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Nonpolynomial spline technique for the solution of ninth order boundary value problems

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Abstract: In this paper, a nonpolynomial spline technique is applied to solve the ninth order linear special case boundary value problems. The end conditions are derived to complete the definition of a spline. Three examples are numerically illustrated to check the efficiency of the method. The comparative analysis shows that the proposed technique gives better results than the homotopy perturbation method and the modified variational iteration method.

Key words: Nonpolynomial spline, boundary value problems, end conditions, homotopy perturbation method, modified variational iteration method

1. Introduction

Boundary value problems play an important role in various physical phenomena. Higher order boundary value problems have been considered due to their mathematical signification and strength in different fields of science. The depiction of ninth order boundary value problems exists rarely in the literature on numerical analysis. Mathematical modeling of AFTI-F16 fighters involves ninth order differential equations [4]. In view of the importance of the application of such problems in aircraft design and modeling, the present paper is devoted to the study of solutions of ninth order boundary value problems. Ninth order boundary value problems also arise in the study of astrophysics, hydrodynamics, and hydromagnetic stability [5, 6].

Siddiqi and Twizell [13–16] presented the solutions of 6th, 8th, 10th, and 12th order boundary value problems using sixth, eighth, tenth, and twelfth degree splines, respectively.

Siddiqi and Akram [8–12] presented the solutions of 5th, 6th, 8th, 10th, and 12th order boundary value problems using nonpolynomial spline techniques.

Wei et al. developed the solution of higher order differential equations by applying a local adaptive differential quadrature method [18].

Viswanadham and Ballem solved the eighth order boundary value problem by presenting a finite element method with the Galerkin method involving quintic B-splines as basis functions [17].

Twizell and Boutayeb proposed the solution of special eighth order boundary value problems by developing a family of finite difference methods [1]. Inc and Evans approximated the solution of eighth order boundary value problems using the Adomian decomposition method [2].

Lamnii et al. illustrated the spline solution of some linear boundary value problems (8th order, 10th order, and 12th order) using the spline collocation method, with the help of spline interpolants [3].

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Mohyud-Din and Yildrim presented the solution of ninth and tenth order boundary value problems using the homotopy perturbation method (HPM)[5] and the modified variational iteration method (MVIM)[6]. Saberi and Zahmatkesh determined the solution of higher order boundary value problems (9th order, 10th order, and 12th order) using the HPM [7].

In this paper, the nonpolynomial spline is introduced to establish the technique for the solution of ninth order boundary value problems.

The following boundary value problem is to be solved:

$$\left.\begin{array}{ll} y^{(9)}(x)+f(x)y(x)=g(x), & x\in[a,b] \\ y(a)=\zeta_0, & y(b)=\zeta_1, \\ y^{(1)}(a)=\eta_0, & y^{(1)}(b)=\eta_1, \\ y^{(2)}(a)=\mu_0, & y^{(2)}(b)=\mu_1, \\ y^{(3)}(a)=\sigma_0, & y^{(3)}(b)=\sigma_0, \\ y^{(4)}(a)=\lambda_0, & \end{array}\right\}, \tag{1}$$

where $\zeta_i,\eta_i,\mu_i,\sigma_i, i=0,1$ and λ_0 are finite real constants and $f(x)$ and $g(x)$ are continuous functions on $[a,b]$.
The nonpolynomial spline function used in the paper has the form
$T_n=\{1,x,x^2,x^3,x^4,x^5,x^6,x^7,x^8,cos(kx),sin(kx)\}$. Here k is the frequency of the trigonometric part, which can be used to increase the precision of the method. It can be real or purely imaginary.

The paper is divided into four sections. By applying the derivative continuities at knots, the consistency relation between the values of the spline and its ninth order derivative at knots is determined in Section 2. The end conditions for the solution of BVP (1) are also derived in Section 2. In Section 3, the nonpolynomial spline solution of BVP (1) is determined.

Three examples are given to observe the efficiency and reliability of the method in Section 4.

2. Consistency relations

For the development of spline approximation to problem (1), the interval $[a,b]$ is divided into N equal subintervals with knots $x_i=a+ih; i=0,1,...,N$, where $h=b-a/N$. The following restriction S_i of S to each subinterval $[x_i,x_{i+1}]$, $i=0,1,...,N-1$, is taken into account:

$$\begin{aligned} S_i(x) &= a_i cosk(x-x_i)+b_i sink(x-x_i)+c_i(x-x_i)^8+d_i(x-x_i)^7+e_i(x-x_i)^6 \\ &\quad +q_i(x-x_i)^5+r_i(x-x_i)^4+u_i(x-x_i)^3+v_i(x-x_i)^2+w_i(x-x_i)+z_i. \end{aligned} \tag{2}$$

Let

$$\left.\begin{array}{ll} y_i=S_i(x_i), & l_i=S_i^{(1)}(x_i), \\ m_i=S_i^{(2)}(x_i), & n_i=S_i^{(3)}(x_i), \\ p_i=S_i^{(4)}(x_i), & t_i=S_i^{(9)}(x_i), \end{array}\right\} \qquad i=0,1,...,N. \tag{3}$$

