Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 282 – 292

doi:10.3906/mat-1511-31

Koebe sets for the class of functions convex in two directions

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Received: 09.11.2015 • **Accepted/Published Online:** 20.04.2016 • **Final Version:** 03.04.2017

Abstract: In this paper, we consider a class K_α of all functions f univalent in the unit disk Δ that are normalized by $f(0)=f'(0)-1=0$ while the sets $f(\Delta)$ are convex in two symmetric directions: $e^{i\alpha\pi/2}$ and $e^{-i\alpha\pi/2}$, $\alpha\in[0,1]$. It means that the intersection of $f(\Delta)$ with each straight line having the direction $e^{i\alpha\pi/2}$ or $e^{-i\alpha\pi/2}$ is either a compact set or an empty set. We find the Koebe set for K_α. Moreover, we perform the same operation for functions in $K_{\beta,\gamma}$, i.e. for functions that are convex in two fixed directions: $e^{i\beta\pi/2}$ and $e^{i\gamma\pi/2}$, $-1\le\beta\le\gamma\le1$.

Key words: Univalent functions, convexity in a direction

1. Introduction

Let $D\subset\mathbb{C}$ be a domain. The set D is said to be convex in the direction of $e^{i\gamma}$ if the intersection of D with each straight line having the direction $e^{i\gamma}$ is either a compact set (i.e. a straight line, a ray, or a segment) or an empty set. Since the convexity of the set D in the direction of $-e^{i\gamma}$ is equivalent to the convexity of D in the direction of $e^{i\gamma}$, one can discuss only the case $\gamma\in[-\pi/2,\pi/2]$.

Let $\mathcal{A}$ be the class of all functions analytic in $\Delta\equiv\{\zeta\in\mathbb{C}:|\zeta|<1\}$ and normalized by $f(0)=f'(0)-1=0$. Denote by K_α the class of those univalent functions $f\in\mathcal{A}$ for which $f(\Delta)$ is a set convex in two symmetric directions: $e^{i\alpha\pi/2}$ and $e^{-i\alpha\pi/2}$, $\alpha\in[0,1]$. For $\alpha=0$ and $\alpha=1$ the set K_α reduces to two well-known families: $K_0=K(1)$ consisting of functions convex in the direction of the real axis and $K_1=K(i)$ consisting of functions convex in the direction of the imaginary axis. These classes were discussed, among others, by Hengartner and Schober, Goodman and Saff, Ciozda, Brown, and Prokhorov (see [1–17]). The set K_α, $\alpha\in(0,1)$ has not been discussed yet.

In this paper, we shall find the Koebe set for K_α. Let us recall that for a given class $A\subset\mathcal{A}$, the Koebe set for A is a set of the form $K(A)=\bigcap_{f\in A}f(\Delta)$. Usually, the Koebe set is a domain; in this case we also call it the Koebe domain.

The main tool used to this end is the technique of subordination. Recall that for two analytic functions f and g, f is subordinated to g if and only if there exists a function ω analytic in Δ, $\omega(0)=0$, $|\omega(z)|<1$ for all $z\in\Delta$, such that $f(z)=g(\omega(z))$. In this case, we write $f\prec g$. Additionally, if the function g is univalent then we have

$$f\prec g \quad \text{if and only if} \quad f(\Delta)\subset g(\Delta).$$

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2010 *AMS Mathematics Subject Classification:* Primary 30C45; Secondary 30C75.

For a given set E, we use the following notation: $\overline{E} = \{\overline{w} : w \in E\}$, $k \cdot E = \{kw : w \in E\}$, $E^c = \{w : w \notin E\}$.

2. Extremal functions for K_α

In this section we describe the extremal functions for the problem of finding the Koebe set for K_α. For such an extremal function f there exist boundary points of $f(\Delta)$ that coincide with the points of $\partial K(K_\alpha)$, i.e. the boundary points of the Koebe set $K(K_\alpha)$.

Lemma 1 *If $f \in K_\alpha$, $\alpha \in [0,1]$, then $\overline{f(\overline{z})}$, $-f(-z)$, and $-\overline{f(-\overline{z})}$ also belong to K_α.*

The proof of this lemma is obvious. From Lemma 1 we conclude that if a set E is the image of Δ under a function belonging to K_α then there exist functions in K_α mapping Δ onto sets $\overline{E}$, $-E$, and $-\overline{E}$. The above three sets are symmetric to E with respect to the real axis, the origin, and the imaginary axis, respectively.

