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Regularity of solutions of the anisotropic hyperbolic heat equation with nonregular heat sources and homogeneous boundary conditions

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Abstract: We study regularity properties for the solution of homogeneous boundary value problems for the anisotropic hyperbolic heat equation in the case of infinitely differentiable coefficients but irregular distributions as internal heat sources.

Key words: Anisotropic hyperbolic heat equation, Sobolev spaces, integral transforms of vector valued distributions

1. Introduction and physical motivation

The hyperbolic heat conduction equation is a fundamental tool in many modern industrial applications such as microelectronics and the processing of materials by irradiation with a laser beam of high intensity and very short application times (see [4, 6, 12, 13] for instance). Usually the mathematical formulation of these problems leads to the study of boundary value problems with data given by irregular distributions such as Heaviside's function or Dirac's δ distribution.

Real industrial materials frequently are neither isotropic (see [16] for instance for some concrete examples) nor homogeneous. Assuming the density ρ and the specific heat c to be constant in order to avoid more complications, the hyperbolic heat equation in the open set Ω occupied by the body is (see [2])

$$-\sum_{h=1}^{3} \frac{\partial}{\partial x_h} \left(\sum_{j=1}^{3} k_{hj}(\mathbf{x}) \frac{\partial T}{\partial x_j}(\mathbf{x}, t) \right) + \rho\, c \left(\frac{\partial T}{\partial t}(\mathbf{x}, t) + \tau \frac{\partial^2 T}{\partial t^2}(\mathbf{x}, t) \right) =$$

$$= \rho \left(S(\mathbf{x}, t) + \tau \frac{\partial S}{\partial t}(\mathbf{x}, t) \right), \qquad (1)$$

where $T(\mathbf{x}, t)$ is the temperature in the point $\mathbf{x}$ at the instant t, $(k_{hj}(\mathbf{x}))$ is the symmetric thermal conductivity tensor of the material, τ is the relaxation parameter, and $S(\mathbf{x}, t)$ denotes the internal heat sources in the body. Moreover, the preservation of the second law of thermodynamics implies that the differential operator in (1) must be strongly elliptic in $\overline{\Omega}$. If $k_{hj} = k$ for every $1 \leq h, j \leq 3$ we obtain the hyperbolic heat equation for an isotropic homogeneous body.

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The study of these problems with full generality is very ambitious and it is expected that a rigorous and complete mathematical treatment will be quite long, difficult, and complex. In a first step to this goal, we have studied in [10] the existence and regularity properties of the solutions of mixed initial and boundary problems related to (1) when all involved data are regular. In a further step, the specific purpose of this paper *is to study existence, uniqueness, and regularity theorems about solutions of homogeneous mixed initial-boundary value problems for the hyperbolic heat equation (1) in the case of infinitely differentiable coefficients up to the closure* $\overline{\Omega}$ *of the spatial domain but with nonregular data distributions in its right side*. Roughly speaking, after the elimination of the temporal variable by Laplace transformation, we are concerned with a family of spatially elliptic operators indexed by the Laplace transform variable. When these operators have "regular" functional spaces as domain and range we can obtain a quite precise estimation of the norm of the inverse operators, an estimation that is essentially conserved after transposition, enlargement of the range space of the transposed operators, and complex interpolation. In this way we arrive at the world of "irregular" domain spaces for our operators in such a way that we can to apply the Laplace inversion formula to obtain our solution functions and to study their properties.

The paper is organized as follows. In this section we set the notation and give an account of the necessary functional spaces to be used as well as some preliminary results. Section 2 settles the selected framework to develop our study and presents various fundamental results in order to find quantitative information about the operators we are concerned with. In section 3 we apply these previous results to get our main theorems on existence and regularity properties of solutions in the case of irregular data.

Notation is standard in general. All used functions and vector spaces are assumed to be complex if not clearly stated otherwise. E' always denotes the *dual* space of the normed space E. Points $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ are denoted by $\mathbf{x}$ in short. A bold Greek letter $\boldsymbol{\alpha} \in (\mathbb{N} \cup \{0\})^n$ will denote a multi-index for derivations. The symbol $\approx$ means "isomorphic". We shall always consider an open *bounded set* $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$ *with boundary* $\partial\Omega$ *being a* $\mathcal{C}^\infty$ *manifold of dimension* $n-1$ *such that* Ω *is locally on the same side of* $\partial\Omega$. We refer to [8] for the definition and properties of the Sobolev spaces $H^r(\Omega), H_0^r(\Omega),\ H^{r,s}(\Omega\times]T_0, T_1[)$ and $H_{0,0}^{r,s}(\Omega\times]T_0, T_1[),\ r, s \in [0, \infty[$ (and analogous spaces defined on the boundary $\partial\Omega$) and its dual spaces as well for basic facts about complex interpolation spaces $[X, Y]_\theta$ between Banach spaces X and Y. If $m \geq k$ in $[0, \infty[$, $I_{m,k} : H^m(\Omega) \longrightarrow H^k(\Omega)$ will denote the continuous inclusion map. For simplicity we will use the notation $\Omega_T := \Omega\times] - T, T[$ for $T > 0$. Unless clearly stated otherwise, the restriction of $G \in \mathcal{D}'(\Omega \times \mathbb{R})$ to any open subset of $\Omega \times \mathbb{R}$ will be denoted by the same symbol, G. We shall need the following lemma.

Lemma 1 *Let* $0 \leq r_2 \leq r_1$ *and* $0 \leq s$ *be real numbers. If* $0 < \rho < 1$ *and* $T > 0$ *one has the continuous inclusion maps*

$$1)\ \ \left[H^{r_1}(] - T, T[, H^s(\Omega)), H^{r_2}(] - T, T[, H^s(\Omega))\right]_\rho \subset H^{(1-\rho)r_1+\rho r_2}(] - T, T[, H^s(\Omega)).$$

$$2)\ \ \left[H^s(] - T, T[, H^{r_1}(\Omega)), H^s(] - T, T[, H^{r_2}(\Omega))\right]_\rho \subset H^s\big(] - T, T[, H^{(1-\rho)r_1+\rho r_2}(\Omega)\big).$$

Proof We apply the technique of complex interpolation (see $\big[$[8], chapter 1, section 14$\big]$). Let $\mathcal{Z} = \big\{\xi + i\eta \in \mathbb{C} \mid \xi \in]0, 1[, \eta \in \mathbb{R}\big\}$ and given Banach spaces $Y_1 \subset Y_2,\ Y_1$ dense in Y_2, let $\mathfrak{H}\big(\mathcal{Z}, Y_1, Y_2)\big)$ be the set of continuous functions $\mathfrak{f} : \overline{\mathcal{Z}} \longrightarrow Y_2$ such that $\mathfrak{f}(\eta i) \in L^\infty(\mathbb{R}, Y_1),\ \mathfrak{f}(1 + \eta i) \in L^\infty(\mathbb{R}, Y_2)$ and $\mathfrak{f}$ is scalarly holomorphic in $\mathcal{Z}$.

1) Take $Y_1 := H^{r_1}(] - T, T[, H^s(\Omega))$ and $Y_2 : H^{r_2}(] - T, T[, H^s(\Omega))$. Given $\varepsilon > 0$ and $z(\mathbf{x}, t) \in Z_1 :=$

$\big[Y_1, Y_2\big]_\rho$ there is $\mathfrak{f} \in \mathfrak{H}\big(\mathcal{Z}, Y_1, Y_2\big)$ such that $z(\mathbf{x}, t) = \mathfrak{f}(\rho)(\mathbf{x}, t)$ and

$$\max\big(\big\|f(\eta i, \mathbf{x}, t)\big\|_{L^\infty(\mathbb{R}, Y_1)}, \big\|f(1+\eta i, \mathbf{x}, t)\big\|_{L^\infty(\mathbb{R}, Y_2)}\big) \leq \big\|z(\mathbf{x}, t)\big\|_{Z_1} + \varepsilon. \tag{2}$$

Defining $\mathfrak{g}(\xi + i\eta, t) = \big\|\mathfrak{f}(\xi + i\eta)(\mathbf{x}, t)\big\|_{H^s(\Omega)} \in \mathfrak{H}(\mathcal{Z}, H^{r_1}(-T, T), H^{r_2}(-T, T))$ and writing $X_1 := H^{r_1}(]-T, T[), X_2 := H^{r_2}(]-T, T[)$ and $r := (1-\rho)r_1 + \rho r_2$ we have $\mathfrak{g}(i\eta, t) \in L^\infty\big(\mathbb{R}, X_1\big)$ and $\mathfrak{g}(1+i\eta, t) \in L^\infty(\mathbb{R}, X_2)$. As a consequence $\mathfrak{g}(\rho, t) = \big\|\mathfrak{f}(\rho)(\mathbf{x}, t)\big\|_{H^s(\Omega)} \in \big[X_1, X_2\big]_\rho \approx H^r(]-T, T[)$ (by $\big[$[8] chapter 1, theorem 9.6$\big]$). Then $z(\mathbf{x}, t) = \mathfrak{f}(\rho)(\mathbf{x}, t) \in Z_2 := H^r\big(]-T, T[, H^s(\Omega)\big)$ and there is $M > 0$ such that by (2) we have

$$\big\|z(\mathbf{x}, t)\big\|_{Z_2} = \big\|\mathfrak{g}(\rho, t)\big\|_{H^r(]-T,T[)} \leq M\big\|\mathfrak{g}(\rho, t)\big\|_{[X_1, X_2]_\rho}$$

$$\leq M \max\big(\big\|\mathfrak{g}(\eta i, t)\big\|_{L^\infty(\mathbb{R}, X_1)}, \big\|\mathfrak{g}(1+\eta i, t)\big\|_{L^\infty(\mathbb{R}, X_2)}\big) \leq M\big(\big\|z(\mathbf{x}, t)\big\|_{Z_1} + \varepsilon\big)$$

and the continuity of the inclusion follows from the arbitrariness of $\varepsilon > 0$.

2) Now consider $Y_1 := H^s(]-T, T[, H^{r_1}(\Omega))$ and $Y_2 := H^s(]-T, T[, H^{r_2}(\Omega))$ and $Z_3 := \big[Y_1, Y_2\big]_\rho$. Since $Y_1 \subset Z_3$, given $z(\mathbf{x}, t) \in Y_1$ there is $f(\xi + \eta i, \mathbf{x}, t) \in \mathfrak{H}\big(\mathcal{Z}, Y_1, Y_2\big)$ such that $z(\mathbf{x}, t) = f(\rho, \mathbf{x}, t)$, $f(i\eta, \mathbf{x}, t) \in L^\infty(\mathbb{R}, Y_1)$, $f(1+\eta i, \mathbf{x}, t) \in L^\infty(\mathbb{R}, Y_2)$ and

$$\max\big(\big\|f(\eta i, \mathbf{x}, t)\big\|_{L^\infty(\mathbb{R}, Y_1)}, \big\|f(1+\eta i, \mathbf{x}, t)\big\|_{L^\infty(\mathbb{R}, Y_2)}\big) \leq \big\|z(\mathbf{x}, t)\big\|_{Z_3} + \varepsilon. \tag{3}$$

Then there is a set $T_0 \subset]-T, T[$ with Lebesgue measure 0 such that for every $t \in]-T, T[\backslash T_0$, the function $g_t(\xi + \eta i, \mathbf{x}) := f(\xi + \eta i, \mathbf{x}, t) \in \mathfrak{H}(\mathcal{Z}, H^{r_1}(\Omega), H^{r_2}(\Omega))$ verifies $g_t(\eta i, \mathbf{x}) \in L^\infty(\mathbb{R}, H^{r_1}(\Omega))$, $g_t(1+\eta i, \mathbf{x}) \in L^\infty(\mathbb{R}, H^{r_2}(\Omega))$. It follows that $g_t(\rho, \mathbf{x}) \in \big[H^{r_1}(\Omega), H^{r_2}(\Omega)\big]_\rho \approx H^r(\Omega)$ (by $\big[$[8] chapter 1, theorem 9.6$\big]$), where $r := (1-\rho)r_1 + \rho r_2$. Then there are $M_1 > 0, M_2 > 0$ such that by (3) $\big\|g_t(\rho, \mathbf{x})\big\|_{H^r(\Omega)} \leq M\big(\big\|z(\mathbf{x}, t)\big\|_{Z_3} + \varepsilon\big)$ and so

$$\big\|g_t(\rho, \mathbf{x})\big\|_{H^s(]-T,T[, H^r(\Omega)} = \Big\|\big\|g_t(\rho, \mathbf{x})\big\|_{H^r(\Omega)}\Big\|_{H^s(]-T,T[)} \leq \Big\|\ \big\|f(\rho, \mathbf{x}, t)\big\|_{H^r(\Omega)}\Big\|_{H^s(]-T,T[)}$$

$$\leq M_1(\big\|z(\mathbf{x}, t)\big\|_{Z_3} + 2T\varepsilon) \leq M_1(M_2\big\|z(\mathbf{x}, t)\big\|_{Y_1} + 2T\varepsilon)$$

and by the arbitrariness of $\varepsilon > 0$ we obtain the continuity of the inclusion map $Y_1 \subset H^s(]-T, T[, H^r(\Omega))$. Since Y_1 is dense in Z_3 ($\big[$[8], chapter 1, remark 2.6$\big]$) the map can be continuously extended to the inclusion into Z_3 (because $Z_3 \subset L^2(\Omega_T)$ and the convergence of a sequence in this space implies almost everywhere punctual convergence of some subsequence). □

We shall need more involved spaces in order to take into account the boundary conditions of our problems. Let $d(\mathbf{x}, \partial\Omega) := \inf_{\mathbf{y} \in \partial\Omega}\big\|\mathbf{x} - \mathbf{y}\big\|$ be the (continuous) distance up to the boundary $\partial\Omega$ of $\mathbf{x} \in \Omega$ function. Let $r \in \mathbb{N} \cup \{0\}$. We define

$$\Xi^r(\Omega) = \Big\{f \in L^2(\Omega)\ \Big|\ \big\|f\big\|_{\Xi^r(\Omega)} = \Big(\sum_{|\boldsymbol{\alpha}| \leq r}\Big\|d(\mathbf{x}, \partial\Omega)^{|\boldsymbol{\alpha}|}\ \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x})\Big\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} < \infty\Big\}. \tag{4}$$

$\Xi^r(\Omega)$ endowed with the norm $\|.\|_{\Xi^r(\Omega)}$ turns out to be a Banach space. We extend this definition to the case $r \in]0, \infty[$ by complex interpolation setting $\Xi^r(\Omega) = \big[\Xi^{k+1}(\Omega), \Xi^k(\Omega)\big]_{k+1-r}$ for $k < r < k+1, k \in \mathbb{N} \cup \{0\}$, endowed with any canonical norm of the interpolated space. It is easy to see that $H^r(\Omega) \subset \Xi^r(\Omega)$ for each $r \geq 0$. Moreover, it can be shown (see $\big[$[8], chapter 2, formula (6.21)$\big]$) that in these cases $\mathcal{D}(\Omega)$ is dense in $\Xi^r(\Omega)$. Hence, if $\Xi^{-r}(\Omega) := (\Xi^r(\Omega))'$ we have a Gelfand triple $\Xi^r(\Omega) \subset L^2(\Omega) \subset \Xi^{-r}(\Omega)$ and the inclusions $\Xi^{-r}(\Omega) \subset H^{-r}(\Omega)$ and $\Xi^{-r}(\Omega) \subset \big(H^r(\Omega)\big)'$.