Using Eq. (3), the coefficients in Eq. (2) are evaluated as:

$$
\begin{aligned}
a_i &= h^9 \frac{t_i \cos\theta - t_{i+1}}{\theta^9 \sin\theta}, \\
b_i &= h^9 \frac{t_i}{\theta^9}, \\
c_i &= \frac{5(y_i - y_{i+1})}{h^8} + 5\frac{l_i}{h^7} + \frac{5(5m_i + 2m_{i+1})}{14h^6} + \frac{13n_i - 8n_{i+1}}{42h^5} + \frac{4p_i + 3p_{i+1}}{168h^4} - 5h\frac{t_i}{\theta^8} \\
&\quad + h\frac{13t_i - 8t_{i+1}}{42\theta^6} + 25h\frac{t_i \cos\theta}{14\theta^7 \sin\theta} + 5h\frac{(t_i - t_{i+1}\cos\theta)}{7\theta^7 \sin\theta} - h\frac{(t_i \cos\theta - t_{i+1})}{42\theta^5 \sin\theta} + h\frac{(t_{i+1}\cos\theta - t_i)}{56\theta^5 \sin\theta} \\
&\quad + 5h\frac{(t_i - t_{i+1}\cos\theta)}{\theta^9 \sin\theta}, \\
d_i &= \frac{20(y_{i+1} - y_i)}{h^7} - 20\frac{l_i}{h^6} - \frac{(51m_i + 19m_{i+1})}{7h^5} - \frac{(55n_i - 29n_{i+1})}{42h^4} - \frac{(9p_i + 5p_{i+1})}{84h^3} + 20h^2\frac{t_i}{\theta^8} \\
&\quad + h^2\frac{(29t_{i+1} - 55t_i)}{42\theta^6} - h^2\frac{(19 + 51\cos\theta)t_i}{7\theta^7 \sin\theta} + 3h^2\frac{(t_i \cos\theta - t_{i+1})}{28\theta^5 \sin\theta} + 5h^2\frac{(t_i - t_{i+1} cos\theta)}{84\theta^5 \sin\theta} \\
&\quad + 20h^2\frac{(\cos\theta - 1)t_i}{\theta^9 \sin\theta} + 20h^2\frac{(\cos\theta - 1)t_{i+1}}{\theta^9 \sin\theta}, \\
e_i &= \frac{28(y_i - y_{i+1})}{h^6} + 28\frac{l_i}{h^5} + \frac{21m_i + 7m_{i+1}}{2h^4} + \frac{12n_i - 5n_{i+1}}{6h^3} + \frac{11p_i + 4p_{i+1}}{60h^2} - 28h^3\frac{t_i}{\theta^8} \\
&\quad + h^3\frac{(12t_i - 5t_{i+1})}{6\theta^6} - 11h^3\frac{(t_i \cos\theta - t_{i+1})}{60\theta^5 \sin\theta} + h^3\frac{(t_i + t_{i+1}\cos\theta)}{15\theta^5 \sin\theta} + h^3\frac{(7 + 21\cos\theta)t_i}{2\theta^7 \sin\theta} \\
&\quad - h^3\frac{(7\cos\theta + 21)t_{i+1}}{2\theta^7 \sin\theta} - 28h^3\frac{(\cos\theta - 1)t_i}{\theta^9 \sin\theta} - 28h^3\frac{(\cos\theta - 1)t_{i+1}}{\theta^9 \sin\theta},
\end{aligned}
$$

$$
\begin{aligned}
q_i &= 14\frac{(y_{i+1} - y_i)}{h^5} - 14\frac{l_i}{h^4} - \frac{(11m_i + 3m_{i+1})}{h^3} - \frac{(7n_i - 2n_{i+1})}{6h^2} - \frac{(17p_i + 3p_{i+1})}{120h} + 14h^4\frac{t_i}{\theta^8} \\
&\quad - h^4\frac{(7t_i - 2t_{i+1})}{6\theta^6} + h^4\frac{(3 + 17\cos\theta)t_i}{120\theta^5 \sin\theta} - h^4\frac{(3 + 17\cos\theta)t_{i+1}}{120\theta^5 \sin\theta} - h^4\frac{(3 + 11\cos\theta)t_i}{2\theta^7 \sin\theta} \\
&\quad + h^4\frac{(3\cos\theta + 11)t_{i+1}}{2\theta^7 \sin\theta} + 14h^4\frac{(\cos\theta - 1)t_i}{\theta^9 \sin\theta} + 14h^4\frac{(\cos\theta - 1)t_{i+1}}{\theta^9 \sin\theta}, \\
r_i &= -h^5\frac{(t_i \cos\theta - t_{i+1})}{24\theta^5 \sin\theta} + \frac{p_i}{24}, \\
u_i &= h^6\frac{t_i}{6\theta^6} + \frac{n_i}{6}, \\
v_i &= h^7\frac{(t_i \cos\theta - t_{i+1})}{2\theta^7} + \frac{m_i}{2}, \\
w_i &= -h^8\frac{t_i}{\theta^8} + l_i, \\
z_i &= -h^9\frac{(t_i \cos\theta - t_{i+1})}{\theta^9 \sin\theta},
\end{aligned}
\tag{4}
$$

where $\theta = kh$ and $i = 0, 1, 2, ..., N - 1$.