Furthermore, from Lemma 1 it follows that it is enough to describe the boundary of the Koebe set for K_α in the first quarter of the complex plane. The whole boundary can be obtained by taking the reflection of this curve in both axes.

Let f be a function in K_α, $\alpha \in (0,1)$ and let w_0 be an arbitrary point that is not in $f(\Delta)$. The convexity of f in the direction $e^{i\alpha\pi/2}$ gives that at least one of two rays emanating from w_0 in the direction $e^{i\alpha\pi/2}$ or $-e^{i\alpha\pi/2}$ is disjoint from $f(\Delta)$. Similarly, from the convexity of f in the direction $e^{-i\alpha\pi/2}$ it follows that at least one of two rays emanating from w_0 in the direction $e^{-i\alpha\pi/2}$ or $-e^{-i\alpha\pi/2}$ is disjoint from $f(\Delta)$.

Without going into detail, it means that $f(\Delta)$ is disjoint from at least one of the following four sectors:

$$D_1 = \{w \in \mathbb{C} \ : \ -\alpha\pi/2 \le \arg(w - w_0) \le \alpha\pi/2\} \tag{1}$$

$$D_2 = \{w \in \mathbb{C} \ : \ \alpha\pi/2 \le \arg(w - w_0) \le (2-\alpha)\pi/2\} \tag{2}$$

$$D_3 = \{w \in \mathbb{C} \ : \ (2-\alpha)\pi/2 \le \arg(w - w_0) \le (2+\alpha)\pi/2\} \tag{3}$$

$$D_4 = \{w \in \mathbb{C} \ : \ -(2-\alpha)\pi/2 \le \arg(w - w_0) \le -\alpha\pi/2\} \tag{4}$$

However, if, for instance, $\arg w_0 = \alpha\pi/2$ then 0 belongs to the boundaries of both sets: D_3 and D_4. Hence, $f(\Delta)$ is not disjoint from either D_3 or D_4. On the other hand, if $\arg w_0 \in (-\alpha\pi/2, \alpha\pi/2)$ then $0 \in D_3$; hence $f(\Delta) \cap D_3 \ne \emptyset$. A similar argument can be given for other values of the argument of w_0.

Now we shall find the functions that map Δ univalently onto $(D_1)^c$ and $(D_2)^c$. We will show that the images of these functions are sufficient to determine the Koebe sets for K_α.

Let α be a fixed number in $(0,1)$. It is known that the function

$$f(z) = a\left[1 - \left(\frac{1-z}{1+z}\right)^{2-\alpha}\right] \ , \ a = \frac{1}{2(2-\alpha)} \quad , \quad z \in \Delta \tag{5}$$

maps Δ univalently onto the complement of the sector $\{w \in \mathbb{C} \ : \ |\arg(w-a)| \le \alpha\pi/2\}$. Consider the Möbius transform of f given by

$$f_t(z) = \frac{f\left(\frac{z-it}{1+itz}\right) - f(-it)}{(1-t^2)f'(-it)} \ , \ t \in (-1,1) \quad , \quad z \in \Delta. \tag{6}$$

Since

$$f'(-it) = \left(\frac{1+it}{1-it}\right)^{1-\alpha} /(1-it)^2 \ , \tag{7}$$

the function f_t maps Δ univalently onto the complement of the sector measuring $\alpha\pi$ with the vertex in $a_t = a\frac{1+t^2}{1-t^2}$. This sector is not necessarily symmetric with respect to the real axis because the expression $f'(-it)$ is not always real. Putting

$$f'(-it) = k(t)e^{i\theta(t)}, \tag{8}$$

we can see that the bisector of the discussed sector has the direction $e^{-i\theta(t)}$. Now

$$F_t(z) = e^{i\theta(t)} f_t(ze^{-i\theta(t)}) \quad , \quad z \in \Delta$$

is the univalent mapping of Δ onto the complement of the sector measuring $\alpha\pi$ with the vertex in $A_t = a_t e^{i\theta(t)}$; its bisector is parallel to the real axis. Hence,

$$F_t(\Delta) = \mathbb{C} \setminus \{w \in \mathbb{C} \ : \ |\arg(w - A_t)| \le \alpha\pi/2\} \, .$$

Applying (7) and (8), we have

$$\theta(t) = \arg f'(-it) = 2(2-\alpha)\arg(1+it) = 2(2-\alpha)\arctan t$$

and consequently

$$t = \tan\frac{\theta(t)}{2(2-\alpha)}.$$

The above yields that

$$t \in (-1,1) \Leftrightarrow \theta(t) \in (-(2-\alpha)\pi/2, (2-\alpha)\pi/2) \, .$$

Combining the above facts, we obtain the following form of A_t:

$$A_t = \frac{1}{2(2-\alpha)\cos^2(\theta(t)/(2-\alpha))} e^{i\theta(t)}. \tag{9}$$

In an analogous way we find the function G_t that maps Δ univalently onto the complement of the set $D_2 = \{w \in \mathbb{C} \ : \ \alpha\pi/2 \le \arg(w - w_0) \le (2-\alpha)\pi/2\}$. The sector D_2 has the measure $(1-\alpha)\pi$.

