

## Characterization of substantially and quasi-substantially efficient solutions in multiobjective optimization problems

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**Abstract:** In this paper, we study the notion of substantial efficiency for a given multiobjective optimization problem. We provide two characterizations for substantially efficient solutions: the first one is based on a scalar problem and the second one is in terms of a stability concept. Moreover, this paper introduces the notion of quasi-substantial efficiency. Similar to those of substantial efficiency, two characterizations for quasi-substantially efficient solutions are obtained.

**Key words:** Substantial efficiency, scalarization function, stable problem, quasi-substantial efficiency

### 1. Introduction and preliminaries

In many real problems of economics, management science, engineering, and industry, decisions are characterized by many criteria and usually these criteria cannot be brought to a common scale by some utility functions. These problems are referred to as multiobjective optimization problems. Multiobjective optimization is one of the most important areas in optimization, which is of great interest because of the large variety of applications. From the large amount of relevant publications about multiobjective optimization, we mention three books [3, 4, 15].

Because of the conflict between objective functions, often there is no solution that optimizes all objective functions simultaneously. Hence, efficient solutions are considered as primary solutions of multiobjective optimization problems. An efficient problem is a feasible solution in which improvement of no objective function is possible without impairing at least one of the others, but the set of efficient solutions is a large set. Thus, selecting a suitable decision for the decision maker among this large set is a difficult task. In order to overcome this difficulty, we should consider some other appropriate factors for selecting decisions (solutions) that are better in some senses. One of the most important factors to exclude anomalous efficient solutions is trade-offs between objective functions. Trade-off analysis is one of the most important elements in quantitative efficiency analysis. A trade-off denotes the amount of giving up one of the objective functions, which leads to improvement of another objective function. There are different concepts of proper efficiency that give different interpretations of trade-offs between objective functions.

As is evident in [5], a special class of efficient solutions that have been defined based on trade-off analysis is the set of properly efficient solutions. Proper efficiency was first introduced by Kuhn and Tucker in 1951 [14]. Geoffrion, considering trade-offs between objective functions, defined the notion of a properly efficient solution [5]. Borwein, utilizing contingent cones, proposed a definition of properly efficient solutions [2]. Benson

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then introduced another definition of proper efficiency [1]. He showed that his definition and Geoffrion's are equivalent where the criterion objective cone is  $\mathbb{R}_+^p$  (the nonnegative orthant of  $\mathbb{R}^p$ ). Henig proposed a more general notion of properness and studied its relations to Benson's and Borwein's properness [8]. Wierzbicki proposed a property scalarization by means of the Chebyshev metric for efficient solutions [16] and then that scalarization was extended by Kaliszewski for Henig's properness [9, 10]. Hartley also introduced a generalization of Geoffrion's properness and provided some appropriate characterizations [7]. Ginchev et al. introduced a characterization of Geoffrion-type higher-order properly efficient solutions in vector optimization [6].

The above-mentioned works show that the notion of proper efficiency is a useful tool for quantitative Pareto analysis from theoretical and computational points of view. However, when the criterion space is a subset of  $\mathbb{R}^p$  ( $p > 2$ ), the number  $M$ , which is finite for each properly efficient element, is not, in general, a common upper bound for all trade-offs. There is no common upper bound for all trade-offs, though such a bound exists for properly efficient elements [10–13]. Motivated by this discussion, it is quite natural to investigate elements for which there is a common upper bound for all trade-offs. Another concept of solution, namely substantially efficient solutions, deals with an efficient solution in which all trade-offs between objective functions are bounded by a common upper bound [10–12]. For illuminating this notion of efficiency, consider a facility location problem. Suppose that this system has three objective functions, cost of raw material, sale market, and manpower, depending on distance, mileage, freight, fare, exhaustion of manpower, and so on. Assume that an efficient solution is available for decision making such that these objective functions are such that if one of these three functions is improved then the others are impaired by the current solution. It is clear that for a good choice of location a decision maker must consider all trade-offs among all objective functions. For example, if the decision maker considers improving the cost of raw material and impairing the cost of both the sale market and manpower in a current efficient solution such that with a large change (improving) in cost of raw material and infinitesimal change (impairing) in sale market and manpower, it means that the decision maker ignores two objective functions, sale market and manpower, in the current efficient solution. Thus, the current efficient solution is not a good choice for the facility location problem.

This paper provides two characterizations for substantially efficient solutions: the first one is based on some scalar functions and the second one is in terms of stability theory.

