d(dV_z) = 2nd\mu|_z$ for any $z \in M$, we get $Vol(\gamma(t)) = Vol(p)$. If $t = 1$, we obtain that $Vol(p) = Vol(q)$, for any $p, q \in M$. □

Recall that, cf. [3], a complex Finsler space is Berwald iff it is Kähler and L^i_{jk} depends only on position. Since the Kähler condition implies weakly Kähler, we can state:

Corollary 2 *Let (M, F) be a complex Berwald manifold. Then $Vol(z)$ is constant.*

Remark. There are two senses regarding the definition of a complex Berwald space. The first definition of the complex Berwald manifold was introduced by Aikou in [2], as a complex Finsler manifold whose Chern–Finsler coefficients L^i_{jk} are functions of position (z^i) alone. However, in a later paper, [3], Aikou approached this problem again and considered the complex Finsler spaces having $L^i_{jk}(z)$ as being modeled on a Minkowski space, and by a complex Berwald manifold, he defined a complex Finsler manifold modeled on a Minkowski space, together with the Kähler condition of its associated Hermitian metric (which is equivalent to the Kähler condition of the complex Finsler function). The latter one is the sense that we are using in this paper, while the first one leads to the generalized Berwald spaces according to [4, 5].

We remark that conditions from Proposition 2 are only sufficient. Next, we want to exemplify this.

First, if we take into consideration that in Example 2 according to [7] the L^i_{jk} nonzero coefficients of the Chern–Finsler connection of the Antonelli–Shimada complex metric are

$$L^1_{11} = L^2_{21} = 2\frac{\partial \sigma}{\partial z}, \quad L^1_{12} = L^2_{22} = 2\frac{\partial \sigma}{\partial w},$$

we observe that the local coefficients L^i_{jk} depend only on z and w, but the metric L_{AS} is not Berwald, because in general, it is not Kähler. However, the metric L_{AS} is one generalized Berwald, since the spray coefficients $G^z = (\frac{\partial\sigma}{\partial z}\eta + \frac{\partial\sigma}{\partial w}\theta)\eta$ and $G^w = (\frac{\partial\sigma}{\partial z}\eta + \frac{\partial\sigma}{\partial w}\theta)\theta$ are holomorphic in η. Thus, we found an example of a generalized Berwald manifold for which $Vol(z)$ is constant, another one being given in the following example.

Example 5. *A complex Finsler space with Randers metric.*

We adjust the example from [6] and we consider a two-dimensional manifold $M = \mathbb{C}^2$ a purely Hermitian metric

$$\alpha^2 = e^{z+\bar{z}}|p|^2 + e^{z+\bar{z}+w+\bar{w}}|q|^2, \tag{13}$$

and we choose the $(1,0)$-differential form β as

$$\beta = e^z p. \tag{14}$$

Then $a_{1\bar{1}} = e^{z+\bar{z}}$, $a_{2\bar{2}} = e^{z+\bar{z}+w+\bar{w}}$, $a_{1\bar{2}} = a_{2\bar{1}} = 0$, $\det(a_{i\bar{j}}) = e^{2(z+\bar{z})+w+\bar{w}}$, $a^{\bar{1}1} = e^{-(z+\bar{z})}$, $a^{\bar{2}2} = e^{-(z+\bar{z}+w+\bar{w})}$, $a^{\bar{1}2} = a^{\bar{2}1} = 0$, $|\beta|^2 = e^{z+\bar{z}}|p|^2$ and thus $b_1 = e^z$, $b^1 = e^{-z}$, $b_2 = b^2 = 0$, and $||b|| = 1$, where we used $b^i = a^{\bar{j}i}b_{\bar{j}}$ and $||b||^2 = a^{\bar{j}i}b_i b_{\bar{j}}$. In addition, we calculate $\Gamma_{l\bar{r}\bar{m}} = \frac{\partial a_{l\bar{m}}}{\partial \bar{z}^r} - \frac{\partial a_{l\bar{r}}}{\partial \bar{z}^m}$ and we obtain $\Gamma_{l\bar{r}\bar{m}} = 0$, except for the coefficients $\Gamma_{2\bar{1}\bar{2}} = -\Gamma_{2\bar{2}\bar{1}} = e^{z+\bar{z}+w+\bar{w}} \neq 0$. Thus, metric (13) is not Kähler.