Taking $\beta = 1-\alpha$, $b = (2(2-\beta))^{-1}$ and

$$g(z) = ib\left[1 - \left(\frac{1-z/i}{1+z/i}\right)^{2-\beta}\right] \quad , \quad z \in \Delta, \tag{10}$$

we obtain

$$g(\Delta) = \mathbb{C} \setminus \{w \in \mathbb{C} \ : \ \alpha\pi/2 \le \arg(w - ib) \le (2-\alpha)\pi/2\} \, .$$

Considering

$$g_t(z) = \frac{g\left(\frac{z-t}{1-tz}\right) - g(-t)}{(1-t^2)g'(-t)} \ , \ t \in (-1,1) \quad , \quad z \in \Delta$$

and

$$G_t(z) = e^{i\psi(t)} g_t(ze^{-i\psi(t)}) \quad , \quad z \in \Delta$$

with

$$\psi(t) = -2(1+\alpha)\arctan t$$

leads to

$$G_t(\Delta) = \mathbb{C} \setminus \{w \in \mathbb{C} \ : \ \alpha\pi/2 \leq \arg(w - B_t) \leq (2-\alpha)\pi/2\} .$$

The vertex of this sector is

$$B_t = \frac{i}{2(1+\alpha)\cos^2(\psi(t)/(1+\alpha))} e^{i\psi(t)}. \tag{11}$$

Moreover,

$$t \in (-1,1) \Leftrightarrow \psi \in (-(1+\alpha)\pi/2, (1+\alpha)\pi/2) .$$

The same result can be achieved using a geometric approach. A set

$$\{w \in \mathbb{C} \ : \ \alpha\pi/2 \leq \arg(w - B_t) \leq (2-\alpha)\pi/2\}$$

is obtained from the set

$$\{w \in \mathbb{C} \ : \ |\arg(w - A_t)| \leq \alpha\pi/2\}$$

substituting α by $1-\alpha$ and applying two transformations: the axial symmetry with respect to the real axis and the rotation through an angle $\pi/2$ about the origin. Consequently, G_t may be obtained as a composition of F_t with some suitably taken transformations.

From an argument similar to the one presented above but given for $\alpha = 0$, it follows that f defined by (5) reduces to the well-known Koebe function $f(z) = \frac{z}{(1+z)^2}$. In this case, the Möbius transform (6) leads to

$$f_t(z) = \frac{z}{\left(1 + \frac{1+it}{1-it} z\right)^2} \left(1 + \frac{2it}{1+t^2} z\right).$$

The set $f_t(\Delta)$ coincides with the complement of a ray having the direction $e^{-i\theta}$, where $\theta = 4\arctan t$. Eventually, the composition of f_t with a suitably taken rotation generates the function

$$F_t(z) = \frac{z}{\left(1 + e^{-i\theta/2} z\right)^2} \cdot \left(1 + i\sin(\theta/2) e^{-i\theta} z\right) \quad \theta = 4\arctan t$$

which maps Δ univalently onto the complement of the horizontal ray.

On the other hand, for $\alpha = 0$ (i.e. for $\beta = 1$) the function g given by (10) is of the form $g(z) = \frac{z}{1+z/i}$. After some easy calculus, we obtain

$$g_t(z) = \frac{z}{1 + ze^{i\psi}/i} \ , \ \psi = -2\arctan t \ \ t \in (-1,1)$$

and hence $G_t(z) = \frac{z}{1+z/i}$ for all $t \in (-1,1)$.