Implementation of the first, fifth, sixth, seventh, and eighth order derivative continuities at knots, i.e. $S_{i-1}^{(\psi)}(x_i) = S_i^{(\psi)}(x_i)$, where $\psi = 1, 5, 6, 7$, and 8, leads to the following consistency relations:

$$\Big[t_{i-1}((1680 - 180\theta^2 + \theta^4)\tan\frac{\theta}{2} + 20\theta(-42 + \theta^2)) + t_i((1680 - 180\theta^2 + \theta^4)\tan\frac{\theta}{2} + 20\theta(-42 + \theta^2)) + \frac{840\theta^9}{h^8}(l_{i-1} + l_i) + \frac{180\theta^9}{h^7}(m_{i-1} - m_i) + \frac{20\theta^9}{h^6}(n_{i-1} + n_i) + \frac{\theta^9}{h^5}(p_{i-1} - p_i) + \frac{1680\theta^9}{h^9}(y_{i-1} - y_i)\Big] = 0, \qquad i = 1, 2, ..., N, \tag{5}$$

$$\Big[-5t_{i-1}(\csc\theta(3(-112 - 20\theta^2 + \theta^4) + (336 - 108\theta^2 + \theta^4)\cos\theta - 16\theta(-21 + \theta^2)\sin\theta)) + 2t_i(20\theta(-42 + \theta^2) + (1680 - 180\theta^2 + \theta^4)\tan\frac{\theta}{2}) + t_{i+1}(\csc\theta(1680 - 660\theta^2 + 17\theta^4 + 3(-560 - 60\theta^2 + \theta^4)\cos\theta)) + \frac{1680\theta^9}{h^8}(l_{i-1} + l_i) + \frac{60\theta^9}{h^7}(9m_{i-1} + 16m_i + 3m_{i+1}) + \frac{40\theta^9}{h^6}(2n_{i-1} + n_i - n_{i+1}) + \frac{\theta^9}{h^5}(5p_{i-1} + 32p_i + 3p_{i+1}) + \frac{1680\theta^9}{h^9}(y_{i-1} - y_{i+1})\Big] = 0, \qquad i = 1, 2, ..., N-1, \tag{6}$$

$$\Big[t_{i-1}(\csc\theta(-20160 - 3240\theta^2 + 108\theta^4 + \theta^6 + 72(280 - 95\theta^2 + \theta^4)\cos\theta - 360\theta(-56 + 3\theta^2)\sin\theta)) - t_i(2\theta\csc\theta(\theta(-5400 + 120\theta^2 + \theta^4)\cos\theta + 60(\theta(-78 + \theta^2) - 4(-42 + 5\theta^2\sin\theta)))) + t_{i+1}(\csc\theta(20160 - 7560\theta^2 + 132\theta^4 + \theta^6 + 24(-840 - 105\theta^2 + 2\theta^4)\cos\theta - 600\theta^3\sin\theta)) + \frac{20160\theta^9}{h^8}(l_i - l_{i-1}) - \frac{360\theta^9}{h^7}(19m_{i-1} - 12m_i - 7m_{i+1}) - \frac{120\theta^9}{h^6}(9n_{i-1} - 20n_i + 5n_{i+1}) + \frac{24\theta^9}{h^5}(-3p_{i-1} + p_i + 2p_{i+1}) - \frac{20160\theta^9}{h^9}(y_{i-1} - 2y_i + y_{i+1})\Big] = 0, \qquad i = 1, 2, ..., N-1, \tag{7}$$

$$\Big[-7t_{i-1}(\csc\theta(-240 - 36\theta^2 + \theta^4 + (240 - 84\theta^2 + \theta^4)\cos\theta - 2\theta(-120 + 7\theta^2)\sin\theta)) + 2t_i(20\theta(-42 + \theta^2) + (1680 - 180\theta^2 + \theta^4)\tan\frac{\theta}{2}) + t_{i+1}(\csc\theta(1680 - 612\theta^2 + 9\theta^4 + (-1680 - 228\theta^2 + 5\theta^4)\cos\theta - 58\theta^3\sin\theta)) + \frac{1680\theta^9}{h^8}(l_{i-1} + l_i) + \frac{12\theta^9}{h^7}(49m_{i-1} + 72m_i + 19m_{i+1}) + \frac{2\theta^9}{h^6}(49n_{i-1} + 20n_i - 29n_{i+1}) + \frac{\theta^9}{h^5}(7p_{i-1} + 16p_i + 5p_{i+1}) + \frac{1680\theta^9}{h^9}(y_{i-1} - y_{i+1})\Big] = 0, \qquad i = 1, 2, ..., N-1, \tag{8}$$