Now we are able to construct the complex Randers metric

$$F = \sqrt{e^{z+\bar{z}}|p|^2 + e^{z+\bar{z}+w+\bar{w}}|q|^2} + \sqrt{e^{z+\bar{z}}|p|^2}, \tag{15}$$

which is nonpurely Hermitian, since $\alpha^2||b||^2 \neq |\beta|^2$. Some computations lead to the conclusion that metric (15) satisfies the condition

$$\left(\bar{\beta} l_{\bar{r}} \frac{\partial b^{\bar{r}}}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r\right) \eta^j = \left(\bar{\beta} l_{\bar{1}} \frac{\partial b^{\bar{1}}}{\partial z^j} + \beta \frac{\partial b_{\bar{1}}}{\partial z^j} \bar{\eta}^1\right) \eta^j = 0,$$

with $l_{\bar{r}} = a_{i\bar{r}}\eta^i$, and so, from [4], Theorem 4.2, this metric is generalized Berwald. However, there exists $\Gamma_{2\bar{2}\bar{1}} b^{\bar{1}} = -e^{z+w+\bar{w}} \neq 0$, and thus the condition $\Gamma_{l\bar{r}\bar{m}} b^{\bar{m}} = 0$ cannot be fulfilled, and according to [6] it is not a complex Douglas metric, even if $\Omega_{\bar{m}} = 0$, $m = 1, 2$.

By direct calculus, using $a_{1\bar{1}}|p|^2 = |\beta|^2$ and $a_{2\bar{2}}|q|^2 = \alpha^2 - |\beta|^2$, we obtain

$$\begin{aligned}
g_{1\bar{1}} &= F^2 \frac{a_{1\bar{1}}}{\sqrt{\alpha^2}|\beta|} - \frac{F^4}{2} \frac{a_{1\bar{1}}}{\alpha^2\sqrt{\alpha^2}|\beta|} + 2F^2 \frac{a_{1\bar{1}}}{\alpha^2}, \\
g_{1\bar{2}} &= \overline{g_{2\bar{1}}} = \frac{F}{2} \frac{a_{1\bar{1}} a_{2\bar{2}} \bar{p} q}{\alpha^2\sqrt{\alpha^2}|\beta|} (\alpha^2 - |\beta|), \\
g_{2\bar{2}} &= F \frac{a_{2\bar{2}}}{\sqrt{\alpha^2}} - \frac{1}{2} \frac{a_{2\bar{2}}|q|^2|\beta|}{\alpha^2\sqrt{\alpha^2}},
\end{aligned}$$

and

$$\mathbf{g} = \det(g_{i\bar{j}}) = \frac{F^4}{2} \frac{\det(a_{i\bar{j}})}{\alpha^2\sqrt{\alpha^2}|\beta|} > 0, \quad i, j = 1, 2.$$

We obtain the volume function $Vol(z)$ as in (9) for the algebraic form of the complex coordinates $p = x + \mathrm{i}y$, $q = u + \mathrm{i}v$, $\mathrm{i} = \sqrt{-1}$ and surface $\Sigma : \sqrt{a_{1\bar{1}}(x^2+y^2) + a_{2\bar{2}}(u^2+v^2)} + \sqrt{a_{1\bar{1}}(x^2+y^2)} = 1$. If we denote $a_{1\bar{1}}(x^2+y^2) + a_{2\bar{2}}(u^2+v^2) = \cos^4\theta$ and $a_{1\bar{1}}(x^2+y^2) = \sin^4\theta$, $\theta \in [0, \frac{\pi}{4}]$, the coordinates of Σ can be changed as $x = a_{1\bar{1}}^{-1/2}\cos\alpha\sin^2\theta$, $y = a_{1\bar{1}}^{-1/2}\sin\alpha\sin^2\theta$, $u = a_{2\bar{2}}^{-1/2}\cos\beta\sqrt{\cos 2\theta}$, $v = a_{2\bar{2}}^{-1/2}\sin\beta\sqrt{\cos 2\theta}$, where $\theta \in [0, \frac{\pi}{4}]$, $\alpha, \beta \in [0, 2\pi]$. Thus, we easily obtain $\mathbf{g} = \frac{1}{2}\frac{a_{1\bar{1}}a_{2\bar{2}}}{\cos^6\theta\sin^2\theta}$ and by direct calculus we get that the volume function has constant value $Vol(z) = 2\pi^2$.