As is expected, the requirement for the existence of a common upper bound for all trade-offs in the notion of substantial efficiency is strong. On the other hand, in this notion we do not have any meaningful interpretation for unbounded trade-offs. Considering the above-mentioned weaknesses of proper efficiency and substantial efficiency, we propose an intermediate notion, namely quasi-substantial efficiency. This new notion considers all trade-offs. In contrast to substantial efficiency, it has a flexible treatment of trade-offs. Like substantial efficiency, quasi-substantial efficiency can be used as an efficient guideline in applications in an interactive procedure where substantially efficient solutions are not available. More precisely, instead of introducing a common upper bound, it considers a certain rate of growth for all trade-offs, including bounded and unbounded trade-offs. This paper also provides two characterizations for quasi-substantially efficient solutions: the first one is based on some scalar functions and the second one is in terms of stability concept.

This paper is organized as follows. Section 2 contains two characterizations for substantially efficient solutions. Section 3 introduces the notion of quasi-substantial efficiency and illustrates this notion by a numerical example, and it gives two characterizations for quasi-substantially efficient solutions.

Throughout this paper we use the following notations:

$$\mathbb{R}_+^p = \{y : y_i \geq 0, \forall i \in \{1, 2, \dots, p\}\},$$

$$\mathbb{R}_{++}^p = \{y : y_i > 0, \forall i \in \{1, 2, \dots, p\}\}.$$

For  $y^1, y^2 \in \mathbb{R}^p$ , we use the following notations:

$$y^1 \leq y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}_+^p,$$

$$y^1 \leq y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}_+^p \text{ and } y_1 \neq y_2,$$

$$y^1 < y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}_{++}^p.$$

In this paper, the following multiobjective programming is considered:

$$\begin{aligned} & \min f(x) \\ & \text{s.t. } x \in X, \end{aligned} \tag{1}$$

where  $X \subseteq \mathbb{R}^n$  is the feasible set and  $f = (f_1, \dots, f_p) : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^p$  is a vector function.

According to the concept of Pareto optimality,  $\hat{x} \in X$  is an efficient solution if

$$(f(X) - f(\hat{x})) \cap (-\mathbb{R}_+^p) = \{0\},$$

where  $f(X)$  is the image of  $X$  under  $f$ . The concept of efficiency plays a useful role in analyzing the vector optimization problem. In order to exclude certain efficient solutions that display an undesirable anomaly and to provide a more satisfactory characterization, the decision maker must use trade-off analysis. From economic points of views, the decision maker is interested in achieving a large value of so-called gain-to-loss by moving from the current solution to another one. Motivated by this important economic idea, boundedness of trade-off or ratio of change for objective functions plays a crucial role in some concepts of enhanced efficiency such as proper efficiency and substantial efficiency. The following definition states the concept of proper efficiency in the sense of Geoffrion.

**Definition 1.1** [5] *An efficient solution  $\hat{x} \in X$  is a properly efficient solution in Geoffrion's sense if there exists a positive number  $M$  such that for all  $x \in X$  and  $i \in \{1, \dots, p\}$  with  $f_i(x) < f_i(\hat{x})$  there exists a  $j \in \{1, \dots, p\}$  with  $f_j(\hat{x}) < f_j(x)$  such that*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M. \tag{2}$$

It is clear that in a properly efficient solution, the trade-off between some objective functions can be unbounded. However, in applicable and natural systems, proper efficiency is usually unusable; consider the facility location problem that was stated in this section. In this regard, Kaliszewski in [10] introduced a concept of optimality that considers the trade-off between all objective functions, and the trade-off between all objective functions can be unbounded, as follows.

**Definition 1.2** [10] *An efficient solution  $\hat{x} \in X$  is said to be a substantially efficient solution for Problem (1) if there exists a positive real number  $M$  such that for all  $x \in X$  and  $i, j \in \{1, \dots, p\}$  with  $f_i(x) < f_i(\hat{x})$  and  $f_j(\hat{x}) < f_j(x)$  the following inequality holds:*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M.$$

The next section proceeds to characterize these solutions.