$$
\begin{aligned}
&\big[t_{i-1}(-\csc\theta(201600+28800\theta^2-720\theta^4+\theta^8-960(210-75\theta^2+\theta^4)\cos\theta+960\theta(-210+13\theta^2)\sin\theta))\\
&+2t_i(\theta\csc\theta(\theta(50400-840\theta^2+\theta^6)\cos\theta+840(-\theta(-60+\theta^2)+12(10+\theta^2)\sin\theta)))\\
&-t_{i+1}(\csc\theta(-201600+72000\theta^2-960\theta^4+\theta^8-720(-280-40\theta^2+\theta^4)\cos\theta+7680\theta^3\sin\theta))\\
&-\frac{201600\theta^9}{h^8}(l_{i-1}-l_i)+\frac{14400\theta^9}{h^7}(5m_{i-1}-3m_i-2m_{i+1})-\frac{960\theta^9}{h^6}(13n_{i-1}-21n_i+8n_{i+1})\\
&-\frac{240\theta^9}{h^5}(4p_{i-1}-p_i-3p_{i+1})+\frac{201600\theta^9}{h^9}(y_{i-1}+y_{i+1})\Big]=0, \qquad i=1,2,...,N-1.
\end{aligned} \tag{9}
$$

With the help of Eqs. (5) − −(9), the following consistency relation regarding the ninth derivative of spline t_i and y_i, $i=0,1,...,N$, is derived:

$$
\begin{aligned}
& h^9t_{i-5}\left(\frac{\cos\theta-1}{\theta^9\sin\theta}+\frac{1}{2\theta^7\sin\theta}-\frac{1}{24\theta^5\sin\theta}+\frac{1}{720\theta^3\sin\theta}-\frac{1}{40320\theta\sin\theta}\right)\\
& +h^9t_{i-4}\left(\frac{7(1-\cos\theta)}{\theta^9\sin\theta}-\frac{5+2\cos\theta}{2\theta^7\sin\theta}+\frac{2\cos\theta-7}{2\theta^5\sin\theta}-\frac{2\cos\theta-55}{720\theta^3\sin\theta}-\frac{247}{40320\theta\sin\theta}\right)\\
& +h^9t_{i-3}\left(\frac{20(\cos\theta-1)}{\theta^9\sin\theta}+\frac{5(1+\cos\theta)}{\theta^7\sin\theta}+\frac{7\cos\theta+13}{2\theta^5\sin\theta}+\frac{19-11\cos\theta}{720\theta^3\sin\theta}+\frac{247\cos\theta-2147}{20160\theta\sin\theta}\right)\\
& +h^9t_{i-2}\left(\frac{28(1-\cos\theta)}{\theta^9\sin\theta}-\frac{9\cos\theta+5}{\theta^7\sin\theta}-\frac{27\cos\theta+13}{12\theta^5\sin\theta}-\frac{95+189\cos\theta}{360\theta^3\sin\theta}+\frac{4293\cos\theta-7933}{20160\theta\sin\theta}\right)\\
& +h^9t_{i-1}\left(\frac{14(\cos\theta-1)}{\theta^9\sin\theta}+\frac{5\cos\theta+2}{\theta^7\sin\theta}+\frac{19\cos\theta+4}{12\theta^5\sin\theta}+\frac{7(35\cos\theta-4)}{360\theta^3\sin\theta}+\frac{15619\cos\theta-9956}{20160\theta\sin\theta}\right)\\
& +h^9t_{i}\left(\frac{14(\cos\theta-1)}{\theta^9\sin\theta}+\frac{5\cos\theta+2}{\theta^7\sin\theta}+\frac{19\cos\theta+4}{12\theta^5\sin\theta}+\frac{7(35\cos\theta-4)}{360\theta^3\sin\theta}+\frac{15619\cos\theta-9956}{20160\theta\sin\theta}\right)\\
& +h^9t_{i+1}\left(\frac{28(1-\cos\theta)}{\theta^9\sin\theta}-\frac{9\cos\theta+5}{\theta^7\sin\theta}-\frac{27\cos\theta+13}{12\theta^5\sin\theta}-\frac{95+189\cos\theta}{360\theta^3\sin\theta}+\frac{4293\cos\theta-7933}{20160\theta\sin\theta}\right)\\
& +h^9t_{i+2}\left(\frac{20(\cos\theta-1)}{\theta^9\sin\theta}+\frac{5(1+\cos\theta)}{\theta^7\sin\theta}+\frac{7\cos\theta+13}{2\theta^5\sin\theta}+\frac{19-11\cos\theta}{720\theta^3\sin\theta}+\frac{247\cos\theta-2147}{20160\theta\sin\theta}\right)\\
& +h^9t_{i+3}\left(\frac{7(1-\cos\theta)}{\theta^9\sin\theta}-\frac{5+2\cos\theta}{2\theta^7\sin\theta}+\frac{2\cos\theta-7}{2\theta^5\sin\theta}-\frac{2\cos\theta-55}{720\theta^3\sin\theta}-\frac{247}{40320\theta\sin\theta}\right)\\
& +h^9t_{i+4}\left(\frac{\cos\theta-1}{\theta^9\sin\theta}+\frac{1}{2\theta^7\sin\theta}-\frac{1}{24\theta^5\sin\theta}+\frac{1}{720\theta^3\sin\theta}-\frac{1}{40320\theta\sin\theta}\right)\\
= \;& -(y_{i-5}-9y_{i-4}+36y_{i-3}-84y_{i-2}+126y_{i-1}-126y_i+84y_{i+1}-36y_{i+2}+9y_{i+3}-y_{i+4}),\\
& \qquad\qquad i=5,6,...,N-4,
\end{aligned} \tag{10}
$$