In order to distinguish the behavior of temporal and spatial variables we introduce another space. Given $0 < T$ we fix a number $T_0 < \frac{T}{2}$. Consider the function $\varphi_{T_0,T} \in \mathcal{C}^\infty(\mathbb{R})$ with compact support $[-T,T]$ defined by

$$\varphi_{T_0,T}(t) := \begin{cases} e^{-\frac{T_0^2}{T_0^2-(t+T-T_0)^2}} & \text{if } \ -T < t \leq -T+T_0 \\ \frac{1}{e} & \text{if } \ -T+T_0 \leq t \leq T-T_0 \\ e^{-\frac{T_0^2}{T_0^2-(t-T+T_0)^2}} & \text{if } \ T-T_0 \leq t < T \\ 0 & \text{if } \ t \in]-\infty,-T] \cup [T,\infty[\end{cases}$$

Clearly $\|\varphi_{T_0,T}\|_{L^\infty(\mathbb{R})} = \frac{1}{e}$ independent on T. For every $s \leq r$ in $\mathbb{N} \cup \{0\}$ we define

$$\Xi^{r,s}(\Omega_T) := \Bigg\{ f \in L^2(]-T,T[, \Xi^r(\Omega)) \ \Bigg|$$

$$\|f\|_{\Xi^{r,s}(\Omega_T)} := \left(\sum_{j=0}^{s} \left\| \ |\varphi_{T_0,T}(t)|^j \ \frac{\partial^j f}{\partial t^j} \right\|^2_{L^2(]-T,T[,\Xi^{r-j}(\Omega))} \right)^{\frac{1}{2}} < \infty \Bigg\}.$$

As above we define $\Xi^{r,s}(\Omega_T)$ for $r \in [0,\infty[$ and $s \in \mathbb{N}$ by interpolation and, in a second step, we define $\Xi^{r,s}(\Omega_T)$, $r \geq 0, s \geq 0$ by interpolation on s, providing all interpolated spaces with any standard interpolation norm. Since $\lim_{t \to -T} \frac{\varphi_{T_0,T}(t)}{T+t} = \lim_{t \to T} \frac{\varphi_{T_0,T}(t)}{T-t} = 0$, from $\big[$[8], chapter 4, proposition 9.1$\big]$, we obtain that $\mathcal{D}(\Omega_T)$ is also dense in $\Xi^{r,s}(\Omega_T)$ and, defining for every $r \geq s \geq 0$ the space $\Xi^{-r,-s}(\Omega_T) := \Xi^{r,s}(\Omega_T)'$, it turns out that we have another Gelfand triple

$$\Xi^{r,s}(\Omega_T) \subset L^2(\Omega_T) \subset \Xi^{-r,-s}(\Omega_T) \subset \mathcal{D}'(\Omega \times \mathbb{R}). \tag{5}$$

An example that will be used later on: Given $\varphi(\mathbf{x}) \in \Xi^r(\Omega)$ and $\alpha > 0$, to estimate $\big\|e^{-\alpha\, t}\varphi(\mathbf{x})\big\|_{\Xi^{r,\frac{1}{2}+\varepsilon}(\Omega_T)}$, $0 < \varepsilon < \frac{1}{2}$ we note that

$$\big\|e^{-\alpha\, t}\varphi(\mathbf{x})\big\|_{\Xi^{r,0}(\Omega_T)} = \left(\int_{-T}^{T} |e^{-2\,\alpha\, t}| \big\|\varphi(\mathbf{x})\big\|^2_{\Xi^r(\Omega)} \, dt \right)^{\frac{1}{2}} \leq \big\|\varphi(\mathbf{x})\big\|_{\Xi^r(\Omega)} \frac{e^{\alpha T}}{\sqrt{2\alpha}}$$

and

$$\big\|e^{-\alpha\, t}\varphi(\mathbf{x})\big\|_{\Xi^{r,1}(\Omega_T)} \leq \left(\int_{-T}^{T} \left(e^{-2\alpha t} + \frac{\alpha^2}{e^2} \, e^{-2\alpha t} \right) \, \big\|\varphi(\mathbf{x})\big\|^2_{\Xi^r(\Omega)} \, dt \right)^{\frac{1}{2}} \leq$$

$$\leq \big\|\varphi(\mathbf{x})\big\|_{\Xi^r(\Omega)} \sqrt{1 + \frac{\alpha^2}{e^2}} \sqrt{\frac{e^{2\alpha T} - e^{-2\alpha T}}{2\alpha}} \leq \big\|\varphi(\mathbf{x})\big\|_{\Xi^r(\Omega)} \sqrt{1 + \frac{\alpha^2}{e^2}} \, \frac{e^{\alpha T}}{\sqrt{2\alpha}}.$$

Since $\Xi^{r,\frac{1}{2}+\varepsilon}(\Omega_T) = \big[\Xi^{r,1}(\Omega_T), \Xi^{r,0}(\Omega_T)\big]_{\frac{1}{2}-\varepsilon}$, there is $Q_\varepsilon > 0$ such that

$$\big\|e^{-\alpha\, t}\varphi(\mathbf{x})\big\|_{\Xi^{r,\frac{1}{2}+\varepsilon}(\Omega_T)} \le Q_\varepsilon \big\|e^{-\alpha\, t}\varphi(\mathbf{x})\big\|^{\frac{1}{2}+\varepsilon}_{\Xi^{r,1}(\Omega_T)} \big\|e^{-\alpha\, t}\varphi(\mathbf{x})\big\|^{\frac{1}{2}-\varepsilon}_{\Xi^{r,0}(\Omega_T)} \le$$

$$\le Q_\varepsilon \big\|\varphi(\mathbf{x})\big\|_{\Xi^{r}(\Omega)} \frac{e^{\alpha\, T}}{\sqrt{2\,\alpha}} \left(\sqrt{1+\frac{\alpha^2}{e^2}}\right)^{\frac{1}{2}+\varepsilon}. \tag{6}$$

Finally we consider spaces of the latter type but with unbounded temporal intervals. We fix an unbounded strictly increasing sequence $\{\mathfrak{T}_m\}_{m=0}^\infty$ in $]0,\infty[$ such that $\mathfrak{T}_0 < \mathfrak{T}_m - \mathfrak{T}_{m-1}$ for each $m \in \mathbb{N}$ and let S_m be the map sending every measurable function in $\Omega \times \mathbb{R}$ to its restriction to $\Omega \times]-\mathfrak{T}_m, \mathfrak{T}_m[$. Given $0 < r, s$ we define

$$\Xi^{r,s}(\Omega\times\mathbb{R}) := \left\{ f : \Omega\times\mathbb{R} \longrightarrow \mathbb{R} \;\middle|\; \big\|f\big\|_{\Xi^{r,s}(\Omega\times\mathbb{R})} := \sup_{m\in\mathbb{N}} \big\|S_m(f)\big\|_{\Xi^{r,s}(\Omega_{\mathfrak{T}_m})} < \infty \right\}. \tag{7}$$

For every $f \in \mathcal{D}(\Omega_{\mathfrak{T}_m})$ let $\widetilde{f} \in \mathcal{D}(\Omega\times\mathbb{R})$ be its extension by 0 on the complement of $\Omega_{\mathfrak{T}_m}$. Then $\widetilde{f} \in \Xi^{r,s}(\Omega\times\mathbb{R})$ and $S_m(\widetilde{f}) = f$. As $\mathcal{D}(\Omega_{\mathfrak{T}_m})$ is dense in $\Xi^{r,s}(\Omega_{\mathfrak{T}_m})$ it turns out that $S_m \in \mathcal{L}\big(\Xi^{r,s}(\Omega\times\mathbb{R}), \Xi^{r,s}(\Omega_{\mathfrak{T}_m})\big)$ has a dense range and so the adjoint map $S'_m \in \mathcal{L}\big(\Xi^{-r,-s}(\Omega_{\mathfrak{T}_m}), (\Xi^{r,s}(\Omega\times\mathbb{R}))'\big)$ is injective. Then we define $\Xi^{-r,-s}(\Omega\times\mathbb{R}) := \bigcup_{m=1}^\infty S'_m(\Xi^{-r,-s}(\Omega_{\mathfrak{T}_m}))$ provided with the topology induced by the norm topology of $(\Xi^{r,s}(\Omega\times\mathbb{R}))'$. We shall identify each $\Psi \in \Xi^{-r,-s}(\Omega_{\mathfrak{T}_m})$ with $S'_m(\Psi) \in \Xi^{-r,-s}(\Omega\times\mathbb{R})$. The following lemma holds:

Lemma 2 *Let $\Psi \in \Xi^{-r,-s}(\Omega,\mathbb{R})$. There are $m \in \mathbb{N}$ and a sequence $\varphi_k(\mathbf{x},t) \in \mathcal{D}(\Omega\times\mathbb{R})$, such that $\Psi = \lim_{k\to\infty} \varphi_k(\mathbf{x},t)$ in $\Xi^{-r,-s}(\Omega,\mathbb{R})$ and $\bigcup_{k=1}^\infty Supp(\varphi_k) \subset \Omega_{\mathfrak{T}_m}$.*

Proof By definition of $\Xi^{-r,-s}(\Omega\times\mathbb{R})$ there are $m \in \mathbb{N}$ and $\varphi \in \Xi^{-r,-s}(\Omega_{\mathfrak{T}_m})$ such that $\Psi = S'_m(\varphi)$. By (5) there is a sequence $\{\varphi_k\}_{k=1}^\infty \subset \mathcal{D}(\Omega_{\mathfrak{T}_m})$ convergent to φ in $\Xi^{-r,-s}(\Omega_{\mathfrak{T}_m})$. Then $\Psi = \lim_{k\to\infty} S'_m(\varphi_k)$ in $\Xi^{-r,-s}(\Omega\times\mathbb{R})$. The fact that $S'_m(\varphi_k) = \widetilde{\varphi}_k, k \in \mathbb{N}$ ends the proof. □

For our concrete applications we note the important result:

Proposition 3 *Let $\mathbf{x}_0 \in \Omega$ and $-\frac{T}{2} < -T_0 < t_0 < T_0 < \frac{T}{2}$. If $r > \frac{n}{2}$ and $r \ge s > \frac{1}{2}$ we have $\delta(\mathbf{x}-\mathbf{x}_0)\otimes\delta(t-t_0) \in \Xi^{-r,-s}(\Omega_T)$ and so $\delta(\mathbf{x}-\mathbf{x}_0)\otimes\delta(t-t_0) \in \Xi^{-r,-s}(\Omega\times\mathbb{R})$. Moreover, for each $k_0 \in \mathbb{N}$ there is $K_{k_0}(\Omega)$ such that*

$$\sup\Big\{ \big\|\delta(\mathbf{x}-\mathbf{x}_0)\otimes\delta(t-t_0)\big\|_{\Xi^{-r,-s}(\Omega_T}, \; d(\mathbf{x}_0,\partial\Omega) > \frac{1}{k_0}, \; \big|t_0\big| < T - \frac{1}{k_0} \Big\} \le K_{k_0}(T,\Omega).$$

Proof Define $\Omega_k := \Big\{\mathbf{x} \in \Omega \quad | \quad d(\mathbf{x},\partial\Omega) > \frac{1}{k}\Big\}$, $k \in \mathbb{N}$ and choose $k_0 \in \mathbb{N}$ such that $\mathbf{x}_0 \in \Omega_{k_0}$ and $|t_0| < T - \frac{1}{k_0}$. Fix functions $\rho_1(\mathbf{x}) \in \mathcal{D}(\Omega)$ and $\rho_2(t) \in \mathcal{D}(]-T,T[)$ such that $\rho_1(\mathbf{x}) = 1$ if $\mathbf{x} \in \Omega_{k_0}$, $\rho_1(\mathbf{x}) = 0$ if $\mathbf{x} \in \Omega\backslash\Omega_{2\,k_0}$ and $\rho_2(t) = 1$ if $t \in \Big]-T+\frac{1}{k_0}, T-\frac{1}{k_0}\Big[$ and $\rho_2(t) = 0$ if $t \in \Big]-T, -T+\frac{1}{2\,k_0}\Big[\cup \Big]T-\frac{1}{2\,k_0}, T\Big[$.

The map $u \in \Xi^{r,s}(\Omega_T) \longrightarrow \rho_1\rho_2 u \in H^s(]-T,T[, H^r(\Omega))$ is continuous and there is $M > 0$ (only depending on $\rho_1(\mathbf{x}), \rho_2(t), k_0, r, s$ and $\varphi_{T_0,T}(t)$) such that

$$\forall\, u \in \Xi^{r,s}(\Omega_T) \quad \Big\|\rho_1\;\rho_2\;u\Big\|_{H^s(]-T,T[,H^r(\Omega))} \le M\;\Big\|u\Big\|_{\Xi^{r,s}(\Omega_T)}. \tag{8}$$

In fact, if r, s are in $\mathbb{N}$ and $u \in \mathcal{D}(\Omega_T)$ it is straightforward to check (8) using Leibnitz's rule on derivative of products. The general result follows by density and can be extended to arbitrary $0 \leq r, s$ by interpolation and application of lemma 1.