**2. Characterization of substantially efficient solutions**

In the previous section, it was stated that substantially efficient solution sets are an important subset of efficient solution sets because of compatibility with natural systems and problems. Hence, investigation and characterization of substantially efficient solutions is important. To this end, this section is devoted to characterizing substantially efficient solutions. First, an equivalent definition of substantial solutions and a geometrical interpretation of these solutions are given. After that, an example is given for illuminating substantial solutions. Next, we propose a characterization of these efficient solutions using a scalar function. At the end of this section a perturbation problem is considered for investigating the stability of a problem with a substantially efficient solution.

**Definition 2.1** [10] *An efficient solution  $\hat{x} \in X$  is called a substantial solution for (1) if there exists a positive real number  $\lambda$  such that*

$$\left( (f_i(\hat{x}), f_j(\hat{x})) - D_{ij}^\lambda \right) \cap \mathcal{Y}_{ij} = \{ (f_i(\hat{x}), f_j(\hat{x})) \},$$

where

$$D_{ij}^\lambda = \left\{ x \in \mathbb{R}^2 : x_j < 0 \text{ and } x_i + \lambda x_j > 0 \right\} \cup \left\{ 0 \right\}, \quad i, j \in \{1, 2, \dots, p\}, \quad i \neq j$$

and  $\mathcal{Y}_{ij}$  is the projection of the set  $f(X)$  on the plane  $f_i$   $f_j$ .

**Example 2.2** *Consider problem  $\min_{-\frac{1}{2} \leq x \leq 1} (f_1(x), f_2(x), f_3(x))$  in which*

$$f_1(x) = \begin{cases} -\ln(-x) & \text{if } -\frac{1}{2} \leq x < 0, \\ -\sin x & \text{if } 0 \leq x \leq 1, \end{cases}$$

$f_2(x) = |x|$  and

$$f_3(x) = \begin{cases} -\exp(-x) & \text{if } -\frac{1}{2} \leq x < 0, \\ -\exp(x) & \text{if } 0 \leq x \leq 1. \end{cases}$$

Set  $\hat{x} := 0$ . Then  $(f_1(\hat{x}), f_2(\hat{x}), f_3(\hat{x})) = (0, 0, 1)$ . It is obvious that  $\hat{x}$  is an efficient solution. It is seen that  $f_1(x) < f_1(\hat{x})$ ,  $f_2(\hat{x}) < f_2(x)$  and  $f_3(x) < f_3(\hat{x})$ , for all  $0 < x \leq 1$ . It can be easily shown that there exists  $M_1 > 0$  such that:

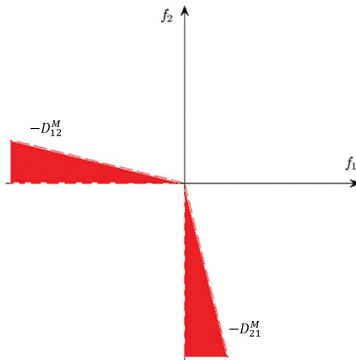
$$\begin{aligned} \frac{f_1(\hat{x}) - f_1(x)}{f_2(x) - f_2(\hat{x})} &= \frac{\sin x}{|x|} \leq M_1, & \text{for all } 0 < x \leq 1, \\ \frac{f_3(\hat{x}) - f_3(x)}{f_2(x) - f_2(\hat{x})} &= \frac{-1 + \exp(x)}{|x|} \leq M_1, & \text{for all } 0 < x \leq 1. \end{aligned}$$

It is also obvious that  $f_1(\hat{x}) < f_1(x)$ ,  $f_2(\hat{x}) < f_2(x)$  and  $f_3(x) < f_3(\hat{x})$ , for all  $-\frac{1}{2} \leq x < 0$ .

It can be easily shown that there exists  $M_2 > 0$  such that:

$$\begin{aligned} \frac{f_3(\hat{x}) - f_3(x)}{f_1(x) - f_1(\hat{x})} &= \frac{-1 + \exp(-x)}{-\ln(-x)} \leq M_2, & \text{for all } -\frac{1}{2} \leq x < 0, \\ \frac{f_3(\hat{x}) - f_3(x)}{f_2(x) - f_2(\hat{x})} &= \frac{-1 + \exp(-x)}{|x|} \leq M_2, & \text{for all } -\frac{1}{2} \leq x < 0. \end{aligned}$$

Set  $M := \max\{M_1, M_2\}$ . Thus,  $\hat{x} = 0$  satisfies Definition 1.2 and it is a substantially efficient solution.