which can be written as

$$\begin{aligned}
& (\alpha h^9 t_{i-5} + \beta h^9 t_{i-4} + \gamma h^9 t_{i-3} + \delta h^9 t_{i-2} + \rho h^9 t_{i-1} + \rho h^9 t_i + \delta h^9 t_{i+1} + \gamma h^9 t_{i+2} + \beta h^9 t_{i+3} + \alpha h^9 t_{i+4}) \\
= \quad & -(y_{i-5} - 9y_{i-4} + 36y_{i-3} - 84y_{i-2} + 126y_{i-1} - 126y_i + 84y_{i+1} - 36y_{i+2} + 9y_{i+3} - y_{i+4}), \\
& \qquad i = 5, 6, ..., N-4, \qquad (11)
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \left(\frac{\cos\theta - 1}{\theta^9 \sin\theta} + \frac{1}{2\theta^7 \sin\theta} - \frac{1}{24\theta^5 \sin\theta} + \frac{1}{720\theta^3 \sin\theta} - \frac{1}{403200\theta \sin\theta} \right), \\
\beta &= \left(\frac{7(1 - \cos\theta)}{\theta^9 \sin\theta} - \frac{5 + 2\cos\theta}{2\theta^7 \sin\theta} + \frac{2\cos\theta - 7}{2\theta^5 \sin\theta} - \frac{2\cos\theta - 55}{720\theta^3 \sin\theta} - \frac{247}{403200\theta \sin\theta} \right), \\
\gamma &= \left(\frac{20(\cos\theta - 1)}{\theta^9 \sin\theta} + \frac{5(1 + \cos\theta)}{\theta^7 \sin\theta} + \frac{7\cos\theta + 13}{2\theta^5 \sin\theta} + \frac{19 - 11\cos\theta}{720\theta^3 \sin\theta} + \frac{247\cos\theta - 2147}{20160\theta \sin\theta} \right), \\
\delta &= \left(\frac{28(1 - \cos\theta)}{\theta^9 \sin\theta} - \frac{9\cos\theta + 5}{\theta^7 \sin\theta} - \frac{27\cos\theta + 13}{12\theta^5 \sin\theta} - \frac{95 + 189\cos\theta}{360\theta^3 \sin\theta} + \frac{4293\cos\theta - 7933}{20160\theta \sin\theta} \right),
\end{aligned}$$

and

$$\rho = \left(\frac{14(\cos\theta - 1)}{\theta^9 \sin\theta} + \frac{5\cos\theta + 2}{\theta^7 \sin\theta} + \frac{19\cos\theta + 4}{12\theta^5 \sin\theta} + \frac{7(35\cos\theta - 4)}{360\theta^3 \sin\theta} + \frac{15619\cos\theta - 9956}{20160\theta \sin\theta} \right).$$

Let y_i be an approximation to $y(x_i)$, obtained by the spline $S(x_i)$, and $y(x)$ be an exact solution of BVP (1).

System (11) forms $(N-8)$ linear equations in $(N-1)$ unknowns $y_i s, i = 1, 2, ..., N-1$, where $t_i = -f_i y_i + g_i, i = 0, 1, ..., N$, is taken from BVP (1), which shows that seven more equations (end conditions) are necessary to obtain a complete solution of $y_i s$ in system (11). Following Siddiqi and Ghazala [9], the end conditions are derived as:

$$\begin{aligned}
t_0 + t_4 &= \frac{1}{h^9} \left(\frac{336581}{75} y_0 - 5760 y_1 + 1800 y_2 - \frac{6400}{9} y_3 + 225 y_4 - \frac{1152}{25} y_5 \right. \\
&\quad \left. + \frac{40}{9} y_6 + \frac{53956}{15} h y_0^{(1)} + 1176 h^2 y_0^{(2)} + 160 h^3 y_0^{(3)} + \frac{2}{21} h^9 y_0^{(9)} \right), \qquad (12) \\
t_1 + t_5 &= \frac{1}{h^9} \left(\frac{285174923040}{1118918657} y_1 - \frac{598239681480}{1118918657} y_2 + \frac{1614131506400}{3356755971} y_3 \right. \\
&\quad - \frac{329341219200}{1118918657} y_4 + \frac{132616764000}{1118918657} y_5 - \frac{94942074920}{3356755971} y_6 \\
&\quad + \frac{3392736480}{1118918657} y_7 + \frac{117588984800}{1118918657} h y_0^{(1)} + \frac{96245251200}{1118918657} h^2 y_0^{(2)} \\
&\quad \left. + \frac{24403713600}{1118918657} h^3 y_0^{(3)} - \frac{361628746}{1118918657} h^9 y_1^{(9)} \right), \qquad (13) \\
t_2 + t_6 &= \frac{1}{h^9} \left(\frac{49126598136}{739120663} y_2 - \frac{415948365952}{2217361989} y_3 + \frac{173291585355}{739120663} y_4 \right.
\end{aligned}$$