On the other hand, since $r > \frac{n}{2}$, by Sobolev's embedding theorem we have $H^r(\Omega) \subset \mathcal{C}(\overline{\Omega})$ and there is $K_1 > 0$ such that for every $u \in \mathcal{D}(\Omega_T)$ we have

$$\forall \ -T < t < T \quad \Big\| \rho_1(\mathbf{x}) \ u(\mathbf{x}, t) \Big\|_{\mathcal{C}(\overline{\Omega})} \leq K_1 \ \Big\| \rho_1(\mathbf{x}) \ u(\mathbf{x}, t) \Big\|_{H^r(\Omega)}. \tag{9}$$

In the same way, since $s > \frac{1}{2}$, by the vector valued version of $\big[$[8], chapter 1, theorem 9.8$\big]$ (the proof is exactly the same as that in the scalar valued case) we obtain $H^s(]-T, T[, H^r(\Omega)) \subset \mathcal{C}([-T, T], H^r(\Omega))$ and there is $K_2 > 0$ such that

$$\forall \ f \in H^s(]-T, T[, H^r(\Omega)) \quad \Big\| f(\mathbf{x}, t) \Big\|_{\mathcal{C}([-T,T], H^r(\Omega))} \leq K_2 \ \Big\| f(\mathbf{x}, t) \Big\|_{H^s(]-T,T[, H^r(\Omega))}. \tag{10}$$

Hence, by (9), (10), and (8) we obtain for every $u \in \mathcal{D}(\Omega_T)$

$$\Big| \Big\langle u, \delta(\mathbf{x} - \mathbf{x}_0) \otimes \delta(t - t_0) \Big\rangle \Big| = |u(\mathbf{x}_0, t_0)| \leq \frac{1}{\rho_1(\mathbf{x}_0)} \ \Big\| \rho_1(\mathbf{x}) \ u(\mathbf{x}, t_0) \Big\|_{\mathcal{C}(\overline{\Omega})} \leq$$

$$\leq \frac{K_1}{\rho_1(\mathbf{x}_0)} \ \Big\| \rho_1(\mathbf{x}) \ u(\mathbf{x}, t_0) \Big\|_{H^r(\Omega)} \leq K_1 \ \frac{1}{\rho_1(\mathbf{x}_0) \ \rho_2(t_0)} \Big\| \rho_1 \ \rho_2 \ u \Big\|_{\mathcal{C}([-T,T], H^r(\Omega))} \leq$$

$$\leq \frac{K_1 \ K_2}{\rho_1(\mathbf{x}_0) \ \rho_2(t_0)} \Big\| \rho_1 \ \rho_2 \ u \Big\|_{H^s(]-T,T[, H^r(\Omega))} \leq M \ K_1 \ K_2 \ \Big\| u \Big\|_{\Xi^{r,s}(\Omega_T)}$$

and hence, by density of $\mathcal{D}(\Omega_T)$ in $\Xi^{r,s}(\Omega_T)$ we obtain finally

$$\sup \Big\{ \big\| \delta(\mathbf{x} - \mathbf{x}_0) \otimes \delta(t - t_0) \big\|_{\Xi^{-r,-s}(\Omega_T)}, \mathbf{x}_0 \in \Omega_{k_0}, |t_0| < T - \frac{1}{k_0} \Big\} \leq K_{k_0}(T, \Omega) \tag{11}$$

where $K_{k_0}(T, \Omega)$ is increasing with Ω. □

In the next theorem we need weighted spaces. Given a measurable real function $g : \Omega \longrightarrow]0, \infty[$ we define the normed weighted space $L^2(\Omega, g)$ as the set of measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that $\big\| f \big\|_{L^2(\Omega, g)} := \big\| f g \big\|_{L^2(\Omega)} < \infty$. If E is a Banach space, the vector E-valued weighted space $L^2(\Omega, g, E)$ is the set of strongly measurable functions $f : \Omega \longrightarrow E$ such that $\big\| f \big\|_{L^2(\Omega, g, E)} := \big\| \| f \|_E \big\|_{L^2(\Omega, g)} < \infty \Big\}$.

Proposition 4 *If $s < r$ and $T > 0$, the inclusion $\Xi^{r,r}(\Omega_T) \subset \Xi^{s,s}(\Omega_T)$ is compact.*

Proof First assume $s \in \mathbb{N} \cup \{0\}, r \in \mathbb{N}$, and $s < r$. Given a bounded sequence $\{f_m\}_{m=1}^{\infty}$ in $\Xi^r(\Omega)$, for every $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = r - 1$ it turns out that the inclusions $\Big\{ d(\mathbf{x}, \partial\Omega)^{|\boldsymbol{\alpha}|} \frac{\partial^{|\boldsymbol{\alpha}|} f_m}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \Big\}_{m=1}^{\infty} \subset L^2(\Omega)$ and $\Big\{ d(\mathbf{x}, \partial\Omega)^{|\boldsymbol{\alpha}|+1} \frac{\partial}{\partial x_j} \Big(\frac{\partial^{|\boldsymbol{\alpha}|} f_m}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \Big) \Big\}_{m=1}^{\infty} \subset L^2(\Omega)$, $j = 1, 2, ..., n$ hold. Then, by $\big[$ [5], theorem 2.7$\big]$ there is a subsequence $\Big\{ \frac{\partial^{|\boldsymbol{\alpha}|} f_{k_m}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \Big\}_{m=1}^{\infty}$ convergent in the weighted space $L^2(\Omega, d(\mathbf{x}, \partial\Omega)^{|\boldsymbol{\alpha}|})$. After a finite and analogous inductive process on $|\boldsymbol{\alpha}| = r - k, 1 \leq k \leq r - 1$ we conclude that the inclusion $\Xi^r(\Omega) \subset \Xi^{r-1}(\Omega)$ is compact.

Now, consider for every $0 \le j < r$ the Banach space

$$W_j^r := \left\{ f \in L^2\Big(]-T,T[,\varphi_{T_0,T}^j, \Xi^{r-j}(\Omega)\Big) \;\Big|\; \frac{\partial f}{\partial t} \in L^2\Big(]-T,T[,\varphi_{T_0,T}^j, \Xi^{r-j-1}(\Omega)\Big) \right\}$$

provided with the norm

$$\left\|f\right\|_{W_j^r} := \left(\left\|\varphi_{T_0,T}^j \, f\right\|^2_{L^2(]-T,T[,\Xi^{r-j}(\Omega))} + \left\|\varphi_{T_0,T}^{j-1} \, \frac{\partial f}{\partial t}\right\|^2_{L^2(]-T,T[,\Xi^{r-j-1}(\Omega))} \right)^{\frac{1}{2}}.$$

As the map $f \longrightarrow \varphi_{T_0,T}^j \, f$ is an isometry from $L^2(]-T,T[,\varphi_{T_0,T}^j, \Xi^{r-j}(\Omega))$ onto $L^2(]-T,T[,\Xi^{r-j}(\Omega))$, by theorem 5.1, chapter 1 in [7] the inclusion $W_j^r \subset L^2(]-T,T[,\varphi_{T_0,T}^j, \Xi^{r-j}(\Omega))$ is compact for every $0 \le j \le s$. Since for every bounded sequence $\{f_m\}_{m=1}^{\infty}$ in $\Xi^{r,r}(\Omega_t)$ and every $0 \le j \le s$ it turns out that $\left\{\frac{\partial^j f_m}{\partial t^j}\right\}_{m=1}^{\infty}$ is bounded in W_j^r the conclusion follows afterwards a finite inductive process, indexed by $j = 0,1,...,s$, selecting in every step j a suitable subsequence of the previous one that converges in $L^2(]-T,T[,\varphi_{T_0,T}^j, \Phi^{r-j}(\Omega))$.

The proof for arbitrary real numbers $0 < s < r$ follows by classical compactness theorems of interpolated operators due to Calderón (see [1]). □

2. Auxiliary technical results

To achieve our results we shall always consider general operators

$$\mathcal{A} := -\mathcal{X} + \frac{1}{\alpha}\left(\frac{\partial}{\partial t} + \tau\,\frac{\partial^2}{\partial t^2}\right), \quad \mathcal{X} := \sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}| \le 1} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\,\mathbf{x}^{\boldsymbol{\alpha}}}\left(a_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\mathbf{x})\,\frac{\partial^{|\boldsymbol{\beta}|}}{\partial\,\mathbf{x}^{\boldsymbol{\beta}}}\right),$$

where $\mathcal{X}$ is a *self-adjoint strongly uniform elliptic operator* in $\overline{\Omega}$ (in the sense of Wloka, [17]) with *real coefficients* $a_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\mathbf{x}) \in \mathcal{C}^\infty(\overline{\Omega})$ verifying

$$\forall\,\mathbf{x} \in \overline{\Omega},\; \forall\,\boldsymbol{\xi} = (\xi_1,...,\xi_n) \in \mathbb{R}^n \qquad \sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=1} a_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\mathbf{x})\boldsymbol{\xi}^{\boldsymbol{\alpha}}\boldsymbol{\xi}^{\boldsymbol{\beta}} \ge K_{\mathcal{X}} \sum_{i=1}^n \left|\xi_i\right|^2 \tag{12}$$

for some $1 > K_{\mathcal{X}} > 0$ (only dependent on Ω and $\mathcal{X}$). Moreover, by the theorem of traces $\big[$[8], chapter 1, theorem 8.3$\big]$, we shall consider differential boundary operators with real coefficients f, g_1, g_2, g_3 in $\mathcal{C}^\infty(\partial\Omega)$ of type

$$\mathfrak{B} : h(\mathbf{x}) \in H^2(\Omega) \longrightarrow \mathfrak{B}(h(\mathbf{x})) := f(\mathbf{x})h(\mathbf{x}) + \sum_{i=1}^3 g_i(\mathbf{x})\frac{\partial h}{\partial x_i}(\mathbf{x}) \in L^2(\partial\Omega). \tag{13}$$

If we take *formally* the Schwartz–Laplace transform $\mathfrak{L}$ with respect to t of a distribution $\mathcal{A}(U)$ we obtain

$$\mathfrak{L}\big[\mathcal{A}(U)\big](p) = \Big(-\mathcal{X} + \frac{1}{\alpha}\left(p + \tau\,p^2\right)\Big)\mathfrak{L}[U](p) = \frac{\tau}{\alpha}\Big(-\frac{\alpha}{\tau}\mathcal{X} - \frac{1}{4\tau^2} + \Big(p + \frac{1}{2\tau}\Big)^2\Big)\mathfrak{L}[U](p),$$

which leads us in a natural way to the introduction of the operators

$$\mathfrak{A} := \frac{\alpha}{\tau}\,\mathcal{X} + \frac{1}{4\tau^2} \qquad \text{and} \qquad \mathcal{X}_p := \frac{\tau}{\alpha}\left(-\mathfrak{A} + \left(p + \frac{1}{2\tau}\right)^2\right), \quad p \in \mathbb{C}. \tag{14}$$

It follows from (12) and the argumentation in $\big[$[17], example 19.1$\big]$ that $\mathfrak{A}$ is $H_0^1(\Omega)$-coercive and so there are constants $\mu_{\mathcal{X}} \geq 1$ and $C_{\mathcal{X}}$ (only dependent on Ω and $\mathcal{X}$) such that $\mathcal{X}_p$ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ when p lies in the half-space $S_{\mu_{\mathcal{X}}} := \left\{p \in \mathbb{C} \mid Re(p + \frac{1}{2\tau}) > \mu_{\mathcal{X}}\right\}$ and by $\big[$[17] theorem 17.10$\big]$

$$\left\|\mathcal{X}_p^{-1} : H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)\right\| \leq \frac{1}{C_{\mathcal{X}}}. \tag{15}$$

Sometimes we shall need to consider the restriction to some Banach space $F \subset H^{-1}(\Omega)$ of $\mathcal{X}_p^{-1}$ or to consider $\mathcal{X}_p^{-1}$ as an operator with range larger than $H_0^1(\Omega)$. These new maps will be denoted by the same symbol $\mathcal{X}_p^{-1}$ unless a more formal representation is necessary.

The linear space

$$\forall\, r \in [0, \infty[\quad \mathfrak{R}^{r+2}(\Omega) := \left\{ f \in H^{r+2}(\Omega) \cap H_0^{r+1}(\Omega) \;\middle|\; \mathfrak{B}(f) = 0 \text{ and } \mathfrak{A}(f) \in H_0^r(\Omega) \right\},$$

provided with the norm induced by $H^{r+2}(\Omega)$, will play an important role in the sequel. Since $\mathcal{D}(\Omega) \subset \mathfrak{R}^{r+2}(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, we have the natural inclusions $\mathfrak{R}^{r+2}(\Omega) \subset L^2(\Omega) \subset \left(\mathfrak{R}^{r+2}(\Omega)\right)' \subset \mathcal{D}'(\Omega)$ forming a Gelfand triple. The continuous inclusion map $\mathfrak{R}^r(\Omega) \subset H^r(\Omega)$ will be denoted by R_r.

Proposition 5 *Let* $r \in [0, \infty[$.

1) $\mathfrak{R}^{r+2}(\Omega)$ *is a Banach space.*

2) $\mathcal{X}_p$ *is an isomorphism from* $\mathfrak{R}^{r+2}(\Omega)$ *onto* $H_0^r(\Omega)$ *for every* $p \in S_{\mu_{\mathcal{X}}}$ *and, in the case* $r - \frac{1}{2} \notin \mathbb{Z}$, *there is* $M_r \geq 0$ *(only dependent on* Ω *and* $\mathcal{X}$*) such that*

$$\left\|R_{r+2}\mathcal{X}_p^{-1}\right\| = \left\|\mathcal{X}_p^{-1}\right\|_{\mathcal{L}(H_0^r(\Omega), H^{r+2}(\Omega))} \leq M_r. \tag{16}$$

Proof 1). It is enough to see that $\mathfrak{R}^{r+2}(\Omega)$ is closed in $H^{r+2}(\Omega)$. Let $f = \lim_{m\to\infty} f_m$ in $H^{r+2}(\Omega)$ with $\{f_m\}_{m=1}^{\infty} \subset \mathfrak{R}^{r+2}(\Omega)$. It turns out that $\{\mathfrak{A}(f_m)\}_{m=1}^{\infty} \subset H_0^r(\Omega)$ must be a Cauchy sequence in $H^r(\Omega)$ because $\mathfrak{A} \in \mathcal{L}\big(H^{r+2}(\Omega), H^r(\Omega)\big)$. Then $\mathfrak{A}(f) = \lim_{m\to\infty} \mathfrak{A}(f_m) \in H_0^r(\Omega)$ since $H_0^r(\Omega)$ is a Banach space. In the same way we have necessarily $f = \lim_{m\to\infty} f_m$ in $H_0^{r+1}(\Omega)$. Finally, for every $m \in \mathbb{N}$ there is $\varphi_m \in \mathcal{D}(\overline{\Omega})$ such that $\left\|f_m - \varphi_m\right\|_{H^{r+2}(\Omega)} \leq \frac{1}{m}$ and $\left\|\mathfrak{B}(\varphi_m)\right\|_{H^{r+\frac{1}{2}}(\partial\Omega)} \leq \frac{1}{m}$. Then $f = \lim_{m\to\infty} \varphi_m$ in $H^{r+2}(\Omega)$ and the continuity of $\mathfrak{B}$ gives us $\mathfrak{B}(f) = 0$ on $\partial\Omega$. Hence $f \in \mathfrak{R}^{r+2}(\Omega)$.

2) Let $p \in S_{\mu_{\mathcal{X}}}$. Obviously we have $\mathcal{X}_p \in \mathcal{L}\big(\mathfrak{R}^{r+2}(\Omega), H_0^r(\Omega)\big)$ and by the $H_0^1(\Omega)$-coerciveness of $\mathcal{X}_p$ it is injective. By the open map theorem we only need to show that $\mathcal{X}_p$ is surjective onto $H_0^r(\Omega)$.