**Figure.** The ordering cones  $-D_{1,2}^M$  and  $-D_{2,1}^M$ .

Note that the number of  $D_{ij}^\lambda$  cones ( $i, j \in \{1, \dots, p\}, i \neq j$ ) in  $\mathbb{R}^p$  is  $p(p-1)$ . Considering Example 2.2, the ordering cones  $-D_{1,2}^M$  and  $-D_{2,1}^M$  are depicted in the Figure. It is clear that  $D_{ij}^\lambda$  cones are pointed, i.e.  $D_{ij}^\lambda \cap -D_{ij}^\lambda = \{0\}$  [12, 13].

It is easy to check that Definitions 1.2 and 2.1 are equivalent [10, 12, 13].

In the sequel, two characterizations for determining substantially efficient solutions are introduced. One of them is in the term of scalarization and the other utilizes the concept of stability.

Corresponding to parameters  $M > 0$  and  $i \in \{1, \dots, p\}$  we define the extended real valued scalar function  $\bar{f}_i : X \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$  as

$$\bar{f}_i(x) = f_i(x) - f_i(\hat{x}) + M \min \left\{ f_j(x) - f_j(\hat{x}) \mid f_j(x) - f_j(\hat{x}) > 0 \text{ and } j \in \{1, \dots, p\} \right\}. \tag{3}$$

In the following theorem, using scalar function (3), substantially efficient solutions are characterized.

**Theorem 2.3** *Let  $\hat{x} \in X$  be an efficient element of Problem (1). Then  $\hat{x}$  is a substantially efficient solution of Problem (1) if and only if there is a positive real number  $M$  such that for any  $i \in \{1, \dots, p\}$*

$$\inf_{x \in X} \bar{f}_i(x) \geq 0, \tag{4}$$

where  $\bar{f}_i$  is defined as (3).

**Proof**

$\Rightarrow$ ) By contradiction assume that for any  $M > 0$  there is an  $x_M$  and an index  $i_M \in \{1, \dots, p\}$  such that  $\bar{f}_{i_M}(x_M) < 0$ . Hence,  $f_{i_M}(x_M) < f_{i_M}(\hat{x})$ . Since  $\hat{x}$  is an efficient solution, there exists an index  $j_0 \in \{1, \dots, p\}$  such that  $f_{j_0}(\hat{x}) < f_{j_0}(x_M)$ . Without loss of generality assume that

$$f_{j_0}(x_M) - f_{j_0}(\hat{x}) = \min \left\{ f_j(x_M) - f_j(\hat{x}) \mid f_j(x_M) - f_j(\hat{x}) > 0 \text{ and } j \in \{1, \dots, p\} \right\}.$$

Therefore,

$$\bar{f}_{i_M}(x_M) = f_{i_M}(x_M) - f_{i_M}(\hat{x}) + M(f_{j_0}(x_M) - f_{j_0}(\hat{x})) < 0.$$

Consequently,

$$\frac{f_{i_M}(\hat{x}) - f_{i_M}(x_M)}{f_{j_0}(x_M) - f_{j_0}(\hat{x})} > M,$$

and this contradicts the substantial efficiency of  $\hat{x}$ . Hence, the proof of the “only if ” part of the theorem is completed.

$\Leftarrow$ ) Suppose that there is a positive real number  $M$  such that, for any  $i \in \{1, \dots, p\}$ , (4) holds. Assume that there are  $i, j \in \{1, \dots, p\}$  and  $x \in X$  such that  $f_i(x) < f_i(\hat{x})$  and  $f_j(\hat{x}) < f_j(x)$ . Then

$$0 \leq f_i(x) - f_i(\hat{x}) + M(f_j(x) - f_j(\hat{x})).$$

Thus,

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M,$$

and it completes the proof. □

At the end of this section, substantially efficient solutions are characterized based on the notion of stability. To this aim, consider problem  $\bar{P}(0)$  as follows:

$$\begin{aligned} \bar{P}(0) : \min \varphi(x) \\ \text{s.t. } x \in X, \\ f(x) \leq 0, \end{aligned}$$

where  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is an arbitrary function. Now, related to any  $y \in \mathbb{R}^p$ , the perturbation Problem  $\bar{P}(y)$  is defined as follows:

$$\begin{aligned} \bar{P}(y) : \min \varphi(x) \\ \text{s.t. } x \in X, \\ f(x) \leq y. \end{aligned}$$

Denote by  $A(y)$  the feasible set of Problem  $\bar{P}(y)$ . Set

$$v(y) = \begin{cases} \inf\{\varphi(x) : x \in A(y)\}, & \text{if } A(y) \neq \emptyset, \\ \infty, & \text{if } A(y) = \emptyset. \end{cases}$$