When α is equal to 1, the situation is reversed. A function f is equal to $f(z) = \frac{z}{1+z}$. Then

$$f_t(z) = \frac{z}{1 + ze^{i\theta}} \ \ \theta = 2\arctan t \ \ t \in (-1,1)$$

and $F_t(z) = \frac{z}{1+z}$ for all $t \in (-1,1)$. On the other hand, g can be written as $g(z) = \frac{z}{(1+z/i)^2}$. This yields that

$$g_t(z) = \frac{z}{\left(1 - i\frac{1-it}{1+it}z\right)^2}\left(1 - \frac{2t}{1+t^2}z\right)$$

and consequently

$$G_t(z) = \frac{z}{\left(1 + e^{-i\psi/2}z/i\right)^2} \cdot \left(1 + i\sin(\psi/2)e^{-i\psi}z/i\right) \quad \psi = -4\arctan t.$$

The function G maps Δ univalently onto the complement of the vertical ray. Moreover, $G_t(z) = iF_{-t}(-iz)$.

The functions F_t (for $\alpha = 0$) and G_t (for $\alpha = 1$) were found by Reade and Zlotkiewicz as well as by Goodman and Saff. They used these functions to determine the Koebe sets for the class of functions convex in the direction of the real axis and the imaginary axis, respectively.

3. Koebe set for K_α

Let A and B be the functions of the form

$$A(\theta) = \frac{1}{2(2-\alpha)\cos(\theta/(2-\alpha))}e^{i\theta} \quad \theta \in (-(2-\alpha)\pi/2, (2-\alpha)\pi/2)\,, \tag{12}$$

$$B(\psi) = \frac{i}{2(1+\alpha)\cos(\psi/(1+\alpha))}e^{i\psi} \quad \psi \in (-(1+\alpha)\pi/2, (1+\alpha)\pi/2)\,. \tag{13}$$

These functions correspond to the vertices of the sectors from the previous section (see formulae (9), (11)).

Observe that for $\theta \in [0, (2-\alpha)\pi/2))$ the function $|A(\theta)|$ is increasing. Moreover,

$$\operatorname{Re} A(\theta) \geq 0 \Leftrightarrow \theta \in [-\pi/2, \pi/2]\,.$$

Analogously, for $\psi \in [0, (1+\alpha)\pi/2)$ a function $|B(\psi)|$ is increasing and

$$\operatorname{Im} B(\psi) \geq 0 \Leftrightarrow \psi \in [-\pi/2, \pi/2]\,.$$

Denote by E_A and E_B two bounded sets that have the boundaries

$$\{A(\theta) : \theta \in [-\pi/2, \pi/2]\} \cup \{-A(\theta) : \theta \in [-\pi/2, \pi/2]\}$$

and

$$\{B(\psi) : \psi \in [-\pi/2, \pi/2]\} \cup \{-B(\psi) : \psi \in [-\pi/2, \pi/2]\}$$

respectively.

Let α_1 be the only solution in $[0,1]$ of the equation

$$(2-\alpha)\left(\cos\frac{\pi}{2(2-\alpha))} - 1\right) + 1 - 2\alpha = 0 \tag{14}$$

and let $\alpha_2 = 1 - \alpha_1$. The exact values of these numbers are: $\alpha_1 = 0.181\ldots$ and $\alpha_2 = 0.818\ldots$.

Lemma 2 *If* $\alpha \in [\alpha_1, \alpha_2]$ *then the curves given by (12) and (13) have one common point in the closed first quarter of the complex plane.*

Proof The complex equation $A(\theta) = B(\psi)$ is equivalent to the system of real equations

$$\begin{cases} \theta = \pi/2 + \psi \\ (2-\alpha)\cos\frac{\theta}{2-\alpha} = (1+\alpha)\cos\frac{\psi}{1+\alpha}. \end{cases}$$

Applying the first condition, the second one takes the form

$$h(\alpha, \theta) = 0 \tag{15}$$

where

$$h(\alpha,\theta) = (2-\alpha)\cos\frac{\theta}{2-\alpha} - (1+\alpha)\cos\frac{\pi/2-\theta}{1+\alpha} \quad , \quad \alpha \in [0,1] \ \ \theta \in [0,\pi/2]. \tag{16}$$

Since

$$\frac{\partial h}{\partial \theta} = -\left(\sin\frac{\theta}{2-\alpha} + \sin\frac{\pi/2-\theta}{1+\alpha}\right)$$

is negative for all $(\alpha,\theta) \in [0,1]\times[0,\pi/2]$, the function h with fixed α is decreasing for $\theta \in [0,\pi/2]$.

Observe that $\min\{h(0,\theta) : \theta \in [0,\pi/2]\} = h(0,\pi/2) = \sqrt{2}-1 > 0$. It means that for $\alpha = 0$ the function h given by (16) does not vanish. Therefore, the curves $A(\theta)$ and $B(\psi)$ have no common points in the first quadrant of the complex plane.