Let γ_0 be the trace operator on $\partial\Omega$. Given $f \in H_0^r(\Omega) \subset L^2(\Omega)$ there is a unique $U_f \in H_0^1(\Omega)$ such that $\mathcal{X}_p(U_f) = f$ and $\gamma_0(U_f) = 0$ (by $\big[$[8], chapter 1, theorem 11.5$\big]$). In the same way, as $\frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \in H^{-1}(\Omega)$ whenever $|\boldsymbol{\alpha}| \leq r + 1$, there is $V_{\boldsymbol{\alpha}} \in H_0^1(\Omega)$ such that $\mathcal{X}_p(V_{\boldsymbol{\alpha}}) = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}$. In such a case, for every $\varphi \in \mathcal{D}(\Omega)$, we have

$$\left\langle V_{\boldsymbol{\alpha}}, \varphi \right\rangle = \left\langle (\mathcal{X}_p)^{-1} \left(\frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right), \varphi \right\rangle = \left\langle \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}, (\mathcal{X}_p^{-1})'(\varphi) \right\rangle =$$

$$= (-1)^{|\boldsymbol{\alpha}|} \left\langle f, \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} (\mathcal{X}_p^{-1})'(\varphi) \right\rangle = (-1)^{|\boldsymbol{\alpha}|} \left\langle \mathcal{X}_p(U_f), \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} (\mathcal{X}_p^{-1})'(\varphi) \right\rangle =$$

$$= \left\langle \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \mathcal{X}_p(U_f), (\mathcal{X}_p^{-1})'(\varphi) \right\rangle = \left\langle \mathcal{X}_p \left(\frac{\partial^{|\boldsymbol{\alpha}|} U_f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right), (\mathcal{X}_p^{-1})'(\varphi) \right\rangle = \left\langle \frac{\partial^{|\boldsymbol{\alpha}|} U_f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}, \varphi \right\rangle$$

and so $\frac{\partial^{|\boldsymbol{\alpha}|} U_f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} = V_{\boldsymbol{\alpha}} \in H_0^1(\Omega)$ and $\gamma_0 \left(\frac{\partial^{|\boldsymbol{\alpha}|} U_f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right) = 0$. In particular, by $\big[$[8], chapter 1, theorem 11.5$\big]$ we obtain $U_f \in H^{r+2}(\Omega) \cap H_0^{r+1}(\Omega)$ and $\frac{\partial U_f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} = V_{\boldsymbol{\alpha}} \in H_0^1(\Omega)$ when $|\boldsymbol{\alpha}| = 1$. By definition of $\mathfrak{B}$ we obtain $\mathfrak{B}(U_f) = 0$, i.e. $U_f \in \mathfrak{R}^{r+2}(\Omega)$.

On the other hand, if $f \in L^2(\Omega)$ by (15) we have

$$\left\| \mathcal{X}_p^{-1}(f) \right\|_{\mathfrak{R}^2(\Omega)}^2 = \left\| \mathcal{X}_p^{-1}(f) \right\|_{H^1(\Omega)}^2 + \sum_{|\boldsymbol{\alpha}|=2} \left\| \frac{\partial^2 \mathcal{X}_p^{-1}(f)}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right\|_{L^2(\Omega)}^2$$

$$\leq \frac{\|f\|_{H^{-1}(\Omega)}^2}{C_{\mathcal{X}}^2} + \sum_{|\boldsymbol{\alpha}|=1} \left\| \frac{\partial \mathcal{X}_p^{-1}(f)}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right\|_{H^1(\Omega)}^2 \leq \frac{\|f\|_{L^2(\Omega)}^2}{C_{\mathcal{X}}^2} + \sum_{|\boldsymbol{\alpha}|=1} \left\| \mathcal{X}_p^{-1} \left(\frac{\partial f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right) \right\|_{H^1(\Omega)}^2$$

$$\leq \frac{\|f\|_{L^2(\Omega)}^2}{C_{\mathcal{X}}^2} + \sum_{|\boldsymbol{\alpha}|=1} \frac{1}{C_{\mathcal{X}}^2} \left\| \frac{\partial f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \right\|_{H^{-1}(\Omega)}^2 \leq \frac{\|f\|_{L^2(\Omega)}^2}{C_{\mathcal{X}}^2} + \frac{n}{C_{\mathcal{X}}^2} \|f\|_{L^2(\Omega)}^2 = M_1^2 \|f\|_{L^2(\Omega)}^2$$

obtaining (16) for $r = 0$. The proof for $r \in \mathbb{N}$ is analogous using induction on r. Finally the general case $r - \frac{1}{2} \notin \mathbb{Z}$ follows by interpolation since the isomorphisms $H^r(\Omega) \approx \left[H^{k+1}(\Omega), H^k(\Omega) \right]_{k+1-r}$ and $H_0^r(\Omega) \approx \left[H_0^{k+1}(\Omega), H_0^k(\Omega) \right]_{k+1-r}$ hold for $r \in]k, k+1[\setminus \left\{ k + \frac{1}{2} \right\}, k \in \mathbb{N} \cup \{0\}$, ($\big[$[8], chapter 1, theorems 9.6 and 11.6$\big]$). □

We have not been able to find fine estimates of $\left\| \mathcal{X}_p^{-1} : H_0^r(\Omega) \longrightarrow \mathfrak{R}^{r+2}(\Omega) \right\|$ as a function of the parameter $p \in S_{\mu_{\mathcal{X}}}$. Fortunately, for our main purposes it will be enough to find these estimates if we replace $\mathfrak{R}^{r+2}(\Omega)$ with some larger spaces. First we note the following result:

Theorem 6 *Let $r \in [0, \infty[\setminus \left(\mathbb{Z} + \frac{1}{2} \right)$. There is $M(r) > 0$ such that if $p \in S_{\mu_{\mathcal{X}}}$ the operator $\mathcal{X}_p^{-1}$ considered an operator from $H_0^r(\Omega)$ into $H^r(\Omega)$ verifies*

$$\left\| (\mathcal{X}_p)^{-1} \right\|_{\mathcal{L}(H_0^r(\Omega), H^r(\Omega))} = \left\| I_{r+2,r} R_{r+2} \mathcal{X}_p^{-1} \right\| \leq \frac{M(r)}{\left| Re \left(p + \frac{1}{2\,\tau} \right) \right| \left| p + \frac{1}{2\,\tau} \right|}. \tag{17}$$

Proof a) Let $p \in S_{\mu_{\mathcal{X}}}$. By theorem 5, $\mathcal{X}_p$ is an isomorphism from $\mathfrak{R}^2(\Omega) \subset L^2(\Omega)$ onto $L^2(\Omega)$. If $Im \left(p + \frac{1}{2\tau} \right) \neq 0$, it follows from $\big[$[3], chapter 8, §1, proposition 2$\big]$ that

$$\left\| \mathcal{X}_p^{-1} \right\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \leq \frac{1}{\left| Im \left(p + \frac{1}{2\,\tau} \right)^2 \right|} = \frac{1}{2 \left| Re \left(p + \frac{1}{2\tau} \right) Im \left(p + \frac{1}{2\tau} \right) \right|}. \tag{18}$$

If $Arg\left(p+\frac{1}{2\tau}\right)\le\frac{\pi}{4}$ we have

$$2\left|Re\left(p+\frac{1}{2\tau}\right)Im\left(p+\frac{1}{2\tau}\right)\right|=\left|p+\frac{1}{2\tau}\right|^2\left|\sin 2Arg\left(p+\frac{1}{2\tau}\right)\right|$$

$$\ge\frac{1}{\sqrt{2}}\left|p+\frac{1}{2\tau}\right|^2\ge\frac{1}{\sqrt{2}}\left|p+\frac{1}{2\tau}\right|\left|Re\left(p+\frac{1}{2\tau}\right)\right|$$

and

$$Arg\left(p+\frac{1}{2\tau}\right)\ge\frac{\pi}{4}\implies\left|Im\left(p+\frac{1}{2\tau}\right)\right|\ge\frac{1}{\sqrt{2}}\left|p+\frac{1}{2\tau}\right|.$$

Then (17) follows from (18) if $r=0$ and $Im\left(p+\frac{1}{2\tau}\right)\neq 0$.

If $p_0\in S_{\mu_{\mathbb{X}}}$ and $Im\left(p_0+\frac{1}{2\tau}\right)=0$, taking a sequence $\{p_m\}_{m=1}^{\infty}\subset S_{\mu_{\mathcal{X}}}\backslash\mathbb{R}$ such that $p_0=\lim_{m\to\infty}p_m$, we remark that, using (16) we can write

$$\left\|I_{2,0}R_2(\mathcal{X}_{p_m}^{-1}-\mathcal{X}_{p_0}^{-1})\right\|=\left\|I_{2,0}R_2\big(\mathcal{X}_{p_0}^{-1}(\mathcal{X}_{p_0}-\mathcal{X}_{p_m})\mathcal{X}_p^{-1}\big)\right\|$$

$$\le\left\|\mathcal{X}_{p_0}^{-1}\right\|\left\|\mathcal{X}_{p_0}-\mathcal{X}_{p_m}\right\|\left\|\mathcal{X}_{p_m}^{-1}\right\|\le M_0^2\frac{\tau}{\alpha}\left|\left(p_m+\frac{1}{2\tau}\right)^2-\left(p_0+\frac{1}{2\tau}\right)^2\right|.$$

This implies that $\lim_{m\to\infty}\left\|I_{2,0}R_2(\mathcal{X}_{p_m}^{-1}-\mathcal{X}_{p_0}^{-1})\right\|=0$ and by the previous result

$$\left\|\mathcal{X}_{p_0}^{-1}\right\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}=\lim_{m\to\infty}\left\|\mathcal{X}_{p_m}^{-1}\right\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}$$

$$\le\lim_{m\to\infty}\frac{M(0)}{\left|Re\left(p_m+\frac{1}{2\,\tau}\right)\right|\left|p_m+\frac{1}{2\,\tau}\right|}=\frac{M(0)}{\left|Re\left(p_0+\frac{1}{2\,\tau}\right)\right|\left|p_0+\frac{1}{2\,\tau}\right|}.$$

b) If $r\in\mathbb{N}$ given $|\boldsymbol{\alpha}|\le r$, for every $f\in H_0^r(\Omega)$, we have $\mathcal{X}_p\left(\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}\mathcal{X}_p^{-1}(f)\right)=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}\Big(\mathcal{X}_p(\mathcal{X}_p^{-1}(f)\Big)=\frac{\partial^{|\boldsymbol{\alpha}|}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}\in H_0^{r-|\boldsymbol{\alpha}|}(\Omega)\subset L^2(\Omega)$ and, by proposition 5

$$\mathcal{X}_p^{-1}\left(\frac{\partial^{|\boldsymbol{\alpha}|}f}{\partial\,\mathbf{x}^{\boldsymbol{\alpha}}}\right)=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}\Big(\mathcal{X}_p^{-1}(f)\Big)\in\mathfrak{R}^{r-|\boldsymbol{\alpha}|+2}(\Omega).\tag{19}$$

By part a) one has

$$\frac{M(0)}{\left|Re\left(p+\frac{1}{2\tau}\right)\right|\left|p+\frac{1}{2\tau}\right|}\left\|\frac{\partial^{|\boldsymbol{\alpha}|}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}\right\|_{L^2(\Omega)}\ge\left\|\mathcal{X}_p^{-1}\left(\frac{\partial^{|\boldsymbol{\alpha}|}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}\right)\right\|_{L^2(\Omega)}$$

and summing over all multi-indexes $|\boldsymbol{\alpha}|\le r$ it turns out that (17) holds.

c) Finally the general case $r\in[1,\infty[\backslash\left(\mathbb{N}+\frac{1}{2}\right)$ follows from part b) by interpolation (as in the second part of proposition 5). □

As a consequence we obtain

Corollary 7 *Let* $0 < r < m$ *in* $\mathbb{R}$ *such that* $r - \frac{1}{2} \notin \mathbb{Z}, m - \frac{1}{2} \notin \mathbb{Z}$. *Then, if* $p \in S_{\mu_{\mathcal{X}}}$ *we have* $\mathcal{X}_p^{-1} \in \mathcal{L}\big(H_0^r(\Omega), H^{r\left(1+\frac{2}{m}\right)}(\Omega)\big)$ *and there is* $A(r,m) > 0$ *independent of* p *such that*

$$\|\mathcal{X}_p^{-1}\|_{\mathcal{L}\left(H_0^r(\Omega), H^{r\left(1+\frac{2}{m}\right)}(\Omega)\right)} \leq \frac{A(r,m)}{\left|Re\left(p + \frac{1}{2\,\tau}\right)\right|^{1-\frac{r}{m}} \left|p + \frac{1}{2\tau}\right|^{1-\frac{r}{m}}}. \tag{20}$$

Proof It follows from $\big[$[8], chapter 1, theorems 11.6 and 12.4$\big]$ that $H_0^r(\Omega) \approx \big[H_0^m(\Omega), L^2(\Omega)\big]_{1-\frac{r}{m}}$ and $\big[H^{m+2}(\Omega), L^2(\Omega)\big]_{1-\frac{r}{m}} \approx H^{r\left(1+\frac{2}{m}\right)}(\Omega)$. Interpolating the operators $R_{m+2}\mathcal{X}_p^{-1} \in \mathcal{L}\big(H_0^m(\Omega), H^{m+2}(\Omega)\big)$ and $I_{2,0}R_2\mathcal{X}_p^{-1} \in \mathcal{L}\big(L^2(\Omega), L^2(\Omega)\big)$ we obtain $\mathcal{X}_p^{-1} \in \mathcal{L}\big(H_0^r(\Omega), H^{r\left(1+\frac{2}{m}\right)}(\Omega)\big)$ and there is $c(r,m) > 0$ such that

$$\|\mathcal{X}_p^{-1}\|_{\mathcal{L}\left(H_0^r(\Omega), H^{r\left(1+\frac{2}{m}\right)}(\Omega)\right)} \leq c(r,m)\|\mathcal{X}_p^{-1}\|^{\frac{r}{m}}_{\mathcal{L}(H_0^m(\Omega), H^{m+2}(\Omega))}\|\mathcal{X}_p^{-1}\|^{1-\frac{r}{m}}_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}$$

and by (16) and (18) the result follows. □

3. Existence and regularity results for the operator $\mathcal{A}$ with nonregular right side

It is known (see [[8], chapter 1,§12.5]) that for every $s \geq 0, m \geq 0$ there is a natural linear embedding $U_{s,m} : f \in H^s(\Omega) \longrightarrow U_{s,m}(f) \in \big(H^m(\Omega)\big)'$, where

$$\forall\, g \in H^m(\Omega) \quad \big\langle U_{s,m}(f), g\big\rangle = \int_\Omega f(\mathbf{x})g(\mathbf{x})\, d\mathbf{x}.$$

There is also a natural inclusion map $J_{s,m} : H_0^s(\Omega) \longrightarrow \big(\mathfrak{R}^m(\Omega)\big)'$ such that $\langle J_{s,m}(f), g\rangle = \langle U_{s,m}(f), R_m(g)\rangle$, i. e. $J_{s,m} = R'_m U_{s,m}$. Then we have

Proposition 8 *Let* $p \in S_{\mu_{\mathcal{X}}}$ *and* $r \in [0, \infty[$.

a) The adjoint isomorphism $(\mathcal{X}_p)' : H^{-r}(\Omega) \longrightarrow \big(\mathfrak{R}^{r+2}(\Omega)\big)'$ *is an extension of* $\mathcal{X}_p : \mathfrak{R}^{r+2}(\Omega) \longrightarrow H_0^r(\Omega)$ *and* $(\mathcal{X}'_p)^{-1} : \big(\mathfrak{R}^{r+2}(\Omega)\big)' \longrightarrow H^{-r}(\Omega)$ *is an extension of* $\mathcal{X}_p^{-1} : H_0^r(\Omega) \longrightarrow \mathfrak{R}^{r+2}(\Omega)$.

b) The adjoint map $(R_{r+2}\mathcal{X}_p^{-1})' \in \mathcal{L}\big((H^{r+2}(\Omega))', H^{-r}(\Omega)\big)$ *is an extension of the map* $R_{r+2}\mathcal{X}_p^{-1} \in \mathcal{L}\big(H_0^r(\Omega), H^{r+2}(\Omega)\big)$.