$$-\frac{129384057600}{739120663}y_5+\frac{178090065160}{2217361989}y_6-\frac{15530368896}{739120663}y_7$$
$$+\frac{1782343269}{739120663}y_8+\frac{9722822060}{739120663}hy_0^{(1)}+\frac{11556052200}{739120663}h^2y_0^{(2)}$$
$$+\frac{4123388640}{739120663}h^3y_0^{(3)}-\frac{676008826}{739120663}h^9y_2^{(9)}\Bigg), \tag{14}$$

$$t_3+t_7 = \frac{1}{h^9}\Bigg(\frac{6285677483034992}{110683870050699}y_3-\frac{4167916332739677}{24596415566822}y_4$$
$$+\frac{2718740555057520}{12298207783411}y_5-\frac{18563415689155960}{110683870050699}y_6$$
$$+\frac{948058565000400}{12298207783411}y_7-\frac{493428834001971}{24596415566822}y_8$$
$$+\frac{252599375937104}{110683870050699}y_9+\frac{107220527018710}{12298207783411}hy_0^{(1)}$$
$$+\frac{202130339338900}{12298207783411}h^2y_0^{(2)}+\frac{144806935763280}{12298207783411}h^3y_0^{(3)}$$
$$+\frac{39579082616640}{12298207783411}h^4y_0^{(4)}\Bigg), \tag{15}$$

$$t_{N-2}+t_{N-6} = \frac{1}{h^9}\Bigg(-\frac{49126598136}{739120663}y_{N-2}+\frac{415948365952}{2217361989}y_{N-3}-\frac{173291585355}{739120663}y_{N-4}$$
$$+\frac{129384057600}{739120663}y_{N-5}-\frac{178090065160}{2217361989}y_{N-6}+\frac{15530368896}{739120663}y_{N-7}$$
$$-\frac{1782343269}{739120663}y_{N-8}+\frac{9722822060}{739120663}hy_N^{(1)}-\frac{11556052200}{739120663}h^2y_N^{(2)}$$
$$+\frac{4123388640}{739120663}h^3y_N^{(3)}-\frac{676008826}{739120663}h^9y_{N-2}^{(9)}\Bigg), \tag{16}$$

$$t_{N-1}+t_{N-5} = \frac{1}{h^9}\Bigg(-\frac{285174923040}{1118918657}y_{N-1}+\frac{598239681480}{1118918657}y_{N-2}-\frac{1614131506400}{3356755971}y_{N-3}$$
$$-\frac{329341219200}{1118918657}y_{N-4}+\frac{132616764000}{1118918657}y_{N-5}+\frac{94942074920}{3356755971}y_{N-6}$$
$$-\frac{3392736480}{1118918657}y_{N-7}+\frac{117588984800}{1118918657}hy_N^{(1)}-\frac{96245251200}{1118918657}h^2y_N^{(2)}$$
$$+\frac{24403713600}{1118918657}h^3y_N^{(3)}-\frac{361628746}{1118918657}h^9y_{N-1}^{(9)}\Bigg), \tag{17}$$

$$t_N+t_{N-4} = \frac{1}{h^9}\Bigg(-\frac{336581}{75}y_N+5760y_{N-1}-1800y_{N-2}+\frac{6400}{9}y_{N-3}-225y_{N-4}$$
$$+\frac{1152}{25}y_{N-5}-\frac{40}{9}y_{N-6}+\frac{53956}{15}hy_N^{(1)}-1176h^2y_N^{(2)}+160h^3y_N^{(3)}$$
$$+\frac{2}{21}h^9y_N^{(9)}\Bigg). \tag{18}$$

The nonpolynomial spline solution of BVP (1) is presented in the next section.

3. Nonpolynomial spline solution

The nonpolynomial spline solution of BVP (1) is determined using Eqs. (12)–(15), (11), and (16)–(18). In consideration of $\mathbf{Y} = [y_1, y_2, ..., y_{N-1}]^T$ and $\mathbf{C} = [c_1, c_2, ..., c_{N-1}]^T$, y_i satisfies the following systems of equations:

$$(\mathbf{A} + h^9 \mathbf{BF})\mathbf{Y} = \mathbf{C},$$

where $\mathbf{A} = (A_1 A_2 A_3)$

$$A_1 = \begin{pmatrix} -5760 & 1800 & \frac{-6400}{9} & 225 \\ \frac{285174923040}{1118918657} & -\frac{598239681480}{1118918657} & \frac{1614131506400}{3356755971} & -\frac{329341219200}{1118918657} \\ 0 & \frac{49126598136}{739120663} & -\frac{415948365952}{2217361989} & \frac{173291585355}{739120663} \\ 0 & 0 & \frac{6285677483034992}{110683870050699} & -\frac{4167916332739677}{24596415566822} \\ -9 & 36 & -84 & 126 \\ 1 & -9 & 36 & -84 \\ \ddots & \ddots & \ddots & \ddots \\ & 1 & -9 & 36 \\ & & 1 & -9 \\ & & 0 & -\frac{1782343269}{739120663} \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\frac{1152}{25} & \frac{40}{9} & 0 \\ \frac{132616764000}{1118918657} & -\frac{94942074920}{3356755971} & \frac{3392736480}{1118918657} \\ -\frac{129384057600}{739120663} & \frac{178090065160}{2217361989} & -\frac{15530368896}{739120663} \\ \frac{2718740555057520}{12298207783411} & -\frac{18563415689155960}{110683870050699} & \frac{948058565000400}{12298207783411} \\ -126 & 84 & -36 \\ 126 & -126 & 84 \\ \ddots & \ddots & \ddots \\ -84 & 126 & -126 \\ 36 & -84 & 126 \\ \frac{15530368896}{739120663} & -\frac{178090065160}{2217361989} & \frac{129384057600}{739120663} \\ -\frac{3392736480}{1118918657} & \frac{94942074920}{3356755971} & -\frac{132616764000}{1118918657} \\ 0 & -\frac{40}{9} & \frac{1152}{25} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \frac{1782343269}{739120663} & 0 & \dots & 0 \\ -\frac{493428834001971}{24596415566822} & \frac{252599375937104}{110683870050699} & 0 \quad \dots & 0 \\ 9 & -1 & & \\ -36 & 9 & -1 & \\ \ddots & \ddots & \ddots & \ddots \\ 84 & -36 & 9 & -1 \\ -126 & 84 & -36 & 9 \\ -\frac{173291585355}{739120663} & \frac{415948365952}{2217361989} & -\frac{49126598136}{739120663} & 0 \\ \frac{329341219200}{1118918657} & -\frac{1614131506400}{3356755971} & \frac{598239681480}{1118918657} & -\frac{285174923040}{1118918657} \\ -225 & \frac{6400}{9} & -1800 & 5760 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{1480547403}{1118918657} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1415129489}{739120663} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ -\beta & -\gamma & -\delta & -\rho & -\rho & -\delta & -\gamma & -\beta & -\alpha & & \\ -\alpha & -\beta & -\gamma & -\delta & -\rho & -\rho & -\delta & -\gamma & -\beta & -\alpha & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & -\alpha & -\beta & -\gamma & -\delta & -\rho & -\rho & -\delta & -\gamma & -\beta & -\alpha \\ & & -\alpha & -\beta & -\gamma & -\delta & -\rho & -\rho & -\delta & -\gamma & -\beta \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \frac{1415129489}{739120663} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \frac{1480547403}{1118918657} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$\mathbf{F} = diag(f_i), \qquad i = 1, 2, ..., N-1.$

Also,

$$c_1 = -\frac{336581}{75}y_0 - \frac{53956}{15}hy_0^{(1)} - 1176h^2y_0^{(2)} - 160h^3y_0^{(3)} + h^9\left[\frac{19}{21}(g_0 - f_0y_0) + g_4\right], \tag{19}$$

$$c_2 = -\frac{117588984800}{1118918657}hy_0^{(1)} - \frac{96245251200}{1118918657}h^2y_0^{(2)} - \frac{24403713600}{1118918657}h^3y_0^{(3)} + \frac{361628746}{1118918657}h^9g_1 + h^9(g_1 + g_5), \tag{20}$$

$$c_3 = -\frac{9722822060}{739120663}hy_0^{(1)} - \frac{11556052200}{739120663}h^2y_0^{(2)} - \frac{4123388640}{739120663}h^3y_0^{(3)} + \frac{676008826}{739120663}h^9g_2 + h^9(g_2 + g_6), \tag{21}$$

$$c_4 = -\frac{107220527018710}{12298207783411}hy_0^{(1)} - \frac{202130339338900}{12298207783411}h^2y_0^{(2)} - \frac{144806935763280}{12298207783411}h^3y_0^{(3)} - \frac{39579082616640}{12298207783411}h^4y_0^{(4)} + h^9(g_3 + g_7), \tag{22}$$

$$c_5 = -h^9(\alpha(g_0 - f_0y_0) + \beta g_1 + \gamma g_2 + \delta g_3 + \rho g_4 + \rho g_5 + \delta g_6 + \gamma g_7 + \beta g_8 + \alpha g_9) - y_0, \tag{23}$$

$$c_i = -h^9(\alpha g_{i-5} + \beta g_{i-4} + \gamma g_{i-3} + \delta g_{i-2} + \rho g_{i-1} + \rho g_i + \delta g_{i+1} + \gamma g_{i+2} + \beta g_{i+3} + \alpha g_{i+4}), \quad i = 6, 7, ..., N-5, \tag{24}$$

$$c_{N-4} = -h^9(\alpha(g_N - f_N y_N) + \beta g_{N-1} + \gamma g_{N-2} + \delta g_{N-3} + \rho g_{N-4} + \rho g_{N-5} + \delta g_{N-6} + \gamma g_{N-7} + \beta g_{N-8} + \alpha g_{N-9}) + y_N, \tag{25}$$

$$c_{N-3} = -\frac{9722822060}{739120663} h y_N^{(1)} + \frac{11556052200}{739120663} h^2 y_N^{(2)} - \frac{4123388640}{739120663} h^3 y_N^{(3)} + \frac{676008826}{739120663} h^9 g_{N-2} + h^9(g_{N-2} + g_{N-6}), \tag{26}$$

$$c_{N-2} = -\frac{117588984800}{1118918657} h y_N^{(1)} + \frac{96245251200}{1118918657} h^2 y_N^{(2)} - \frac{24403713600}{1118918657} h^3 y_N^{(3)} + \frac{361628746}{1118918657} h^9 g_{N-1} + h^9(g_{N-1} + g_{N-5}), \tag{27}$$

and

$$c_{N-1} = \frac{336581}{75} y_N - \frac{53956}{15} h y_N^{(1)} + 1176 h^2 y_N^{(2)} - 160 h^3 y_N^{(3)} + h^9 \left[\frac{19}{21}(g_N - f_N y_N) + g_{N-4}\right]. \tag{28}$$

Three examples are considered to demonstrate the implementation of the method in the following section.