Assume that  $\hat{x}$  is a minimizer of Problem  $\bar{P}(0)$ ; that is,

$$v(0) = \inf\{\varphi(x) | x \in A(0)\} = \varphi(\hat{x}).$$

In this case, we say that Problem  $\bar{P}(0)$  is stable at  $\hat{x}$  if there exists an  $M > 0$  such that

$$\frac{v(y) - v(0)}{\min\{y_j | y_j > 0 \text{ and } j \in \{1, \dots, p\}\}} \geq -M, \text{ for all } y \neq 0.$$

**Theorem 2.4** Assume that  $\hat{x} \in X$  is an efficient solution of Problem (1). Then  $\hat{x}$  is a substantially efficient solution if and only if for any  $i \in \{1, \dots, p\}$ ,  $P_i(0)$  is stable at  $\hat{x}$ , where  $P_i(y)$  is defined as follows:

$$P_i(y) : v_i(y) = \min(f_i(x) - f_i(\hat{x}))$$

$$s.t. x \in X,$$

$$f(x) - f(\hat{x}) \leq y.$$

**Proof**  $\Rightarrow$ ) By contradiction assume that  $P_i(0)$  is not stable at  $\hat{x}$ . Thus, for any  $M > 0$  there are  $y^M \in \mathbb{R}^p$  and  $x_M \in X$  with  $f(x_M) - f(\hat{x}) \leq y^M$  such that

$$\frac{v_i(y^M) - v_i(0)}{\min\{y_j^M | y_j^M > 0 \text{ and } j \in \{1, \dots, p\}\}} < -M,$$

and

$$\frac{f_i(x_M) - f_i(\hat{x})}{\min\{y_j^M | y_j^M > 0 \text{ and } j \in \{1, \dots, p\}\}} < -M. \tag{5}$$

Since  $\hat{x}$  is an efficient solution and  $f_i(x_M) < f_i(\hat{x})$ , there is an index  $j \in \{1, \dots, p\}$  with  $f_j(\hat{x}) < f_j(x_M)$ . Choose  $j_0 \in \{1, \dots, p\}$  such that

$$f_{j_0}(x_M) - f_{j_0}(\hat{x}) = \min\{f_j(x_M) - f_j(\hat{x}) | f_j(x_M) - f_j(\hat{x}) > 0 \text{ and } j \in \{1, \dots, p\}\}.$$

Then

$$f_{j_0}(x_M) - f_{j_0}(\hat{x}) \leq \min\{y_j^M | y_j^M > 0 \text{ and } j \in \{1, \dots, p\}\}. \tag{6}$$

Consequently, by (5) and (6), we have

$$f_i(\hat{x}) - f_i(x_M) > M(f_{j_0}(x_M) - f_{j_0}(\hat{x})).$$

This inequality contradicts the substantial efficiency of  $\hat{x}$ . Hence, the “only if” part of the theorem is proven.

$\Leftarrow$ ) By contradiction assume that  $\hat{x}$  is not a substantial solution of Problem (1). Hence, by Theorem 2.3, for any unbounded sequence of positive real numbers  $\{M_k\}$  there are sequences  $\{x_k\}$  and  $\{i_k\} \in \{1, \dots, p\}$  such that

$$f_{i_k}(x_k) - f_{i_k}(\hat{x}) + M_k \min\{(f_j(x_k) - f_j(\hat{x})) | (f_j(x_k) - f_j(\hat{x})) > 0 \text{ and } j \in \{1, \dots, p\}\} < 0.$$

Define  $y^k := f(x_k) - f(\hat{x})$ . Since  $\hat{x}$  is efficient and  $f_{i_k}(x_k) < f_{i_k}(\hat{x})$ , there is  $j \in \{1, \dots, p\}$  such that  $y_j^k > 0$ . Hence,

$$\begin{aligned} \frac{v_i(y^k) - v_i(0)}{\min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}} &\leq \frac{f_i(x_k) - f_i(\hat{x})}{\min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}} \\ &\leq \frac{-M_k \min\{y_j^k : y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}}{\min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}} \\ &= -M_k \rightarrow -\infty, \text{ as } k \rightarrow \infty \end{aligned}$$

and it contradicts the stability of  $P_i(0)$  at  $\hat{x}$ . This contradiction completes the “if” part of the proof.  $\square$