Similarly, $\max\{h(1,\theta) : \theta \in [0,\pi/2]\} = h(1,0) = 1-\sqrt{2} < 0$ means that for $\alpha = 1$ the curves $A(\theta)$ and $B(\psi)$ do not intersect each other in the first quadrant.

From the above we conclude that the discussed curves have a point of intersection if and only if (15) has a solution; it holds only for some range of variability of α.

For a fixed $\theta \in [0,\pi/2]$, the function

$$[0,1] \ni \alpha \mapsto (2-\alpha)\cos\frac{\theta}{2-\alpha}$$

decreases and a function

$$[0,1] \ni \alpha \mapsto (1+\alpha)\cos\frac{\pi/2-\theta}{1+\alpha}$$

increases. Hence, for a fixed θ, the function h, as a function of a variable α, is decreasing.

Consequently, there exists only one number $\alpha \in (0,1)$, which is the solution of $h(\alpha,\pi/2) = 0$; let us denote it by α_1. The number α_1 satisfies equation (14). Analogously, there exists only one number $\alpha \in (0,1)$ such that $h(\alpha_2, 0) = 0$; let us denote it by α_2. It is easy to check that $\alpha_2 = 1-\alpha_1$.

It follows from the monotonicity of $h(\alpha,\theta)$ with respect to θ that if $\alpha \in (\alpha_1,\alpha_2)$ then there exists only one $\theta_\alpha \in (0,\pi/2)$ satisfying $h(\alpha,\theta_\alpha) = 0$. Moreover, for $\theta \in [0,\theta_\alpha)$ we have $h(\alpha,\theta) > 0$ and for $\theta \in (\theta_\alpha,\pi/2]$ there is $h(\alpha,\theta) < 0$. □

Now we are ready to prove the main result of the paper. Let $r_1(\theta)$, $r_2(\theta)$ be given as follows

$$r_1(\theta) = \left[2(2-\alpha)\cos\frac{\theta}{2-\alpha}\right]^{-1}$$

$$r_2(\theta) = \left[2(1+\alpha)\cos\frac{\theta - \pi/2}{1+\alpha}\right]^{-1}.$$

Let α_1 be the solution of (14), $\alpha_2 = 1-\alpha_1$, and θ_α be the number from the proof of Lemma 2.

Theorem 1 *The Koebe set for the class K_α, $\alpha \in [0,1]$ is a bounded domain that is symmetric with respect to both axes of the complex plane. Its boundary in the first quadrant of the complex plane has the polar equation $w = r(\theta)e^{i\theta}$, where:*

a) $r(\theta) = r_1(\theta)$ $\theta \in [0, \pi/2]$ *for* $\alpha \in [0, \alpha_1]$

b) $r(\theta) = \begin{cases} r_1(\theta) & \theta \in [0, \theta_\alpha] \\ r_2(\theta) & \theta \in [\theta_\alpha, \pi/2] \end{cases}$ *for* $\alpha \in [\alpha_1, \alpha_2]$

c) $r(\theta) = r_2(\theta)$ $\theta \in [0, \pi/2]$ *for* $\alpha \in [\alpha_2, 1]$.

The Koebe sets for two selected values of α are presented in Figure 1.

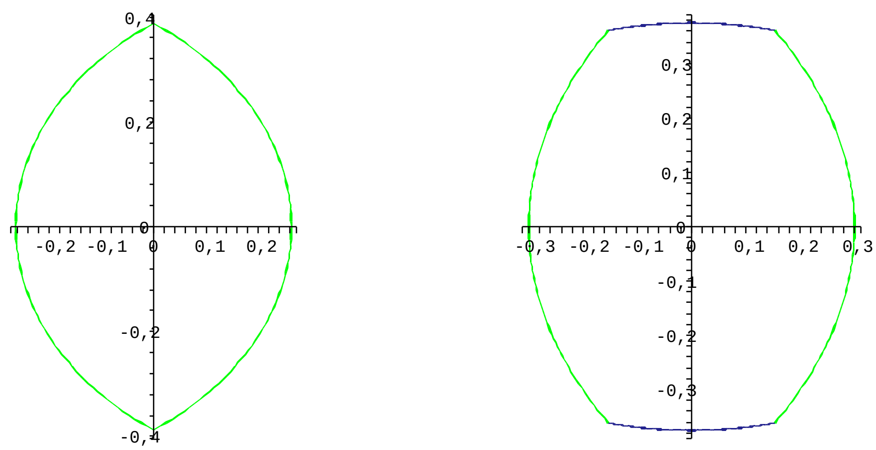

Figure 1. Koebe sets $K(K_\alpha)$ for $\alpha = 1/10$ (the left one) and $\alpha = 1/3$ (the right one).