Remark. In [6] it can be found that any complex Randers–Douglas space of dimension two is a complex Berwald space, and thus, according to Corollary 2, we can state:

Corollary 3 *Let* (M, F) *be a complex Randers–Douglas space with* $\dim_C M = 2$*. Then the volume function* $Vol(z)$ *of its complex indicatrix is constant.*

Application: The length of the complex indicatrix

A *Riemann surface* is a connected one-complex dimensional analytic manifold, i.e. a two-real-dimensional connected manifold M, with a complex structure on it. Take the local chart (U, z), the holomorphic tangent space determined by (z, η), where the η coordinate comes from z through $\eta\frac{\partial}{\partial z}$, and a Finsler complex function F with $g = \frac{\partial^2 L}{\partial\eta\partial\bar{\eta}}$. By the $(0,0)$-homogeneity condition (1) of the metric tensor g, it follows that for the Riemannian surface (M, F) the complex Finsler metric is Hermitian. Then we consider the indicatrix as a one-real-dimensional curve of the one-complex-dimensional Finsler space, $I = \{\eta \in T'M : g\eta\bar{\eta} = 1\}$, and we introduce the notion of the indicatrix length as the volume function $Vol(z)$ from (8) with $n = 1$ in Proposition 1.

The Hermitian metric of the Minkowski plane $T'_z M$ is $\mathcal{G} = g d\eta \otimes d\bar{\eta}$, which induces $G = \mathrm{Re}\mathcal{G}$ on I, $N = \frac{1}{F}\eta\frac{\partial}{\partial\eta} + \frac{1}{F}\bar{\eta}\frac{\partial}{\partial\bar{\eta}}$, and the indicatrix volume form $dV = \frac{\mathrm{i}}{2F^2} g(\eta d\bar{\eta} - \bar{\eta} d\eta)$. Using it, we define the element of arc length on the standard unit circle $\mathbb{S} = \{\eta \in \mathbb{C} : |\eta|^2 = 1\}$, $ds = \frac{g}{F^2}\mathrm{Re}(\mathrm{i}\eta d\bar{\eta})$ and it can be easily verified that

$$\mathcal{L} = \int_{\mathbb{S}} \frac{\mathrm{i}}{2F^2} g(\eta d\bar{\eta} - \bar{\eta} d\eta) = \int_{x^2+y^2=1} \frac{g}{F^2}(x dy - y dx) = 2\pi,$$

where we used the algebraic form of the complex coordinate $\eta = x + \mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$.

Then the length of the indicatrix I is $\mathcal{L}(z) = \int_I \frac{\mathrm{i}}{2F^2} g(\eta d\bar{\eta} - \bar{\eta} d\eta)$. Since from (1) we have $F^2 = g\eta\bar{\eta}$, the length formula becomes

$$\mathcal{L}(z) = \int_{F=1} \frac{\mathrm{i}}{2|\eta|^2}(\eta d\bar{\eta} - \bar{\eta} d\eta). \tag{16}$$

As $dV|_\eta = \frac{\mathrm{i}}{2|\eta|^2}(\eta d\bar{\eta} - \bar{\eta} d\eta)$ is constant along any complex ray that emanates from the origin, i.e. it is invariant under rescaling in η, $dV|_{\lambda\eta} = dV|_\eta$, $\forall\lambda \in \mathbb{C}$, we expect its integrals over I and $\mathbb{S}$ to give the same answer. Our intuition is borne out by the fact that the new integrand is a closed 1-form on $\mathbb{C}^*$ and, together with an application of Stokes' theorem, by integrating dV over I and $\mathbb{S}$ the same answer of the length function is obtained. By direct calculus, if we take $\eta = x + \mathrm{i}y$, the formula of the indicatrix length becomes

$$\mathcal{L}(z) = \int_{g(z,x,y)(x^2+y^2)=1} \frac{1}{(x^2+y^2)}(x dy - y dx),$$

and if $x = \frac{\cos\theta}{\sqrt{g(z,x,y)}}$, $y = \frac{\sin\theta}{\sqrt{g(z,x,y)}}$, $\theta \in [0, 2\pi)$, we get $\mathcal{L}(z) = 2\pi$, as deduced above. Thus, we can state:

Proposition 3 *The length of the complex indicatrix of any Riemannian Finsler surface has constant value equal to* 2π*.*

4. On the volume of submersed manifolds of I_zM

In [18] we find that a submersion from a CR-submanifold $\tilde{M}$ of a Kähler manifold M onto an almost Hermitian manifold M' is a *Riemannian submersion* $\phi : \tilde{M} \to M'$ with: (i) $\mathcal{D}^\perp$ is the kernel of ϕ_*; (ii) $\phi_* : \mathcal{D}_u \to T_{\phi(u)}M'$ is complex isometry for every $u \in \tilde{M}$. This definition is given for the case $(T\tilde{M})^\perp = J(\mathcal{D}^\perp)$.