### 3. Characterization of quasi-substantially efficient solutions

In the previous section, substantial solutions were investigated. The main weaknesses of substantial efficiency are due to its interpretation for unbounded trade-offs. Indeed, it does not distinguish different types of unbounded trade-offs. In order to overcome these weaknesses, this paper proposes a new concept of efficient solutions, namely quasi-substantial efficiency (a generalization of substantial efficiency), which analyzes the treatment of both bounded and unbounded trade-off. The rate of growth of unbounded trade-offs is indicated by a certain expression denoting the order of quasi-substantial efficiency. In applications, whenever the decision maker cannot use the substantially efficient solutions or whenever substantially efficient solutions are not available, for example inaccessible points, then the decision maker can use some efficient solutions for which the rates of growth of unbounded trade-offs are less than others. In this regard, this section gives the definition and characterization of quasi-substantially efficient solutions.

**Definition 3.1** *Let  $s$  be a nonnegative real number. An efficient solution  $\hat{x} \in X$  is said to be a quasi-substantially efficient solution of order  $s$  for Problem (1), if there exists a positive number  $M$  such that for all  $x \in X$  and  $i, j \in \{1, \dots, p\}$  with  $f_i(x) < f_i(\hat{x})$  and  $f_j(\hat{x}) < f_j(x)$  the following inequality holds:*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq \frac{M}{\|x - \hat{x}\|^s}.$$

**Example 3.2** *Consider problem  $\min_{-\frac{1}{2} \leq x \leq 1} (f_1(x), f_2(x), f_3(x))$  in which*

$$f_1(x) = \begin{cases} \ln(-x) & \text{if } -\frac{1}{2} \leq x < 0, \\ \sin x & \text{if } 0 \leq x \leq 1, \end{cases}$$

$f_2(x) = x^2$  and

$$f_3(x) = \begin{cases} -\exp(-x) & \text{if } -\frac{1}{2} \leq x < 0, \\ -\exp(x) & \text{if } 0 \leq x \leq 1. \end{cases}$$

Set  $\hat{x} := 0$ . Then  $(f_1(\hat{x}), f_2(\hat{x}), f_3(\hat{x})) = (0, 0, 1)$ . Let  $s > 2$ . It is obvious that  $\hat{x}$  is an efficient solution.

It is seen that  $f_1(\hat{x}) < f_1(x)$ ,  $f_2(\hat{x}) < f_2(x)$  and  $f_3(x) < f_3(\hat{x})$ , for all  $0 < x \leq 1$ . It can be easily shown that there exists  $M_1 > 0$  such that:

$$\frac{f_3(\hat{x}) - f_3(x)}{f_2(x) - f_2(\hat{x})} = \frac{-1 + \exp(x)}{x^2} \leq \frac{M_1}{\|x - \hat{x}\|^s}, \quad \text{for all } 0 < x \leq 1,$$

$$\frac{f_3(\hat{x}) - f_3(x)}{f_1(x) - f_1(\hat{x})} = \frac{-1 + \exp(x)}{\sin x} \leq \frac{M_1}{\|x - \hat{x}\|^s}, \quad \text{for all } 0 < x \leq 1.$$

Also,  $f_1(x) < f_1(\hat{x})$ ,  $f_2(\hat{x}) < f_2(x)$  and  $f_3(x) < f_3(\hat{x})$ , for all  $-\frac{1}{2} \leq x < 0$ .

It can be easily shown that there exists  $M_2 > 0$  such that:

$$\frac{f_1(\hat{x}) - f_1(x)}{f_2(x) - f_2(\hat{x})} = \frac{\ln(-x)}{x^2} \leq \frac{M_2}{\|x - \hat{x}\|^s}, \quad \text{for all } -\frac{1}{2} \leq x < 0,$$

$$\frac{f_3(\hat{x}) - f_3(x)}{f_2(x) - f_2(\hat{x})} = \frac{-1 + \exp(-x)}{x^2} \leq \frac{M_2}{\|x - \hat{x}\|^s}, \quad \text{for all } -\frac{1}{2} \leq x < 0.$$

Set  $M := \max\{M_1, M_2\}$ . Thus,  $\hat{x} = 0$  is a quasi-substantially efficient solution of order  $s > 2$ .