Proof Let $w_0 \in \mathbb{C}$ be an arbitrary boundary point of $K(K_\alpha)$ with a fixed $\alpha \in [0,1]$. There exists a function $f_{w_0} \in K_\alpha$ such that $w_0 \in \partial f_{w_0}(\Delta)$.

By virtue of Lemma 1, the following functions $\overline{f_{w_0}(\overline{z})}$, $-f_{w_0}(-z)$, and $-\overline{f_{w_0}(\overline{-z})}$ also belong to K_α, and so the points $\overline{w_0}$, $-w_0$, and $-\overline{w_0}$ belong to the boundary of the set $K(K_\alpha)$. It means that $K(K_\alpha)$ is symmetric with respect to both axes of the complex plane.

From the convexity of f_{w_0} in the directions $e^{i\alpha\pi/2}$ and $-e^{i\alpha\pi/2}$ we conclude that at least one of the four possibilities holds:

$$f_{w_0}(\Delta) \subset (D_k)^c \quad , \quad k = 1,2,3,4$$

where D_k are defined by (1)–(4). We shall discuss each case separately.

At the beginning, assume that $f_{w_0}(\Delta) \subset (D_1)^c$. For this reason, $f_{w_0}(\Delta) \subset F_t(\Delta)$ for some $t \in (-1,1)$. However, $F_t(0) = 0$ and F_t is univalent; it yields that $f_{w_0} \prec F_t$. On the other hand, $1 = f'_{w_0}(0) \le F'_t(0) = 1$, which implies that $f_{w_0} \equiv F_t$. Hence, $w_0 = A(\theta)$, where $\theta = \arg w_0 \in (-(2-\alpha)\pi/2, (2-\alpha)\pi/2)$. It means that the points $A(\theta)$ may belong to the boundary of the Koebe set for K_α only if the extremal functions $f_{w_0} = f_{A(\theta)}$ satisfy $f_{A(\theta)}(\Delta) = (D_1)^c$.

From Lemma 1 it follows that the points $-A(\theta)$, $\theta \in (-(2-\alpha)\pi/2, (2-\alpha)\pi/2)$ coincide with the boundary of $K(K_\alpha)$ only if the functions $f_{-A(\theta)}$ satisfy the condition $f_{-A(\theta)}(\Delta) = (D_3)^c$.

Suppose now that $f_{w_0}(\Delta) \subset (D_2)^c$. Hence, $f_{w_0}(\Delta) \subset G_t(\Delta)$ for some $t \in (-1,1)$. Consequently, $f_{w_0} \prec G_t$. Since $1 = f'_{w_0}(0) \leq G'_t(0) = 1$, there is $f_{w_0} \equiv G_t$. It means that $w_0 = B(\psi)$, where $\psi = \arg w_0 - \pi/2 \in (-(1+\alpha)\pi/2, (1+\alpha)\pi/2)$. This shows that the points $B(\psi)$ belong to the Koebe set for K_α only if the functions $f_{w_0} = f_{B(\psi)}$ satisfy $f_{B(\psi)}(\Delta) = (D_2)^c$.

By virtue of Lemma 1, the points $-B(\psi)$, $\psi \in (-(1+\alpha)\pi/2, (1+\alpha)\pi/2)$ are the boundary points of $K(K_\alpha)$ only if the complements of the images of Δ under $f_{-B(\psi)}$ are equal to D_4.

The above considerations prove that, in order to find the boundary of $K(K_\alpha)$, it is sufficient to discuss only the curves $A(\theta)$, $B(\psi)$ defined by (12), (13), and the curves obtained from these two curves as a result of the axial symmetry with respect to both axes of the complex plane, or equivalently as a result of the symmetry with respect to the origin. We have proved

$$K(K_\alpha) = E_A \cap E_B.$$

The final step is finding a solution of

$$\min\{|A(\theta)|, |B(\psi)| : \arg A(\theta) = \arg B(\psi) = \varphi\} \tag{17}$$

for each fixed number $\varphi \in [0, \pi/2]$.