Since I_zM is a CR-hypersurface of the Kähler manifold T'_zM, from [12], we deduce $\mathcal{D}^\perp = span\{JN\}$ on I_zM, with N the unit normal vector field from (4), $J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k$, $J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}$, $\mathrm{i} := \sqrt{-1}$ the complex structure defined on T'_zM, and $\mathcal{D}$ the maximal J-invariant subspace, such that $T_R(I_zM) = \mathcal{D} \oplus \mathcal{D}^\perp$. Then we take the tangent unit vector

$$\xi = JN = \mathrm{i}\left(l^k\dot{\partial}_k - l^{\bar{k}}\dot{\partial}_{\bar{k}}\right), \quad \mathrm{i} := \sqrt{-1}.$$

Consider then the submersion $\phi' : I_zM \to M'$ onto an almost Hermitian manifold M', named *submersed manifold*, such that $\phi'_* : T_R(I_zM) \to T_RM'$ fulfills $\phi_*(\xi) = 0$ and $\phi_* : \mathcal{D}_\eta \to T_{\phi(\eta)}M'$ is a complex isometry for every $\eta \in I_zM$.

Considering that $F = 1$ on the complex indicatrix, $\xi = \mathrm{i}\left(\eta^k\dot{\partial}_k - \eta^{\bar{k}}\dot{\partial}_{\bar{k}}\right)$ is the outward-pointing normal to any submersed manifold M' of I_zM, with $\dim_R M' = 2n-2$, and thus the volume element $d\omega$ of the induced metric on M' is given by the contraction of the indicatrix volume form dV_z with ξ in the first slot $dV_z(N, ...)$, as

$$\begin{aligned} d\omega &= dV_z\left(\mathrm{i}\left(\eta^k\dot{\partial}_k - \eta^{\bar{k}}\dot{\partial}_{\bar{k}}\right), ...\right) = \iota_\xi(dV_z) \\ &= \frac{\mathbf{g}}{2^{n-1}}\mathrm{Re}[(-1)^{\frac{n^2+1}{2}}\sum_{\substack{j,k=1\\k\neq j}}^{n}(-1)^{j+k}\eta^j\eta^k d\eta^1\wedge..\wedge\widehat{d\eta^k}\wedge..\wedge\widehat{d\eta^j}\wedge..\wedge d\eta^n\wedge d\bar{\eta}^1\wedge..\wedge d\bar{\eta}^n \\ &\quad + (-1)^{\frac{(n+1)^2}{2}}\sum_{j,k=1}^{n}(-1)^{j+k}\eta^j\bar{\eta}^k d\eta^1\wedge..\wedge\widehat{d\eta^j}\wedge..\wedge d\eta^n\wedge d\bar{\eta}^1\wedge..\wedge\widehat{d\bar{\eta}^k}\wedge..\wedge d\bar{\eta}^n]. \end{aligned}$$

Since the volume element is quite complicated, it is difficult to calculate its volume. However, we can construct $\phi : I_zM \to P_zM$, $\phi(\eta/F) := [\eta]$ a submersion onto the punctured complex projective bundle P_zM, with the tangential map

$$\phi_* : T_R(I_zM) \to T_R(P_zM), \qquad v_\alpha \mapsto e_\alpha := \mathrm{Re}\{(v^j_\alpha - \frac{\mathrm{i}\sqrt{2}}{2}\eta^j)\dot{\partial}_j\},$$

where $T_R(I_zM) = span\{v_\alpha\}$, $v_\alpha = \frac{1}{F}(v^j_\alpha\dot{\partial}_j + v^{\bar{j}}_\alpha\dot{\partial}_{\bar{j}})$, with v^j_α, $v^{\bar{j}}_\alpha$ 1-homogeneous functions in η variables, with $\alpha \in \{1, .., 2n-1\}$, which verify $G_R(v_\alpha, v_\beta) = \frac{1}{L}\mathrm{Re}\{v^j_\alpha v^{\bar{k}}_\beta g_{j\bar{k}}\}$. We take $\xi = v_{2n-1}$, which implies $v^j_{2n-1} = \mathrm{i}\eta^j$, $v^j_\alpha\eta_j = 0$, $v^{\bar{j}}\eta_{\bar{j}} = 0$, and $\phi_*(\xi) = 0$. Since the Sasaki type lift of the metric tensor $g_{i\bar{j}}$ on T'_zM

$$G = g_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^k + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j$$

descends on the complex projective space $PM = T'M/_{\mathbb{C}^*}$ to the metric [13]

$$\tilde{G} = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + (\log L)_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j,$$

we can state that the Hermitian metric on T'_zM given by $\mathcal{G}$ descends to the metric $h = (\log L)_{j\bar{k}} d\eta^j \otimes d\bar{\eta}^k$ on P_zM, which is equivalent to

$$h = (\frac{1}{L} g_{j\bar{k}} - \frac{1}{L^2} \eta_j \eta_{\bar{k}}) d\eta^j \otimes d\bar{\eta}^k.$$

Therefore, we obtain $h(e_a, e_b) = \frac{1}{L}\mathrm{Re}\{v^j_\alpha v^{\bar{k}}_\beta g_{j\bar{k}}\}$, so that $h(\phi_*(v_a), \phi_*(v_b)) = G_R(v_a, v_b)$, $\forall a, b \in \{1, .., 2n-2\}$, i.e. ϕ_* is an isometry.