Proof a) Let $f \in \mathfrak{R}^{r+2}(\Omega)$ and $g \in \mathcal{D}(\Omega)$. We have, $\mathfrak{A}$ being self-adjoint,

$$\Big\langle (\mathcal{X}_p)'(f), g\Big\rangle = \Big\langle f, \mathfrak{A}(g) + \left(p + \frac{1}{2\tau}\right)^2 g\Big\rangle = \int_\Omega f\left(\mathfrak{A}(g) + \left(p + \frac{1}{2\tau}\right)^2 g\right)\, d\mathbf{x}$$

$$= \int_\Omega \left(\mathfrak{A}(f) + \left(p + \frac{1}{2\tau}\right)^2 f\right) g\, d\mathbf{x} = \Big\langle \mathcal{X}_p(f), g\Big\rangle.$$

Thus

$$\Big|\Big\langle \mathcal{X}'_p(f), g\Big\rangle\Big| = \Big|\Big\langle \mathcal{X}_p(f), g\Big\rangle\Big| \leq \Big\|\mathcal{X}_p(f)\Big\|_{H_0^r(\Omega)}\, \Big\|g\Big\|_{H^{-r}(\Omega)}$$

and by density of $\mathcal{D}(\Omega)$ in $H^{-r}(\Omega)$ it follows that $\mathcal{X}_p'(f) \in H_0^r(\Omega)$ and $\mathcal{X}_p'(f) = \mathcal{X}_p(f) \in H_0^r(\Omega)$.

b) Recall that $R_{r+2}' U_{r,r+2} = J_{r,r+2}$. Then it is enough to remark that, by the result in a), for every $f \in H_0^r(\Omega)$ and $g \in H^{r+2}(\Omega)$ we have

$$\big\langle (R_{r+2}\mathcal{X}_p^{-1})' U_{r,r+2}(f), g \big\rangle = \big\langle (\mathcal{X}_p^{-1})' J_{r,r+2}(f), g \big\rangle = \big\langle \mathcal{X}_p^{-1}(f), g \big\rangle.$$

□

Theorem 9 *Let $0 < \alpha < m$ in $\mathbb{R}$ such that $\alpha - \frac{1}{2} \notin \mathbb{Z}, m - \frac{1}{2} \notin \mathbb{Z}$, $0 \le r < \alpha\left(1 + \frac{2}{m}\right)$, $\pm\left(\frac{2\alpha}{m} - r\right) - \frac{1}{2} \notin \mathbb{N} \cup \{0\}$ and $p \in S_{\mu_{\mathcal{X}}}$. Then $(\mathcal{X}_p^{-1})' \in \mathcal{L}\big(\big(H^r(\Omega)\big)', H^{\frac{2\alpha}{m} - r}(\Omega)\big)$ and there is $B(\alpha, m, r) > 0$ independent of p such that*

$$\forall\, p \in S_{\mu_{\mathcal{X}}} \quad \big\|(\mathcal{X}_p^{-1})'\big\|_{\mathcal{L}((H^r(\Omega))', H^{\frac{2\alpha}{m} - r}(\Omega))} \le \frac{B(\alpha, m, r)}{\big|Re\,\big(p + \frac{1}{2\,\tau}\big)\big|^{1 - \frac{\alpha}{m}}\, \big|p + \frac{1}{2\tau}\big|^{1 - \frac{\alpha}{m}}}. \tag{21}$$

Proof We have $H^{\alpha+2}(\Omega) \subset H^{\alpha\left(1+\frac{2}{m}\right)}(\Omega) \subset L^2(\Omega) \subset H^{-\alpha}(\Omega)$. The map $\big(I_{\alpha+2, \alpha\left(1+\frac{2}{m}\right)} R_{\alpha+2} \mathcal{X}_p^{-1}\big)' \in \mathcal{L}((H^{\alpha\left(1+\frac{2}{m}\right)})', H^{-\alpha}(\Omega))$ is nothing other than $(\mathcal{X}_p^{-1})'$ considered as an operator from $(H^{\alpha\left(1+\frac{2}{m}\right)})'$ into $H^{-\alpha}(\Omega)$. It follows from the result of part b) in proposition 8 that its restriction to $H_0^\alpha(\Omega)$ is $I_{\alpha+2, \alpha\left(1+\frac{2}{m}\right)} R_{\alpha+2}$ $\mathcal{X}_p^{-1} \in \mathcal{L}(H_0^\alpha(\Omega), H^{\alpha\left(1+\frac{2}{m}\right)}(\Omega))$, i.e. the operator $\mathcal{X}_p^{-1}$ considered as a map from $H_0^\alpha(\Omega)$ into $H^{\alpha\left(1+\frac{2}{m}\right)}(\Omega)$. Then $\big\|(\mathcal{X}_p^{-1})'\big\|_{\mathcal{L}(H_0^\alpha(\Omega), H^{\alpha(1+\frac{2}{m})}(\Omega))} = \big\|\mathcal{X}_p^{-1}\big\|_{\mathcal{L}(H_0^\alpha(\Omega), H^{\alpha(1+\frac{2}{m})}(\Omega))}$ and we can interpolate the operators $\mathcal{X}_p^{-1} \in \mathcal{L}\big(H_0^\alpha(\Omega),\ H^{\alpha(1+\frac{2}{m})}(\Omega)\big)$ and $(\mathcal{X}_p^{-1})' \in \mathcal{L}((H^{\alpha(1+\frac{2}{m})}(\Omega))', H^{-\alpha}(\Omega))$. By $\big[$[8], chapter 1, theorem 12.6$\big]$ we have $\big(H^r(\Omega)\big)' \approx \big[H_0^\alpha(\Omega), \big(H^{\alpha(1+\frac{2}{m})}(\Omega)\big)'\big]_\eta$ where $\eta = \frac{m(\alpha+r)}{2\alpha(m+1)}$. As $(1-\eta)\alpha(1 + \frac{2}{m}) - \eta\alpha = \frac{2\alpha}{m} - r$, it follows from $\big[$[8], chapter 1, theorem 12.4$\big]$ that $(\mathcal{X}_p^{-1})' \in \mathcal{L}\big(\big(H^r(\Omega)\big)', H^{\frac{2\alpha}{m} - r}(\Omega)\big)$. By interpolation properties we have

$$\big\|(\mathcal{X}_p^{-1})'\big\|_{\mathcal{L}((H^r(\Omega))', H^{\frac{2\alpha}{m} - r}(\Omega))} \le$$

$$\le c_2(m, \alpha, r) \big\|(\mathcal{X}_p^{-1})'\big\|^{1-\eta}_{\mathcal{L}(H_0^\alpha(\Omega), H^{\alpha(1+\frac{2}{m})}(\Omega))} \big\|(\mathcal{X}_p^{-1})'\big\|^{\eta}_{\mathcal{L}((H^{\alpha(1+\frac{2}{m})}(\Omega))', H^{-\alpha}(\Omega))}.$$

Since an operator and its adjoint map have the same norm the result follows easily from (20). □

Since $\frac{2\alpha}{m} - r$ can be positive, theorem 9 is the key to obtain regularity results for the solutions U of the equation $\mathcal{X}_p'(U) = \Psi$ with irregular elements $\Psi \in \big(H^r(\Omega)\big)'$. However, to assure moreover the condition $\mathfrak{B}(U) = 0$ we need to consider a space smaller than $\big(H^r(\Omega)\big)'$. That is the reason to deal with the space $\Xi^{-r}(\Omega) \subset \big(H^r(\Omega)\big)'$ considered in section 1. Note that $\mathfrak{R}^{r+2}(\Omega) \subset H^{r+2}(\Omega) \subset H^r(\Omega) \subset \Xi^r(\Omega), r \ge 0$ with continuous inclusions. As $\mathcal{D}(\Omega) \subset \mathfrak{R}^{r+2}(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $\Xi^r(\Omega)$, we obtain the inclusion $\Xi^{-r}(\Omega) \subset \big(\mathfrak{R}^{r+2}(\Omega)\big)'$. Then by $\big[$[8], chapter 2, theorems 6.5 and 7.3$\big]$, for every $0 < r$, the space

$$\mathcal{J}_p^{-r+2}(\Omega) = \Big\{ U \in H^{-r+2}(\Omega) \ \Big|\ \mathcal{X}_p'(U) \in \Xi^{-r}(\Omega) \subset (\mathfrak{R}^{r+2}(\Omega))' \text{ and } \mathfrak{B}(U) = 0 \Big\}$$

is well defined. We provide $\mathcal{J}_p^{-r+2}(\Omega)$ with the norm

$$\forall\, U \in \mathcal{J}_p^{-r+2}(\Omega) \quad \big\|U\big\|_{\mathcal{J}_p^{-r+2}(\Omega)} = \big\|U\big\|_{H^{-r+2}(\Omega)} + \big\|\mathcal{X}_p'(U)\big\|_{\Xi^{-r}(\Omega)}. \tag{22}$$

Remark that $\left\| \mathcal{X}'_p : \mathcal{J}_p^{-r+2}(\Omega) \longrightarrow \Xi^{-r}(\Omega) \right\| \leq 1$. We have

Theorem 10 *Let* $p \in S_{\mu_{\mathcal{X}}}$ *and* $r \in [0, \infty[$. *Then* $\mathcal{J}_p^{-r+2}(\Omega)$ *is a Banach space and the restriction* $\mathcal{H}_p$ *to* $\mathcal{J}_p^{-r+2}(\Omega)$ *of* $\mathcal{X}'_p$ *is an isomorphism from* $\mathcal{J}_p^{-r+2}(\Omega)$ *onto* $\Xi^{-r}(\Omega)$. *Moreover, if* $0 < \alpha < m$ *in* $[0, \infty[\setminus \left(\mathbb{N} + \frac{1}{2}\right)$, $0 < r < \alpha\left(1 + \frac{2}{m}\right)$ *and* $\pm\left(\frac{2\alpha}{m} - r\right) - \frac{1}{2} \notin \mathbb{N} \cup \{0\}$ *there is* $M(\alpha, m, r) > 0$ *independent of* p *such that*

$$\left\| (\mathcal{X}'_p)^{-1} \right\|_{\mathcal{L}(\Xi^{-r}(\Omega), H^{-r+\frac{2\alpha}{m}}(\Omega))} \leq \frac{M(\alpha, m, r)}{\left| Re\left(p + \frac{1}{2\tau}\right) \right|^{1-\frac{\alpha}{m}} \left| p + \frac{1}{2\tau} \right|^{1-\frac{\alpha}{m}}}. \tag{23}$$

Proof Let $\{f_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $\mathcal{J}_p^{-r+2}(\Omega)$. There exists $f = \lim_{m\to\infty} f_m$ in $H^{-r+2}(\Omega)$ and $g = \lim_{m\to\infty} \mathcal{X}'_p(f_m)$ in $\Xi^{-r}(\Omega)$. Then $g = \lim_{m\to\infty} \mathcal{X}'_p(f_m)$ in $\left(\mathfrak{R}^{r+2}(\Omega)\right)'$. As $H^{-r+2}(\Omega) \subset H^{-r}(\Omega)$, by proposition 8 we have $f = \lim_{m\to\infty} f_m$ in $H^{-r}(\Omega)$ and $\mathcal{X}'_p(f) = \lim_{m\to\infty} \mathcal{X}'_p(f_m)$ in $\left(\mathfrak{R}^{r+2}(\Omega)\right)'$. It follows that $g = \mathcal{X}'_p(f)$ and so $\mathcal{X}'_p(f) \in \Xi^{-r}(\Omega)$, i.e. $f \in \mathcal{J}_p^{-r+2}(\Omega)$ and $\mathcal{J}_p^{-r+2}(\Omega)$ becomes a Banach space.

The continuity of $\mathcal{H}_p$ follows from (22). Given $f \in \Xi^{-r}(\Omega)$, by $\big[$[8], chapter 2, theorems 5.4 (for the case $r = 0$), 6.7 and 7.4$\big]$ there is $U \in H^{-r+2}(\Omega)$ such that $\mathcal{X}_p(U) = f$ and $\mathfrak{B}(U) = 0$. By proposition 8 $\mathcal{X}_p(U) = \mathcal{X}'_p(U)$ and then $U \in \mathcal{J}_p^{-r+2}(\Omega)$ and $\mathcal{H}_p : \mathcal{J}_p^{-r+2}(\Omega) \longrightarrow \Xi^{-r}(\Omega)$ is bijective and continuous. By the open map theorem $\mathcal{H}_p$ is an onto isomorphism. Since $\Xi^{-r}(\Omega) \subset (H^r(\Omega))'$, an application of theorem 9 finishes the proof. □

Proposition 8 can be improved in the following way:

Corollary 11 *If* $p \in S_{\mu_{\mathcal{X}}}$ *and* $r \in [0, \infty[$, *the adjoint isomorphism* $(\mathcal{X}_p)' : H^{-r}(\Omega) \longrightarrow \left(\mathfrak{R}^{r+2}(\Omega)\right)'$ *is an extension of* $\mathcal{X}_p : \mathcal{J}_p^{-r+2}(\Omega) \longrightarrow \Xi^{-r}(\Omega)$.