Let  $s \geq 0$ . Corresponding to parameters  $M > 0$  and  $i \in \{1, \dots, p\}$  we define the extended real valued function  $f : X \rightarrow \hat{\mathbb{R}}$  as

$$\hat{f}_i(x) = \|x - \hat{x}\|^s (f_i(x) - f_i(\hat{x})) + M \min \left\{ f_j(x) - f_j(\hat{x}) \mid f_j(x) - f_j(\hat{x}) > 0 \text{ and } j \in \{1, \dots, p\} \right\}. \quad (7)$$

The following theorem (by using scalar function (7)) provides a necessary and sufficient condition for characterizing quasi-substantial efficient solutions.

**Theorem 3.3** *Let  $s \geq 0$  and  $\hat{x} \in X$  be an efficient element of Problem (1). Then  $\hat{x}$  is a quasi-substantially efficient solution of order  $s$  for Problem (1) if and only if there is a positive real number  $M$  such that for any  $i \in \{1, \dots, p\}$ ,*

$$\inf_{x \in X} \hat{f}_i(x) \geq 0, \quad (8)$$

where  $\hat{f}_i$  is defined as (7).

**Proof**  $\Rightarrow$ ) By contradiction assume that for any  $M > 0$  there is an  $x_M$  and an index  $i_M \in \{1, \dots, p\}$  such that  $\hat{f}_{i_M}(x_M) < 0$ . Hence,  $f_{i_M}(x_M) < f_{i_M}(\hat{x})$ . Since  $\hat{x}$  is an efficient solution, there exists a  $j_0 \in \{1, \dots, p\}$  such that  $f_{j_0}(\hat{x}) < f_{j_0}(x_M)$ . Without loss of generality, assume that

$$f_{j_0}(x_M) - f_{j_0}(\hat{x}) = \min \left\{ f_j(x_M) - f_j(\hat{x}) \mid f_j(x_M) - f_j(\hat{x}) > 0 \text{ and } j \in \{1, \dots, p\} \right\}.$$

Therefore,

$$\hat{f}_{i_M}(x_M) = \|x - \hat{x}\|^s (f_{i_M}(x_M) - f_{i_M}(\hat{x})) + M(f_{j_0}(x_M) - f_{j_0}(\hat{x})) < 0.$$

Consequently,

$$\frac{f_{i_M}(\hat{x}) - f_{i_M}(x_M)}{f_{j_0}(x_M) - f_{j_0}(\hat{x})} > \frac{M}{\|x - \hat{x}\|^s},$$

and this contradicts the quasi-substantial efficiency of  $\hat{x}$ . Hence, the proof of the “only if” part of the theorem is completed.

$\Leftarrow$ ) Suppose that there is a positive real number  $M$  such that for any  $i \in \{1, \dots, p\}$ , (8) holds. Assume that there are  $i, j \in \{1, \dots, p\}$  and  $x \in X$  such that  $f_i(x) < f_i(\hat{x})$  and  $f_j(\hat{x}) < f_j(x)$ . Then

$$0 \leq \|x - \hat{x}\|^s (f_i(x) - f_i(\hat{x})) + M(f_j(x) - f_j(\hat{x})).$$

Thus,

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq \frac{M}{\|x - \hat{x}\|^s}.$$

□

The following theorem is devoted to the stability of Problem (9) and quasi-substantial efficiency.

**Theorem 3.4** Assume that  $\hat{x} \in X$  is an efficient solution of Problem (1) and  $s \geq 0$ . Then  $\hat{x}$  is a quasi-substantially efficient solution of order  $s$  if and only if for any  $i \in \{1, \dots, p\}$ ,  $Q_i(0)$  is stable at  $\hat{x}$ , where  $Q_i(y)$  is as follows:

$$Q_i(y) : u_i(y) = \min \|x - \hat{x}\|^s (f_i(x) - f_i(\hat{x})) \tag{9}$$

$$s.t. \ x \in X, \tag{10}$$

$$f(x) - f(\hat{x}) \leq y. \tag{11}$$

**Proof**  $\Rightarrow$ ) By contradiction assume that  $Q_i(0)$  is not stable at  $\hat{x}$ . Thus, for any  $M > 0$  there are  $y^M \in \mathbb{R}^p$  and  $x_M \in X$  with  $f(x_M) - f(\hat{x}) \leq y^M$  such that

$$\frac{u_i(y^M) - u_i(0)}{\min\{y_j^M | y_j^M > 0 \text{ and } j \in \{1, \dots, p\}\}} < -M,$$

and

$$\frac{\|x - \hat{x}\|^s (f_i(x_M) - f_i(\hat{x}))}{\min\{y_j^M | y_j^M > 0 \text{ and } j \in \{1, \dots, p\}\}} < -M. \tag{12}$$