Thus, between the tangent bundle of any submersed manifold M' of I_zM and the tangent bundle of P_zM exists the isometry $\phi'_* \circ \phi_*^{-1}$. Considering that according to [23] the volume function of the projectivized tangent bundle P_zM is constant, we state:

Proposition 4 *Let (M, F) be a complex Finsler manifold with I_zM the complex indicatrix. The volume function of any almost Hermitian submersed manifold from I_zM has constant value.*

References

[1] Abate M, Patrizio G. Finsler Metrics — A Global Approach. Berlin, Germany: Springer-Verlag, 1994.

[2] Aikou T. On complex Finsler manifolds. Rep Fac Sci Kagoshima Univ Math Phys Chem 1991; 24: 9-25.

[3] Aikou T. Some remarks on locally conformal complex Berwald spaces. In: Finsler Geometry (Seattle, WA, 1995). Providence, RI, USA: American Mathematical Society, 1996, pp. 109-120.

[4] Aldea N, Munteanu G. On complex Landsberg and Berwald spaces. J Geom Phys 2012; 62: 368-380.

[5] Aldea N, Munteanu G. On projective invariants of the complex Finsler spaces. Differential Geom Appl 2012; 30: 562-575.

[6] Aldea N, Munteanu G. On complex Douglas spaces. J Geom Phys 2013; 66: 80-93.

[7] Aldea N, Munteanu G. The main invariants of a complex Finsler space. Acta Math Sci Ser B Engl Ed 2014; 34: 995-1011.

[8] Bao D, Chern SS, Shen Z. An Introduction to Riemann-Finsler Geometry. New York, NY, USA: Springer-Verlag, 2000.

[9] Bao D, Shen Z, On the volume of unit tangent spheres in a Finsler manifold. Results Math 1994; 26: 1-17.

[10] Bejancu A. CR submanifolds of a Kaehler manifold. I. P Am Math Soc 1978; 69: 135-142.

[11] Bejancu A. CR submanifolds of a Kaehler manifold. II. T Am Math Soc 1979; 250: 333-345.

[12] Bejancu A. Geometria CR-subvarietăţilor. Mem Secţ Ştiinţ Acad Repub Soc România Ser IV 1982; 5: 53-101.

[13] Bland J, Kalka M. Variations of holomorphic curvature for Kähler Finsler metrics. In: Finsler Geometry (Seattle, WA, 1995). Providence, RI, USA: American Mathematical Society, 1996, pp. 121-132.

[14] Chen B, Shen Y. Kähler Finsler metrics are actually strongly Kähler. Chin Ann Math Ser B 2009; 30: 173-178.

[15] Chen BY. Geometry of Submanifolds and Its Applications. Tokyo, Japan: Science University of Tokyo, 1981.

[16] Dragomir S. Cauchy-Riemann submanifolds of Kaehlerian Finsler spaces. Collect Math 1989; 40: 225-240.

[17] Dragomir S, Grimaldi R. On Rund's connection. Note Mat 1995; 15: 85-98.

[18] Kobayashi S. Submersions of CR-submanifolds. Tohoku Math J 1987; 39: 95-100.

[19] Matsumoto M. Foundations of Finsler Geometry and Special Finsler Spaces. Shiga-ken, Japan: Kaiseisha Press, 1986.

[20] Munteanu G. Complex Spaces in Finsler, Lagrange and Hamilton Geometries. Dordrecht, the Netherlands: Kluwer Academic Publishers, 2004.

[21] Nomizu K, Yano K. On circles and spheres in Riemannian geometry. Math Ann 1974; 210: 163-170.

[22] Rund H. The Differential Geometry of Finsler Spaces. Berlin, Germany: Springer-Verlag, 1959.

[23] Yan R. On the volume of the projectivized tangent bundle in a complex Finsler manifold. Arch Math (Basel) 2006; 86: 458-463.

[24] Zhao W. A Gauss-Bonnet-Chern theorem for complex Finsler manifold. Sci China Math 2016; 59: 515-530.