Proof As $\Xi^r(\Omega) \subset L^2(\Omega) \subset \Xi^{-r}(\Omega)$ is a Gelfand triple, given $f \in \mathcal{J}_p^{-r+2}(\Omega)$ there is a sequence $\{\varphi_m\}_{m=1}^{\infty} \subset \mathcal{D}(\Omega)$ such that $g := \mathcal{X}'_p(f) = \lim_{m\to\infty} \varphi_m$ in $\Xi^{-r}(\Omega)$. As a consequence $g = \lim_{m\to\infty} \varphi_m$ in $H^{-r}(\Omega)$ holds. We have $\varphi_m \in H_0^k(\Omega)$ for every k and $m \in \mathbb{N}$. By proposition 5 $\mathcal{X}_p^{-1}(\varphi_m) \in H^{k+2}(\Omega)$, i.e. $\mathcal{X}_p^{-1}(\varphi_m) \in \mathcal{C}^{\infty}(\overline{\Omega})$ (Sobolev's embedding theorem). By theorem 10 we have $f = (\mathcal{X}'_p)^{-1}(g) = \lim_{m\to\infty} (\mathcal{X}'_p)^{-1}(\varphi_m)$ in $\mathcal{J}_p^{-r+2}(\Omega)$ and in $H^{-r+2}(\Omega)$ indeed. By proposition 8 we obtain $f = \lim_{m\to\infty} (\mathcal{X}'_p)^{-1}(\varphi_m) = \lim_{m\to\infty} (\mathcal{X}_p)^{-1}(\varphi_m)$ in $H^{-r+2}(\Omega)$. Then $\mathcal{X}_p(f) = \lim_{m\to\infty} \mathcal{X}_p(\mathcal{X}_p)^{-1}(\varphi_m) = \lim_{m\to\infty} \varphi_m$ in $H^{-r}(\Omega)$ obtaining $\mathcal{X}_p(f) = g = \mathcal{X}'_p(f)$. □

Lemma 12 *Let* $0 < \alpha < m$ *in* $[0, \infty[\setminus \left(\mathbb{N} + \frac{1}{2}\right)$, $0 < r < \alpha\left(1 + \frac{2}{m}\right)$ *and* $\pm\left(\frac{2\alpha}{m} - r\right) - \frac{1}{2} \notin \mathbb{N} \cup \{0\}$. *The map* $\mathfrak{G} : p \in S_{\mu_{\mathcal{X}}} \longrightarrow (\mathcal{X}'_p)^{-1} \in \mathcal{L}(\Xi^{-r}(\Omega), H^{\frac{2\alpha}{m} - r}(\Omega))$ *is holomorphic.*

Proof Let $p_1 \in S_{\mu_{\mathcal{X}}}$ and $\delta > 0$ such that if $|p - p_1| \leq \delta$ then $p \in \mathbb{S}_{\mathcal{X}}$. Let $I_p : \mathcal{J}_p^{-r+2}(\Omega) \longrightarrow H^{-r+\frac{2\alpha}{m}}(\Omega)$ be the inclusion map. The composition map $I_p \mathcal{H}_p$ is nothing other than the operator $(\mathcal{X}'_p)^{-1}$ considered as a map from $\Xi^{-r}(\Omega)$ into $H^{-r+\frac{2\alpha}{m}}(\Omega)$. Recall that $\mathcal{X}'_p \in \mathcal{L}\big(H^{-r}(\Omega), (\mathfrak{R}^{r+2}(\Omega))'\big)$ is an isomorphism (proposition 8) and that $H^{-r+\frac{2\alpha}{m}}(\Omega) \subset H^{-r}(\Omega)$. Since $\|I_p\| \leq 1$, by (22), (14) and directly by definition of $\mathcal{X}_p$ one has

$$\left\| \mathcal{X}'_{p_1} - \mathcal{X}'_p \right\|_{\mathcal{L}(H^{-r}(\Omega), (\mathfrak{R}^{r+2}(\Omega))')} = \left\| \mathcal{X}_{p_1} - \mathcal{X}_p \right\|_{\mathcal{L}(\mathfrak{R}^{r+2}(\Omega), H_0^r(\Omega))} \leq$$

$$\leq \left|\frac{\tau}{\alpha}\right| \left|\left(p+\frac{1}{2\,\tau}\right)^2 - \left(p_1+\frac{1}{2\,\tau}\right)^2\right|. \tag{24}$$

Then we have

$$\left\|I_p\mathcal{H}_p - I_{p_1}\mathcal{H}_{p_1}\right\| = \left\|(\mathcal{X}'_p)^{-1} - (\mathcal{X}'_{p_1})^{-1}\right\|_{\mathcal{L}(\Xi^{-r}(\Omega),H^{-r+\frac{2\alpha}{m}}(\Omega))} =$$

$$= \left\|(\mathcal{X}'_{p_1})^{-1}(\mathcal{X}'_{p_1} - \mathcal{X}'_p)I_p\mathcal{H}_p\right\|_{\mathcal{L}(\Xi^{-r}(\Omega),H^{-r+\frac{2\alpha}{m}}(\Omega))}$$

$$\leq \left\|(\mathcal{X}'_{p_1})^{-1}\right\| \left\|\mathcal{X}'_p - \mathcal{X}'_{p_1}\right\|_{\mathcal{L}(H^{-r}(\Omega),(\mathfrak{R}^{r+2}(\Omega))')} \sup_{|q-p_1|\leq\delta}\left\|I_q\mathcal{H}_q\right\|$$

and the continuity of $\mathfrak{G}$ in $p_1 \in S_{\mu_{\mathcal{X}}}$ follows easily from (24) and theorem 10. Then our result is a consequence of resolvent's identity and the chain rule (see $\big[$[8], chapter 4, theorem 3.1$\big]$ for details). □

The next theorem contains the main results of the paper about regularity properties of the solutions of $\mathcal{A}(Z) = \Psi$ when Ψ is an irregular distribution.

Theorem 13 *Let $T > 0$ and let $0 < \alpha < m$ in $\mathbb{R}$ such that $\{\alpha, m\} \subset [0,\infty[\setminus\left(\mathbb{N}+\frac{1}{2}\right)$, $0 < r < \alpha\left(1+\frac{2}{m}\right)$ and $\pm\left(\frac{2\alpha}{m} - r\right) - \frac{1}{2} \notin \mathbb{N}\cup\{0\}$. Assume $\Psi \in \Xi^{-r,-r}(\Omega\times\mathbb{R})$, and that there are $K > 0$ and $-\infty < \rho < \frac{1}{2} - \frac{\alpha}{m}$ such that the Schwartz–Laplace transform $\mathfrak{L}[\Psi](\mathbf{x},p)$ verifies*

$$\forall\, p \in S_{\mu_{\mathcal{X}}} \quad \mathfrak{L}[\Psi](\mathbf{x},p) \in \Xi^{-r}(\Omega) \quad \textit{and} \quad \left\|\mathfrak{L}\big[\Psi\big](\mathbf{x},p)\right\|_{\Xi^{-r}(\Omega)} \leq K\big|p\big|^{\rho}. \tag{25}$$

Then there exists $V \in \mathbb{R}$ and a unique $Z \in \mathcal{D}'(\Omega\times\mathbb{R})$ such that

$$a)\quad \mathcal{A}(Z)(\mathbf{x},t) = \Psi(\mathbf{x},t) \quad \textit{in}\ \Omega\times\mathbb{R}. \tag{26}$$

$$b)\quad \mathfrak{B}\big(Z(\mathbf{x},t)\big) = 0 \quad \textit{in}\ \partial\Omega\times\mathbb{R} \quad \textit{and} \quad Z(\mathbf{x},t) = 0 \quad \textit{in}\ \Omega\times]-\infty,V[. \tag{27}$$

c) The restriction to Ω_T of Z verifies $Z \in H^s\big(]-T,T[,H^{-r+\frac{2\alpha}{m}}(\Omega)\big)$ for every $0 \leq s < \frac{1}{2} - \frac{\alpha}{m} - \rho$.

Proof **Part 1. Existence and uniqueness of Z.** By theorem 10 and corollary 11, for every $p \in S_{\mu_{\mathcal{X}}}$ there is a unique $\mathcal{G}(\mathbf{x},p) \in \mathcal{J}_p^{-r+2}(\Omega) \subset H^{-r+2}(\Omega) \subset H^{\frac{2\alpha}{m}-r}(\Omega) \subset H^{-r}(\Omega)$ such that

$$\mathcal{X}_p(\mathcal{G}) = \mathcal{X}'_p(\mathcal{G}) = \mathfrak{A}(\mathcal{G}) + \frac{1}{\alpha}\big(p+\tau\,p^2\big)\mathcal{G} = \mathfrak{L}[\Psi] \tag{28}$$

and $\mathfrak{B}(\mathcal{G}(\mathbf{x},p) = 0$ in $\partial\Omega$. By lemma 12 the map $\mathfrak{G} : p \in S_{\mu_{\mathcal{X}}} \longrightarrow (\mathcal{X}'_p)^{-1} \in \mathcal{L}\big(\Xi^{-r}(\Omega), H^{\frac{2\alpha}{m}-r}(\Omega)\big)$ is holomorphic in $S_{\mu_{\mathcal{X}}}$ and moreover, by theorem 10, $\mathfrak{G}$ is slowly increasing with respect to p. As a consequence (see $\big[$[15], chapter I, §3, page 79, proposition 22$\big]$ and $\big[$[14], chapter 8, section 4, remark 2$\big]$) there exists the inverse Laplace–Schwartz transform $\mathfrak{H} := \big(\mathfrak{L}^{-1}[\mathfrak{G}]\big)(\mathbf{x},t) \in \mathcal{D}'\big(\mathbb{R}, \mathcal{L}\big(\Xi^{-r}(\Omega), H^{\frac{2\alpha}{m}-r}(\Omega)\big)\big)$ as well as some $V \in \mathbb{R}$ such that $\mathfrak{H} = 0$ on $]-\infty,V[$. Denote the convolution with respect to the t-variable by the symbol $*$. By definition of $\Xi^{-r,-v}(\Omega\times\mathbb{R})$ there is $a \in \mathbb{N}$ such that $\Psi \in \Xi^{-r,-v}(\Omega_{\mathfrak{T}_a})$. Then the convolution product $\mathfrak{H}*\Psi$

of vector valued distributions is well defined. As it follows from (28) that $\mathcal{G} = (\mathcal{X}_p')^{-1}(\mathfrak{L}[\Psi](\mathbf{x}, p)\big) \in \mathcal{J}_p^{-r+2}(\Omega)$, by the convolution theorem (see $\big[$[15], chapter II, §7, proposition 43$\big]$) and corollary 11 we have

$$\mathfrak{L}[\mathcal{A}(\mathfrak{H} * \Psi)](p) = \frac{\tau}{\alpha}\mathcal{X}_p\big(\mathfrak{L}[\mathfrak{H} * \Psi](p)\big) = \frac{\tau}{\alpha}\mathcal{X}_p\Big(\mathfrak{L}[\mathfrak{H}](p)\Big(\mathfrak{L}[\Psi](\mathbf{x}, p)\Big)\Big)$$

$$= \frac{\tau}{\alpha}\Big(\mathcal{X}_p\ \mathfrak{G}(p)\big(\mathfrak{L}[\Psi](\mathbf{x}, p)\Big) = \frac{\tau}{\alpha}\mathcal{X}_p\big((\mathcal{X}_p')^{-1}(\mathfrak{L}[\Psi](\mathbf{x}, p)\big)$$

$$= \frac{\tau}{\alpha}\mathcal{X}_p'\big((\mathcal{X}_p')^{-1}(\mathfrak{L}[\Psi](\mathbf{x}, p)\big) = \mathfrak{L}[\Psi](\mathbf{x}, p) \tag{29}$$

and by the uniqueness of the Laplace–Schwartz transform we obtain $\mathcal{A}(\mathfrak{H} * \Psi) = \Psi$, i.e. $Z := \mathfrak{H} * \Psi$ verifies (26) and (27). Moreover, from proposition 8, (29), and (28) we deduce

$$\mathfrak{L}[Z] = \mathcal{G} = \big(\mathcal{X}_p'\big)^{-1}(\mathfrak{L}[\Psi](\mathbf{x}, p)) \in \mathcal{J}_p^{-r+2}(\Omega) \subset H^{-r+2}(\Omega) \subset H^{\frac{2\alpha}{m}-r}(\Omega). \tag{30}$$

Take $p = \psi + i\ \nu \in S_{\mu_{\mathcal{X}}}$. As $\alpha < m$ we have $H^{-r+2}(\Omega) \subset H^{\frac{2\alpha}{m}-r}(\Omega)$ continuously. By theorem 10 we obtain $(\mathcal{X}_p^{-1})' \in \mathcal{L}\big(\Xi^{-r}(\Omega), H^{\frac{2\alpha}{m}-r}(\Omega)\big)$ and the map $\nu \in \mathbb{R} \longrightarrow \Big\|\big(\mathcal{X}_{\psi+i\ \nu}^{-1}\big)'\Big\|_{\mathcal{L}(\Xi^{-r}(\Omega), H^{\frac{2\alpha}{m}-r}(\Omega))}$ becomes continuous.

Let $\mathcal{F}[.](p)$ denote the Fourier transform with respect to t. If $0 \le s < \frac{1}{2} - \frac{\alpha}{m} - \rho$, by theorems 10 and 6 and (25) we have for some $C > 0$ independent of p

$$\int_{-\infty}^{\infty}\Big(1 + |\psi + i\ \nu|^2\Big)^s \Big\|\mathcal{F}\Big[e^{-\psi\ t}Z(\mathbf{x}, t)\Big](\nu)\Big\|^2_{H^{\frac{2\alpha}{m}-r}(\Omega)}\ d\nu =$$

$$= \int_{-\infty}^{\infty}\Big(1 + |\psi + i\ \nu|^2\Big)^s \Big\|\mathfrak{L}\Big[Z(\mathbf{x}, t)\Big](\psi + i\ \nu)\Big\|^2_{H^{\frac{2\alpha}{m}-r}(\Omega)}\ d\nu =$$

$$= \int_{-\infty}^{\infty}\Big(1 + |\psi + i\ \nu|^2\Big)^s \Big\|\Big(\mathcal{X}_{\psi+i\ \nu}^{-1}\Big)'\Big(\mathfrak{L}[\Psi](\psi + i\ \nu)\Big)\Big\|^2_{H^{\frac{2\alpha}{m}-r}(\Omega)}\ d\nu \le$$

$$\le \int_{-\infty}^{\infty}\Big(1 + |\psi + i\ \nu|^2\Big)^s \Big\|\Big(\mathcal{X}_{\psi+i\ \nu}^{-1}\Big)'\Big\|^2_{\mathcal{L}(\Xi^{-r}(\Omega), H^{\frac{2\alpha}{m}-r}(\Omega))}\Big\|\mathfrak{L}[\Psi](\psi + i\ \nu)\Big)\Big\|^2_{\Xi^{-r}(\Omega)}\ d\nu \le$$

$$\le C\int_{-\infty}^{\infty}\frac{\Big(1 + |\psi + i\ \nu|^2\Big)^s}{\big|\psi + \frac{1}{2\ \tau}\big|^{2\left(1-\frac{\alpha}{m}\right)}}\frac{|\psi + i\ \nu|^{2\ \rho}}{\big|\psi + i\ \nu + \frac{1}{2\ \tau}\big|^{2\left(1-\frac{\alpha}{m}\right)}}\ d\nu \tag{31}$$

and since $\frac{1}{2} - \frac{\alpha}{m} - \rho > s$, it turns out that the integral (31) is convergent. By the vector valued version of Plancherel's theorem (see $\big[$[8], chapter 4, §3.2$\big]$ for instance) one has $e^{-\psi\ t}Z \in H^s(\mathbb{R}, H^{\frac{2\alpha}{m}-r}(\Omega))$ and so $Z = e^{\psi\ t}\Big(e^{-\psi\ t}Z\Big) \in H^s\Big(\big] - T, T\big[, H^{-r+\frac{2\alpha}{m}}(\Omega)\Big)$.

Concerning the uniqueness of Z, if there would be Z^1 and Z^2 verifying the previous conditions, by (30) we would have $\mathfrak{L}\big[Z^1 - Z^2\big] = 0$ and hence, by the uniqueness of the inverse Laplace transform $Z^1 = Z^2$.

Part 2). Existence and computation of $\mathfrak{B}(Z)(\mathbf{x}, t)$ in $\partial\Omega \times \mathbb{R}$. By lemma 2 there are $a \in \mathbb{N}$ and a sequence $\{\Psi_k(\mathbf{x}, t)\}_{k=1}^{\infty} \subset \mathcal{D}(\Omega \times \mathbb{R})$ such that $\Psi = \lim_{k\to\infty}\Psi_k$ in $\Xi^{-r,-r}(\Omega \times \mathbb{R})$ and $\bigcup_{k=1}^{\infty} Supp(\Psi_k) \subset \Omega_{\mathfrak{T}_a}$.