From Lemma 2 we know that if $\alpha \in (\alpha_1, \alpha_2)$ then the curves given by (12) and (13) intersect each other only in one point in the open first quadrant of the complex plane. For this reason, if $\alpha \in (\alpha_1, \alpha_2)$, then it is enough to derive the minimum (17) for $\varphi = 0$. From (12) and (13) it follows that $\arg A(\theta) = 0$ for $\theta = 0$ and $\arg B(\psi) = 0$ for $\psi = -\pi/2$. We shall compare two values: $A(0) = [2(2-\alpha)]^{-1}$ and $B(-\pi/2) = [2(1+\alpha)\cos(\pi/2(1+\alpha))]^{-1}$. It follows from (14) for $\alpha = \alpha_1$ that $\cos(\pi/2(2-\alpha_1)) = \frac{1+\alpha_1}{2-\alpha_1}$, or equivalently $\cos(\pi/2(1+\alpha_2)) = \frac{2-\alpha_2}{1+\alpha_2}$. From the monotonicity of $\cos(\pi/2(1+\alpha))$ and $\frac{2-\alpha}{1+\alpha}$, we conclude that taking α less than α_2 we obtain

$$\cos\frac{\pi}{2(1+\alpha)} < \frac{2-\alpha}{1+\alpha}$$

which means that

$$A(0) < B(-\pi/2).$$

Hence, for $\alpha \in (\alpha_1, \alpha_2)$,

$$\min\{|A(\theta)|, |B(\psi)| : \arg A(\theta) = \arg B(\psi) = \varphi\} = \begin{cases} r_1(\theta)\ , & \theta \in [0, \theta_\alpha] \\ r_2(\theta)\ , & \theta \in [\theta_\alpha, \pi/2] \end{cases}$$

The minimum value of (17) for $\alpha \in [0, \alpha_1]$ and for $\alpha \in [\alpha_2, 1]$ we can find in a similar way. □

In particular, for $\alpha = 1/2$, we have $r_1(\pi/2 - \theta) = r_2(\theta)$, and consequently

Corollary 1 *The Koebe set* $K(K_{1/2})$ *is a bounded domain that is symmetric with respect to both axes of the complex plane. Moreover, it satisfies* $iK(K_{1/2}) = K(K_{1/2})$. *Its boundary is described by the polar equation* $w = \frac{1}{3\cos(\frac{2}{3}\theta)}e^{i\theta}$, $\theta \in [0, \pi/4]$.

A set D satisfying the property $iD = D$ is called 4-fold symmetric. From Corollary 1 it follows that there exist 4 symmetry axes of $K(K_{1/2})$: $\zeta = e^{ik\pi/4}t$, $t \in \mathbb{R}$, $k = 0, 1, 2, 3$. It means that in order to describe the whole boundary of $K(K_{1/2})$ it is sufficient to give it only for $\theta \in [0, \pi/4]$. The set $K(K_{1/2})$ is shown in Figure 2.

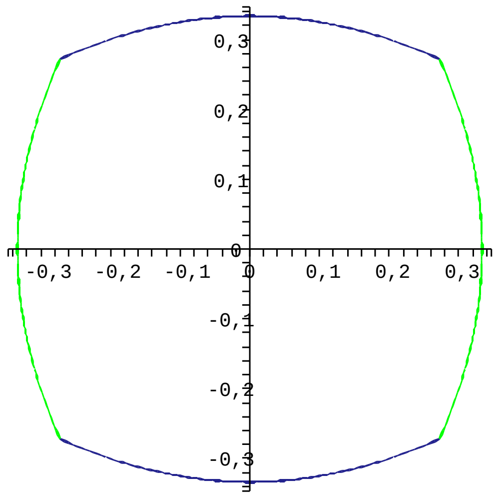

Figure 2. Koebe set $K(K_{1/2})$.

For $\alpha = 0$ the result stated in Theorem 1 coincides with the known result for the class $K_0 = K(1)$ consisting of univalent functions that are convex in the direction of the real axis. Namely,

Corollary 2 *The Koebe set* $K(K_1)$ *is a bounded domain that is symmetric with respect to both axes of the complex plane. Its boundary in the first quadrant of the complex plane has the polar equation* $w = \frac{1}{4\cos(\theta/2-\pi/4)}e^{i\theta}$, $\theta \in [0, \pi/2]$.

Furthermore,

Corollary 3 $K(K_0) = iK(K_1)$.

This result was obtained by Reade and Złotkiewicz [16], and independently by Goodman and Saff [6].

4. The class $K_{\beta,\gamma}$

In the first section of this paper the class K_α was defined. It consists of all univalent functions that are convex in two directions: $e^{i\alpha\pi/2}$ and $e^{-i\alpha\pi/2}$, $\alpha \in [0, 1]$. The definition of K_α can be generalized in the following way.