Since  $\hat{x}$  is an efficient solution and  $f_i(x_M) < f_i(\hat{x})$ , there is an index  $j \in \{1, \dots, p\}$  with  $f_j(\hat{x}) < f_j(x_M)$ . Choose  $j_0 \in \{1, \dots, p\}$  such that

$$f_{j_0}(x_M) - f_{j_0}(\hat{x}) = \min \left\{ f_j(x_M) - f_j(\hat{x}) \mid f_j(x_M) - f_j(\hat{x}) > 0, \text{ and } j \in \{1, \dots, p\} \right\}.$$

Then

$$0 < f_{j_0}(x_M) - f_{j_0}(\hat{x}) \leq \min\{y_j^M | y_j^M > 0 \text{ and } j \in \{1, \dots, p\}\}. \tag{13}$$

Consequently, by (12) and (13) we have

$$\|x - \hat{x}\|^s (f_i(\hat{x}) - f_i(x_M)) > M(f_{j_0}(x_M) - f_{j_0}(\hat{x}));$$

that is,

$$\frac{f_i(\hat{x}) - f_i(x_M)}{f_{j_0}(x_M) - f_{j_0}(\hat{x})} > \frac{M}{\|x - \hat{x}\|^s}.$$

This inequality contradicts the quasi-substantial efficiency of order  $s$  of  $\hat{x}$ . Hence, the “only if” part of the theorem is proven.

$\Leftarrow$ ) By contradiction assume that  $\hat{x}$  is not a quasi-substantial solution of order  $s$  of Problem (1). Hence, by Theorem 2.3, for any unbounded sequence of positive real numbers  $\{M_k\}$  there are sequences  $\{x_k\}$  and  $\{i_k\} \in \{1, \dots, p\}$  such that

$$\|x - \hat{x}\|^s (f_{i_k}(x_k) - f_{i_k}(\hat{x})) + M_k \min \left\{ (f_j(x_k) - f_j(\hat{x})) \mid (f_j(x_k) - f_j(\hat{x})) > 0, \text{ and } j \in \{1, \dots, p\} \right\} < 0.$$

Define  $y^k := f(x_k) - f(\hat{x})$ . Since  $\hat{x}$  is efficient and  $f_{i_k}(x_k) < f_{i_k}(\hat{x})$ , there is  $j \in \{1, \dots, p\}$  such that  $y_j^k > 0$ . Hence,

$$\begin{aligned} \frac{u_i(y^k) - u_i(0)}{\min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}} &\leq \frac{\|x - \hat{x}\|^s (f_i(x_k) - f_i(\hat{x}))}{\min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}} \\ &\leq \frac{-M_k \min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}}{\min\{y_j^k | y_j^k > 0 \text{ and } j \in \{1, \dots, p\}\}} \\ &= -M_k \rightarrow -\infty, \text{ as } k \rightarrow \infty \end{aligned}$$

and it contradicts the stability of  $Q_i(0)$  at  $\hat{x}$ . This contradiction completes the “if” part of the proof.  $\square$

#### 4. Conclusion

In this paper, we present two characterizations of substantially efficient solutions and we also introduce a new concept of efficiency, namely quasi-substantially efficient solutions, and characterize it. In the interactive optimization literature substantially efficient solutions play an important role and can be used as an efficient guideline in applications. Therefore, it is interesting to notice this point in interactive optimization. However, the definition of substantial efficiency is strong in theory but we consider quasi-substantial efficiency to overcome this problem. Like substantial efficiency, quasi-substantial efficiency, in the interactive optimization literature, can be used as an efficient guideline in applications where we can not use substantially efficient solutions or substantially efficient solutions are not available.

Because of the natural uncertainty in real-world situations, studying substantial efficiency and quasi-substantial efficiency in the presence of uncertainty can be considered as a topic for further research [17].

It should be noted that the scalar problems given in this paper just propose some characterizations for substantially and quasi-substantially efficient solutions. In order to have a computational procedure, by comparing available approximation of efficient frontiers (if any exist), we can consider some certain values of “ $M$ ” and determine so-called “ $M$ -substantially” and “ $M$ -quasi-substantially” efficient solutions.

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