For every $|\boldsymbol{\beta}| \geq 0$ and $h \in \mathbb{N}$ we have $W_k^{|\boldsymbol{\beta}|,h} := \frac{\partial^{|\boldsymbol{\beta}|+h}\Psi_k}{\partial \mathbf{x}^{\boldsymbol{\beta}} \partial t^h} \in \mathcal{D}(\Omega_{\mathfrak{T}_a})$ and $\mathfrak{L}\big[W_k^{|\boldsymbol{\beta}|,h}\big](\mathbf{x},p) \in \mathcal{D}(\Omega) \subset H_0^v(\Omega)$ for every $v \geq 0$ and $p \in S_{\mu_{\mathcal{X}}}$. Moreover,

$$\mathfrak{L}[W_k^{|\boldsymbol{\beta}|,h}](\mathbf{x},p) = \mathfrak{L}\left[\int_{-\infty}^{t} W_k^{|\boldsymbol{\beta}|,h+1}(\mathbf{x},\xi)\, d\xi\right](\mathbf{x},p) = \frac{1}{p}\mathfrak{L}[W_k^{|\boldsymbol{\beta}|,h+1}](\mathbf{x},p). \tag{32}$$

In particular, (32) implies that $\mathfrak{L}\left[\frac{\partial \Psi_k}{\partial t}\right](\mathbf{x},p) \in \Xi^{-r}(\Omega)$ if $\mathfrak{L}\big[\Psi_k\big](\mathbf{x},p) \in \Xi^{-r}(\Omega)$.

By theorem 6, using the same argumentation of part a) it turns out that there exists $Z_k^{\boldsymbol{\beta},h} := \mathfrak{H} * W_k^{|\boldsymbol{\beta}|,h}$ verifying $\mathcal{A}(Z_k^{\boldsymbol{\beta},h})(\mathbf{x},t) = W_k^{|\boldsymbol{\beta}|,h}$ in $\Omega \times \mathbb{R}$ and

$$\mathfrak{L}[Z_k^{\boldsymbol{\beta},h}](\mathbf{x},p) = \mathcal{X}_p^{-1}\left(\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{x},p)\right) \in \mathfrak{R}^{v+2}(\Omega) \subset H^{v+2}(\Omega) \tag{33}$$

for every $p \in S_{\mu_{\mathcal{X}}}$ and $v \in \mathbb{N}$. As a consequence we have

$$\mathfrak{B}(\mathfrak{L}[Z_k^{\boldsymbol{\beta},h}](\mathbf{x},p)) = \mathfrak{L}[\mathfrak{B}(Z_k^{\boldsymbol{\beta},h})](\mathbf{x},p)) = 0 \quad \text{in } \partial\Omega. \tag{34}$$

In the sequel, to simplify, we put $Z_k := Z_k^{0,0}$ and $W_k := W_k^{0,0} = \Psi_k, k \in \mathbb{N}$. □

Claim 1. For every $v \in \mathbb{N}$, $\mathfrak{B}(Z_k)(\mathbf{x},t) = 0$ in $\partial\Omega \times \mathbb{R}$.

Proof As $v \geq 0$ is arbitrary, Sobolev's embedding theorem gives us $\mathfrak{L}[Z_k^{\boldsymbol{\beta},h}](\mathbf{x},p) \in \mathcal{C}^\infty(\overline{\Omega}) \subset H^v(\Omega) \subset \mathcal{C}(\overline{\Omega})$ for every $v > \frac{n}{2}$ with continuous inclusions. It follows that there is $C_1 > 0$ such that for every $p = \psi + i\nu$ as above in part a), for every $\mathbf{x} \in \Omega$, $t \in \mathbb{R}$, and every $m_v \in \mathbb{N}$ such that $v < m_v$ we obtain

$$\left|e^{pt}\, \mathcal{X}_p^{-1}\left(\mathfrak{L}\left[W_v^{|\boldsymbol{\beta}|,h}\right](\mathbf{x},p)\right)\right| \leq e^{\psi\, t}\left\|\mathcal{X}_p^{-1}(\mathfrak{L}\left[W_v^{|\boldsymbol{\beta}|,h}\right](\mathbf{y},p))\right\|_{\mathcal{C}(\overline{\Omega})} \leq$$

$$\leq C_1 e^{\psi\, t}\left\|\mathcal{X}_p^{-1}\left(\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{y},p)\right)\right\|_{H^{v\left(1+\frac{2}{m_v}\right)}(\Omega)} \leq$$

$$\leq C_1\, e^{\psi\, t}\left\|\mathcal{X}_p^{-1}\right\|_{\mathcal{L}(H_0^v(\Omega), H^{v\left(1+\frac{2}{m_v}\right)}(\Omega))}\left\|\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{y},p)\right\|_{H_0^v(\Omega)}. \tag{35}$$

Now remark that for every $\boldsymbol{\rho} \in \mathbb{N}^n$ such that $|\boldsymbol{\rho}| \leq v$, by Fubini's theorem and Hölder's inequality we have

$$\left\|\frac{\partial^{|\boldsymbol{\rho}|}}{\partial \mathbf{x}^{\boldsymbol{\rho}}}\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{y},p)\right\|_{L^2(\Omega)} = \sup_{\|g\|_{L^2(\Omega)}\leq 1}\left|\int_\Omega \mathfrak{L}\left[W_k^{|\boldsymbol{\beta}+\boldsymbol{\rho}|,h}\right](\mathbf{y},p)g(\mathbf{y})d\mathbf{y}\right| =$$

$$= \sup_{\|g\|_{L^2(\Omega)}\leq 1}\left|\int_\Omega \frac{1}{p}\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}+\boldsymbol{\rho}|,h+1}\right](\mathbf{y},p)g(\mathbf{y})d\mathbf{y}\right| \leq$$

$$\leq \sup_{\|g\|_{L^2(\Omega)}\leq 1}\frac{1}{|p|}\int_{-\mathfrak{T}_a}^{\infty} e^{-\psi\, t}\left(\int_\Omega\left|W_k^{|\boldsymbol{\beta}+\boldsymbol{\rho}|,h+1}(\mathbf{y},t)g(\mathbf{y})\right|d\mathbf{y}\right)dt \leq$$

$$\leq \sqrt{|\Omega|}\,\frac{e^{\psi\, \mathfrak{T}_a}}{\psi|p|}\,\left\|W_k^{|\boldsymbol{\beta}+\boldsymbol{\rho}|,h+1}\right\|_{\mathcal{C}(\Omega\times\mathbb{R})}$$

where $\left|\Omega\right|$ denotes the Lebesgue measure of Ω. It follows that there is $M^{\psi}_{kv} > 0$ such that

$$\sup_{\nu\in\mathbb{R}} \left\| \mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{y},\psi+i\nu)\right\|_{H_0^v(\Omega)} \leq \frac{M^{\psi}_{kvh}}{|\psi+i\,\nu|} \tag{36}$$

and then, by (35), (20), since $v < m_v$

$$\int_{-\infty}^{\infty}\left|e^{(\psi+i\nu)t}\mathcal{X}_p^{-1}\left(\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{x},\psi+i\nu)\right)\right| \, d\nu \leq$$

$$\leq \int_{-\infty}^{\infty} \frac{M^{\psi}_{kvh}\, e^{\psi t}}{\left|\psi+i\,\nu\right| \, \left|\psi+i\nu+\frac{1}{2\tau}\right|^{1-\frac{v}{m_v}}}\, d\nu < \infty \tag{37}$$

turns out to be a uniformly convergent integral in every bounded closed neighborhood of $(\mathbf{x},t)\in\Omega\times\mathbb{R}$. That means we can apply the inversion formula for Laplace–Schwartz transforms $\left[[18]\ \right]$ obtaining for every $k\in\mathbb{N}$ and $t\geq V$

$$\forall|\boldsymbol{\beta}|\geq 0, \forall h\geq 0 \quad Z_k^{\boldsymbol{\beta},h} = \frac{1}{2\,\pi\, i}\int_{-\infty}^{\infty} e^{t\,(\psi+i\,\nu)}\,\mathcal{X}^{-1}_{\psi+i\,\nu}\Big(\mathfrak{L}\left[W_k^{|\boldsymbol{\beta}|,h}\right](\mathbf{x},\psi+i\,\nu)\Big)\,d\nu =$$

$$= \frac{1}{2\,\pi\, i}\int_{-\infty}^{\infty}(\psi+i\,\nu)^h e^{t\,(\psi+i\,\nu)}\,\mathcal{X}^{-1}_{\psi+i\,\nu}\Big(\mathfrak{L}\left[\frac{\partial^{|\boldsymbol{\beta}|}\Psi_k}{\partial\mathbf{x}^{\boldsymbol{\beta}}}\right](\mathbf{x},\psi+i\,\nu)\Big)\,d\nu = \frac{\partial^{|\boldsymbol{\beta}|+h}Z_k}{\partial\mathbf{x}^{\boldsymbol{\beta}}\partial t^h}(\mathbf{x},t) \tag{38}$$

by the uniform convergence of the involved integrals and (19). Then by (33) and (34) we obtain easily

$$\forall(\mathbf{x},t)\in\partial\Omega\times\mathbb{R} \quad \mathfrak{B}(Z_k)(\mathbf{x},t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\psi+i\nu)t}\mathfrak{L}\left[\mathfrak{B}(Z_k)\right](\mathbf{x},\psi+i\nu)\,d\nu = 0. \tag{39}$$

□

Define β by the equality $\alpha\left(1+\frac{2}{m}\right)+1 = \beta\left(1+\frac{2}{m+1}\right)$. Then, by the assumptions about α and m, after elementary operations, we see that $\alpha<\beta$ and

$$r+1 < \beta\left(1+\frac{2}{m+1}\right), \quad \beta = \frac{(m+1)(m+\alpha(m+2))}{m(m+3)} < m+1, \quad \frac{2\alpha}{m} > -1+\frac{2\beta}{m+1}. \tag{40}$$

Remark that by (37) and the last computations, $Z_k\in\mathcal{C}^{\infty}(\overline{\Omega\times\mathbb{R}})\subset H^{-r-1,-r-2}(\Omega_T)$ and, moreover, $\mathcal{A}(Z_k) = \Psi_k\in\Xi^{-r,-r}(\Omega\times\mathbb{R})\subset\Xi^{-(r+3),-(r+3)}(\Omega\times\mathbb{R})$. On the other hand, by (40) we have the continuous inclusions

$$H^s\big(]-T,T[,H^{-r+\frac{2\alpha}{m}}(\Omega)\big)\subset H^s\big(]-T,T[,H^{-r-1+\frac{2\beta}{m+1}}(\Omega)\big)\subset$$

$$\subset L^2\big(]-T,T[,H^{-(r+1)}(\Omega)\big)\subset H^{-(r+1),-(r+2)}(\Omega_T)$$

because, for every $P\in L^2\big(]-T,T[,H^{-(r+1)}(\Omega)\big)$, by Hölder's inequality one has

$$\forall\, f(\mathbf{x},t)\in H^{r+1,r+2}_{0,0}(\Omega_T) \qquad \left|\int_{-T}^{T}\big\langle P(\mathbf{x},t),f(\mathbf{x},t)\big\rangle_{\Omega}\,dt\right| \leq$$

$$\leq \int_{-T}^{T} \big\| f(\mathbf{x},t) \big\|_{H_0^{r+1}(\Omega)} \big\| P(\mathbf{x},t) \big\|_{H^{-(r+1)}(\Omega)} \, dt \leq \big\| f \big\|_{H^{r+1,r+2}(\Omega_T)} \big\| P \big\|_{L^2\left( ]-T,T[, H^{-r-1}(\Omega) \right)}.$$

Since $\mathcal{A}(Z) = \Psi \in \Xi^{-r,-r}(\Omega_T) \subset \Xi^{-(r+3),-(r+3)}(\Omega_T)$, by $\big[$[11], theorem 4$\big]$ in order that $\mathfrak{B}(Z)$ be defined it is enough to see that $\lim_{k\to\infty} Z_k = Z$ in the Lebesgue–Bochner space $L^2\big( ]-T,T[, H^{-r-1+\frac{2\beta}{m+1}}(\Omega) \big)$ and then part b) will follow directly from (34).

Claim 2. Define $D_{kh}(\psi) := \sup_{\nu\in\mathbb{R}} \Big\| \mathfrak{L}\big[\Psi_k^{0,1}\big](\mathbf{x},\psi+i\nu) - \mathfrak{L}\big[\Psi_h^{0,1}\big](\mathbf{x},\psi+i\nu) \Big\|_{\Xi^{-r-1}(\Omega)}$. Then $\lim_{k,h\to\infty} D_{kh}(\psi) = 0$.

Proof It follows from the definition of Ψ and $\{\Psi_k\}_{k=1}^{\infty}$ in $\Xi^{-r-1,-s}(\Omega\times\mathbb{R})$ that $\frac{\partial\Psi}{\partial t} = \lim_{k\to\infty} W_k^{0,1}$ in $\Xi^{-r-1,-s-1}(\Omega_{\mathfrak{T}_a})$. Then there is $K_1(\mathfrak{T}_a) > 0$ such that for every $\varphi(\mathbf{x})$ in the closed unit ball of $\Xi^{r+1}(\Omega)$

$$\sup_{\nu\in\mathbb{R}} \Big| \Big\langle \mathfrak{L}\big[W_k^{0,1} - W_h^{0,1}\big](\mathbf{x},\psi+i\nu), \varphi(\mathbf{x}) \Big\rangle \Big| =$$

$$= \sup_{\nu\in\mathbb{R}} \Big| \Big\langle (W_k^{0,1} - W_h^{0,1})(\mathbf{x},t), \varphi(\mathbf{x}) e^{-(\psi+i\nu)\,t} \chi_{[-\mathfrak{T}_a,\mathfrak{T}_a]}(t) \Big\rangle \Big|$$

$$\leq \Big\| W_k^{0,1} - W_h^{0,1} \Big\|_{\Xi^{-r-1,-r-1}(\Omega_{\mathfrak{T}_a})} \sup_{\nu\in\mathbb{R}} \Big\| \varphi(\mathbf{x})\, e^{-(\psi+i\nu)\,t} \chi_{[-\mathfrak{T}_a,\mathfrak{T}_a]}(t) \Big\|_{\Xi^{r+1,r+1}(\Omega_{\mathfrak{T}_a})}$$

$$\leq K_1(\mathfrak{T}_a) \, \Big\| W_k^{0,1} - W_h^{0,1} \Big\|_{\Xi^{-r-1,-r-1}(\Omega_{\mathfrak{T}_a})} \Big\| \varphi \Big\|_{\Xi^{r+1}(\Omega)} \leq$$

$$\leq K_1(\mathfrak{T}_a) \, \Big\| W_k^{0,1} - W_h^{0,1} \Big\|_{\Xi^{-r-1,-r-1}(\Omega_{\mathfrak{T}_a})} \tag{41}$$

and the claim follows from the density of $\mathcal{D}(\Omega_{\mathfrak{T}_a})$ in $\Xi^{r+1,r+1}(\Omega_{\mathfrak{T}_a})$. □

Claim 3. We have $\lim_{k\to\infty} Z_k = Z$ in $L^2\big( ]-T,T[, H^{-r-1+\frac{2\beta}{m+1}}(\Omega) \big)$.