4. Numerical results

Example 1. For $0 \leq x \leq 1$, consider the following boundary value problem:

$$\left.\begin{array}{ll} y^{(9)}(x) + y(x) = e^x(16(-1+x)cos(x) + (127+17x)sin(x)), & \\ y(0) = 0 = y(1), & \\ y^{(1)}(0) = -1, & y^{(1)}(1) = esin(1), \\ y^{(2)}(0) = 0, & y^{(2)}(1) = 2esin(1) + 2ecos(1), \\ y^{(3)}(0) = 4, & y^{(3)}(1) = 6ecos(1), \\ y^{(4)}(0) = 8. & \end{array}\right\} \tag{29}$$

The analytical solution of the above-mentioned problem is:

$$y(x) = e^x(x-1)sin(x).$$

The maximum errors associated with $y_i s$ for Example 1, relative to different values of N, are recorded in Table 1.

Table 1. Maximum absolute errors for Example 1 in $y_i s$.

N	Max. absolute errors
10	5.13×10^{-8}
15	2.72×10^{-8}
20	1.58×10^{-8}
25	1.03×10^{-8}
30	6.54×10^{-9}

It is confirmed from Table 1 that if h is reduced by a factor of $1/2$, then $\|E\|$ is reduced by a factor of $1/4$, which indicates that the present method gives second-order results.

Example 2. For $-1 \leq x \leq 1$, consider the following boundary value problem:

$$\left.\begin{array}{l} y^{(9)}(x) + y(x) = (-1 + 18x + x^2)\cos(x) - (-73 + x^2)sin(x), \\ y(-1) = 0 = y(1), \\ y^{(1)}(-1) = 2cos(1) = -y^{(1)}(1), \\ y^{(2)}(-1) = 2cos(1) - 4sin(1) = y^{(2)}(1), \\ y^{(3)}(-1) = 6cos(1) + 6sin(1) = -y^{(3)}(1), \\ y^{(4)}(-1) = -12cos(1) + 8sin(1). \end{array}\right\} \quad (30)$$

The analytical solution of the above-mentioned problem is:

$$y(x) = (x^2 - 1)cos(x).$$

The maximum errors associated with $y_i s$ for Example 2, relative to different values of N, are recorded in Table 2.

Table 2. Maximum absolute errors for Example 2 in $y_i s$.

N	Max. absolute errors
10	1.43×10^{-6}
15	1.15×10^{-6}
20	7.58×10^{-7}
25	5.00×10^{-7}
30	3.56×10^{-7}

It is confirmed from Table 2 that if h is reduced by a factor of $1/2$, then $\|E\|$ is reduced by a factor of $1/4$, which indicates that the present method gives second-order results.

Example 3. For $0 \leq x \leq 1$, consider the following boundary value problem:

$$\left.\begin{array}{ll} y^{(9)}(x) - y(x) = -9e^x, & \\ y(0) = 1, & y(1) = 0, \\ y^{(1)}(0) = 0, & y^{(1)}(1) = -e, \\ y^{(2)}(0) = -1, & y^{(2)}(1) = -2e, \\ y^{(3)}(0) = -2, & y^{(3)}(1) = -3e, \\ y^{(4)}(0) = -3. & \end{array}\right\} \quad (31)$$

The analytical solution of the above-mentioned problem is:

$$y(x) = (1 - x)e^x.$$

The maximum absolute error associated with y_i for Example 3 of the presented method is compared with the HPM in Table 3.
It may be noted from Table 3 that the presented method is more efficient.

Remark. In the above examples, α, β, γ, δ, and ρ are chosen such that $\alpha + \beta + \gamma + \delta + \rho = 1/2$. The values of α, β, γ, δ, and ρ are taken as $14/360$, $36/360$, $25/360$, $35/360$, and $70/360$, respectively.

Table 3. Maximum absolute error for Example 3 in y_i.

N	Presented method	Mohyud-Din and Yildrim [5]	Mohyud-Din and Yildrim [6]	Saberi and Zahmatkesh [7]
10	1.82×10^{-9}	3.4×10^{-9}	3.4×10^{-9}	6.0×10^{-6}

5. Conclusion

For the solution of ninth order linear special case boundary value problems, a numerical method has been established using a nonpolynomial spline technique. Three examples have been considered to illustrate the preciseness of the method. The comparison of the maximum absolute error of the presented method with those of Mohyud-Din and Yildrim [5, 6] and Saberi and Zahmatkesh [7] is given in Table 3, which shows that the presented method is more efficient than the HPM and MVIM. Conclusively, the presented method is a reliable technique, as the numerical solutions are in very good agreement with the exact solutions.

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