A function f is said to be in a class $K_{\beta,\gamma}$ if it maps Δ univalently onto a set convex in two fixed directions: $e^{i\beta\pi/2}$ and $e^{i\gamma\pi/2}$. It can be assumed that $-1 \le \beta \le \gamma \le 1$. In particular, $K_{\alpha,-\alpha} = K_\alpha$, $\alpha \in [0,1]$.

It is easy to observe that if $f \in K_{\beta,\gamma}$, $-1 \le \beta \le \gamma \le 1$ then

$$F(z) = e^{-i\delta\pi/2} f(e^{i\delta\pi/2} z) \quad , \quad \delta = \frac{1}{2}(\beta+\gamma) \tag{18}$$

belongs to K_α, $\alpha = \frac{1}{2}(\gamma-\beta)$. It means that knowledge of the extremal functions in the problem of finding the Koebe sets for K_α is sufficient to obtain the extremal functions in this problem for $K_{\beta,\gamma}$ with the help of (18).

Theorem 2 *Let β, γ be fixed numbers such that $-1 \le \beta \le \gamma \le 1$. Then*

$$K(K_{\beta,\gamma}) = e^{i\delta\pi/2} K(K_\alpha) \ \text{ where } \delta = \frac{1}{2}(\beta+\gamma) \ \ \alpha = \frac{1}{2}(\gamma-\beta).$$

In particular, the class $K_{0,1}$ consists of univalent functions that are convex in the direction of both the real and imaginary axes. In this case $\delta = \alpha = 1/2$, and so

Corollary 4

$$K(K_{0,1}) = e^{i\pi/4} K(K_{1/2}).$$

It means that the Koebe set for $K_{0,1}$ can be obtained from the set in Figure 2 by rotation about the origin through an angle of $\pi/4$.

References

[1] Brown JE. Level sets for functions convex in one direction. Proc Am Math Soc 1987; 100: 442-446.

[2] Bshouty D, Lyzzaik A. Univalent convex functions in the positive direction of the real axis. Complex analysis and dynamical systems III. Contemp Math 2008; 455: 41-51.

[3] Ciozda K. Sur la classe des fonctions convexes vers laxe réel négatif. Bull Acad Polon Sci Ser Sci Math 1979; 27: 255-261.

[4] Ciozda K. Sur quelques problémes extremaux dans les classes des fonctions convexes vers laxe réel négatif. Ann Polon Math 1980; 38: 311-317.

[5] Elin M, Shoikhet D, Yacobzon F. A distortion theorem for functions convex in one direction. Complex Anal Oper Theory 2011; 5: 751-758.

[6] Goodman AW, Saff EB. On univalent functions convex in one direction. Proc Am Math Soc 1979; 73: 183-187.

[7] Hengartner W, Schober G. On schlicht mappings to domains convex in one direction. Comment Math Helv 1970; 45: 303-314.

[8] Koczan L. Typically real functions convex in the direction of the real axis. Ann Univ Mariae Curie-Skłodowska Sect A 1989; 43: 23-29.

[9] Koczan L, Zaprawa P. On functions convex in the direction of the real axis with real coefficients. Bull Belg Math Soc - Simon Stevin 2011; 18: 321-335.

[10] Koczan L, Zaprawa P. On functions convex in the direction of the real axis with a fixed second coefficient. Appl Math Comput 2011; 217: 6644-6651.

[11] Koczan L, Zaprawa P. Covering problems for functions n-fold symmetric and convex in the direction of the real axis. Appl Math Comput 2012; 219: 947-958.

[12] Koczan L, Zaprawa P. Covering problems for functions n-fold symmetric and convex in the direction of the real axis II. Bull Malays Math Sci Soc (2) 2015; 38: 1637-1655.

[13] Krzyz J, Reade MO. Koebe domains for certain classes of analytic functions. J Anal Math 1967; 18: 185-195.

[14] Lecko A. On the class of functions convex in the negative direction of the imaginary axis. J Aust Math Soc 2002; 73: 110.

[15] Prokhorov DV. Level curves of functions convex in the direction of an axis. Math Notes 1988; 44: 767-769.

[16] Reade MO, Złotkiewicz E. On univalent functions with two preassigned values. Proc Am Math Soc 1971; 30: 539-544.

[17] Royster WC, Ziegler M. Univalent functions convex in one direction. Publ Math Debrecen 1976: 23: 340-345.