Proof We have $\mathfrak{L}(\Psi_k - \Psi_h)(\mathbf{x},\psi+\nu i) \in \mathcal{D}(\Omega) \subset \Xi^{-r-1}(\Omega)$ for every k,h in $\mathbb{N}$ and $\nu\in\mathbb{R}$. On the other hand, it follows from (40) corollary 11 that for every $\nu\in\mathbb{R}$, the restriction to $\Xi^{-r-1}(\Omega)$ of $(\mathcal{X}_{\psi+\nu i}^{-1})' \in \mathcal{L}\big((\mathfrak{R}^{r+3}(\Omega))', H^{-r-1+\frac{2\beta}{m+1}} \big)$ is the map $\mathcal{X}_{\psi+\nu i}^{-1} \in \mathcal{L}(\Xi^{-r-1}(\Omega), H^{-r-1+\frac{2\beta}{m+1}}(\Omega))$. Let $\mathcal{B}$ be the closed unit ball of $\big(H^{-r-1+\frac{2\beta}{m+1}}(\Omega) \big)'$. By (32), since $\mathcal{D}(\overline{\Omega})$ is dense in $H^{-r-1+\frac{2\beta}{m+1}}(\Omega) \subset \big(H^{-r-1+\frac{2\beta}{m+1}}(\Omega) \big)'$, by $\big[$[17], example 17.2$\big]$ for every $t \in]-T,T[$ and k and h in $\mathbb{N}$ there is Υ_t^{kh} in $\mathcal{B}$ and a sequence $\{g_b^t(\mathbf{x})\}_{b=1}^{\infty} \subset \mathcal{D}(\overline{\Omega})$ in $\mathcal{B}$ such that $\Upsilon_t^{kh} = \lim_{b\to\infty} g_b(\mathbf{x})$ in $\big(H^{-r-1+\frac{2\beta}{m+1}}(\Omega) \big)'$ and

$$\big\| (Z_k - Z_h)(\mathbf{x},t) \big\|_{H^{-r-1+\frac{2\beta}{m+1}}(\Omega)} = \big| \big\langle (Z_k - Z_h)(\mathbf{x},t), \Upsilon_t^{kh} \big\rangle \big| = \lim_{b\to\infty} \big| \big\langle (Z_k - Z_h)(\mathbf{x},t), g_b^t(\mathbf{x}) \big\rangle \big|$$

$$= \lim_{b\to\infty} \frac{1}{2\pi} \left| \int_\Omega \left(\int_{-\infty}^{\infty} e^{(\psi+i\,\nu)t} \mathcal{X}_{\psi+i\nu}^{-1} \big(\mathfrak{L}[\Psi_k - \Psi_h](\mathbf{x},\psi+i\nu) \big) \, d\nu \right) g_b^t(\mathbf{x}) \, d\mathbf{x} \right|$$

$$\leq \lim_{b\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\psi t} \left| \int_\Omega \mathcal{X}_{\psi+i\nu}^{-1} \big(\mathfrak{L}[\Psi_k - \Psi_h](\mathbf{x},\psi+i\nu) \big) g_b^t(\mathbf{x}) \, d\mathbf{x} \right| \, d\nu$$

$$= \lim_{b\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\psi t} \left| \int_{\Omega} (\mathcal{X}_{\psi+i\nu}^{-1})' \big(\mathfrak{L}[\Psi_k - \Psi_h](\mathbf{x}, \psi+i\nu) \big) g_b^t(\mathbf{x}) \, d\mathbf{x} \right| \, d\nu$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\psi t} \big\| (\mathcal{X}_{\psi+i\nu}^{-1})' \big(\mathfrak{L}[\Psi_k - \Psi_h](\mathbf{x}, \psi+i\nu) \big\|_{H^{-r-1+\frac{2\beta}{m+1}}(\Omega)} \, d\nu$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\psi t} \left\| \frac{1}{\psi+i\nu} (\mathcal{X}_{\psi+i\nu}^{-1})' \left(\mathfrak{L}\left[\frac{\partial \Psi_k}{\partial t} - \frac{\partial \Psi_h}{\partial t} \right] (\mathbf{x}, \psi+i\nu) \right) \right\|_{H^{-r-1+\frac{2\beta}{m+1}}(\Omega)} \, d\nu$$

$$\leq \frac{e^{\psi t}}{2\pi} D_{kh}(\psi) \int_{-\infty}^{\infty} \frac{1}{|\psi+i\nu|} \big\| (\mathcal{X}_{\psi+i\nu}^{-1})' \big\|_{\mathcal{L}(\Xi^{-r-1}(\Omega), H^{-r-1+\frac{2\beta}{m+1}}(\Omega))} \, d\nu < \infty$$

(by (32), (23) and the fact that $\beta < m+1$). It follows easily from claim 2 that $\{Z_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2\big(]-T,T[, H^{-r-1+\frac{2\beta}{m+1}}(\Omega)\big)$ and so there exists $W = \lim_{k\to\infty} Z_k$ in $L^2\big(]-T,T[, H^{-r-1+\frac{2\beta}{m+1}}(\Omega)\big)$.

A similar argumentation to the used one in (41) shows that $\mathfrak{L}[W](\mathbf{x},p) = \lim_{k\to\infty} \mathfrak{L}[Z_k](\mathbf{x},p)$ and $\mathfrak{L}[\Psi](\mathbf{x},p) = \lim_{k\to\infty} \mathfrak{L}[\Psi_k](\mathbf{x},p)$ in $\Xi^{-r}(\Omega)$ and so in $\Xi^{-r-1}(\Omega)$. Arguing as in (29), as $\Xi^{-r-1}(\Omega) \subset (H^{r+1}(\Omega))'$, by theorem 9 (with parameters $r+1, \beta, m+1$) and (30), taking the limits in $H^{-r-1+\frac{2\beta}{m+1}}(\Omega)$ we have

$$\mathfrak{L}[W](\mathbf{x},p) = \lim_{k\to\infty} \mathfrak{L}[Z_k](\mathbf{x},p) = \lim_{k\to\infty} \mathfrak{L}[\mathfrak{H} * \Psi_k](\mathbf{x},p) = \lim_{k\to\infty} \mathfrak{G}(p)\big(\mathfrak{L}[\Psi_k](\mathbf{x},p)\big)$$

$$= \lim_{k\to\infty} \mathcal{X}_p^{-1}\big(\mathfrak{L}[\Psi_k](\mathbf{x},p)\big) = \lim_{k\to\infty} (\mathcal{X}_p')^{-1}\big(\mathfrak{L}[\Psi_k](\mathbf{x},p)\big) = (\mathcal{X}_p')^{-1}\big(\mathfrak{L}[\Psi]((\mathbf{x},p)\big) = \mathfrak{L}[Z](\mathbf{x},p)$$

and so $Z = W$. □

We present some examples of application of theorem 13:

Corollary 14 *Let $0 < \eta < \xi < \frac{1}{2}$, $\varepsilon > 0$ and $T > 0$. If $(\mathbf{x}_0, t_0) \in \Omega \times \mathbb{R}$ and $\Psi := \delta(\mathbf{x} - \mathbf{x}_0) \otimes \delta(t - t_0)$, there is a unique $G \in \mathcal{D}'(\Omega \times \mathbb{R})$ verifying (26) and (27) for some $V \in \mathbb{R}$ and such that*

a) If $n = 1$ we have $G(\mathbf{x},t) \in H^{\frac{1}{2}-\eta}\big(]-T,T[, H^{\frac{1}{2}-\xi}(\Omega)\big)$. In particular $G(\mathbf{x},t) \in L^2(\Omega_T)$.

b) If $n \in \mathbb{N}$, we have $G(\mathbf{x},t) \in H^{\eta}\big(]-T,T[, H^{-\left(\frac{n}{2}-1+\varepsilon+2\xi\right)}(\Omega)\big)$.

c) If $n \in \mathbb{N}$, and $\Psi := \delta(\mathbf{x} - \mathbf{x}_0) \otimes H(t - t_0)$ (where $H(t)$ is the Heaviside function) we have $G(\mathbf{x},t) \in H^{1+\eta}\big(]-T,T[, H^{-\left(\frac{n}{2}-1+\varepsilon+2\xi\right)}(\Omega)\big)$.

Proof We use the same notations of theorem 13. By proposition 3 we can take $r = \frac{n}{2} + \varepsilon$, and clearly $\rho = 0$. We can choose $\alpha = r$ and $m = 2\,\alpha + \iota$ with arbitrary $\iota > 0$ in order that the hypothesis of theorem 13 be fulfilled.

a) Let $n = 1$. We have $\lim_{\iota\to 0} \left(\frac{1}{2} - \frac{\alpha}{m}\right) = 0$ uniformly with respect to $\varepsilon > 0$ and

$$\lim_{\iota\to 0} \left(r - \frac{2\alpha}{m} \right) = \frac{2\,\varepsilon^2 - \frac{1}{2}}{1 + 2\,\varepsilon}.$$

Hence, choosing $\varepsilon > 0$ small enough, by theorem 13 it turns out that $G(\mathbf{x},t) \in H^{\frac{1}{2}-\eta}\big(]-T,T[, H^{\frac{1}{2}-\xi}(\Omega)\big)$ and so $G(\mathbf{x},t) \in L^2(\Omega_T)$.

b) Let $n \geq 2$. If we fix $\varepsilon > 0$, $\frac{1}{2} - \frac{\alpha}{m}$ is an increasing function of ι convergent to 0 if ι approaches 0 and with limit $\frac{1}{2}$ if ι approaches ∞. Then we can write $\frac{1}{2} - \frac{\alpha}{m} = \xi$ for arbitrary $0 < \xi < \frac{1}{2}$. On the other hand, we obtain $r - \frac{2\,\alpha}{m} = \frac{n}{2} - 1 + \varepsilon + 2\,\xi$ and by theorem 13, $G(\mathbf{x}, t) \in H^{\eta}\big(]-T, T[, H^{-\left(\frac{n}{2}-1+\varepsilon+2\xi\right)}(\Omega)\big)$ for every $0 < \eta < \xi < \frac{1}{2}$.

c) The proof is analogous noting that in this case we can take $\rho = -1$.

□

For the proof of the following corollary we introduce some new notation in order to simplify the expression of the formulas that will appear. For every $(\mathbf{x}_0, t_0) \in \Omega_T$ we set $\Psi_{\mathbf{x}_0,t_0} := \delta(\mathbf{x}-\mathbf{x}_0) \otimes \delta(t-t_0)$, $\Psi_{\mathbf{x}_0} := \delta(\mathbf{x}-\mathbf{x}_0)$ and $A_{\mathbf{x}_0} := \left\|\Psi_{\mathbf{x}_0}\right\|_{\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)}$. First, we establish a lemma

Lemma 15 *Let* $(\overline{\mathbf{x}}_0, \bar{t}_0) \longrightarrow (\mathbf{x}_0, t_0)$ *in* $\Omega \times \mathbb{R}$ *and* $0 < \varepsilon < \frac{1}{2}$. *Then*

$$\lim_{(\overline{\mathbf{x}}_0,\bar{t}_0)\to(\mathbf{x}_0,t_0)} \Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} = \Psi_{\mathbf{x}_0,t_0} \quad \textit{and} \quad \lim_{(\overline{\mathbf{x}}_0,\bar{t}_0)\to(\mathbf{x}_0,t_0)} \mathfrak{L}\big[\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0}\big](p,\mathbf{x}) = \mathfrak{L}\big[\Psi_{\mathbf{x}_0,t_0}\big](p,\mathbf{x})$$

in the spaces $\Xi^{-\left(\frac{n}{2}+\varepsilon\right),-\left(\frac{1}{2}+\varepsilon\right)}(\Omega_T)$ *and* $\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)$ *respectively.*

Proof a) Choose $0 < \varepsilon' < \varepsilon$. By proposition 3, $\Psi_{\mathbf{x}_0,t_0} \in \Xi^{-\left(\frac{n}{2}+\varepsilon'\right),-\left(\frac{1}{2}+\varepsilon'\right)}(\Omega \times \mathbb{R})$ for each $(\mathbf{x}_0, t_0) \in \Omega \times \mathbb{R}$. Fixed $(\mathbf{x}_0, t_0)$ and $T > 0$, by proposition 3 there is a neighborhood W of $(\mathbf{x}_0, t_0)$ such that $\{\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} \mid (\overline{\mathbf{x}}_0, \bar{t}_0) \in W\}$ is bounded in $\Xi^{-\left(\frac{n}{2}+\varepsilon'\right),-\left(\frac{1}{2}+\varepsilon'\right)}(\Omega_T)$. Since for each $\varphi \in \mathcal{D}(\Omega_T)$ we have

$$\lim_{(\overline{\mathbf{x}}_0,\bar{t}_0)\to(\mathbf{x}_0,t_0)} \big\langle \varphi, \Psi_{\overline{\mathbf{x}}_0,\bar{t}_0}(\mathbf{x}, t)\big\rangle = \lim_{(\overline{\mathbf{x}}_0,\bar{t}_0)\to(\mathbf{x}_0,t_0)} \varphi(\overline{\mathbf{x}}_0, \bar{t}_0) = \varphi(\mathbf{x}_0, t_0) = \big\langle \varphi, \Psi_{\mathbf{x}_0,t_0}\big\rangle,$$

we obtain by density that $\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} \longrightarrow \Psi_{\mathbf{x}_0,t_0}$ weakly in $\Xi^{-\left(\frac{n}{2}+\varepsilon'\right),-\left(\frac{1}{2}+\varepsilon'\right)}(\Omega_T)$. The inclusion $\Xi^{-\left(\frac{n}{2}+\varepsilon'\right),-\left(\frac{1}{2}+\varepsilon'\right)}(\Omega_T) \subset \Xi^{-\left(\frac{n}{2}+\varepsilon\right),-\left(\frac{1}{2}+\varepsilon\right)}(\Omega_T)$ is compact by proposition 4 and Schauder's theorem and so $\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} \longrightarrow \Psi_{\mathbf{x}_0,t_0}$ in the norm of $\Xi^{-\left(\frac{n}{2}+\varepsilon\right),-\left(\frac{1}{2}+\varepsilon\right)}(\Omega_T)$.

b) For every $p \in S_{\mu_{\mathcal{X}}}$ and every $\kappa > 0$ there is $\varphi_{\overline{\mathbf{x}}_0,\mathbf{x}_0}(\mathbf{x})$ in the open unit ball of $\Xi^{\left(\frac{n}{2}+\varepsilon\right)}(\Omega)$ such that

$$\left\|\mathfrak{L}\big(\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} - \Psi_{\mathbf{x}_0,t_0}\big)(p,\mathbf{x})\right\|_{\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)}$$

$$\leq \left\|e^{-p\,\bar{t}_0}\big(\Psi_{\overline{\mathbf{x}}_0} - \Psi_{\mathbf{x}_0}\big)\right\|_{\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)} + \left\|\big(e^{-p\,\bar{t}_0} - e^{-p\,t_0}\big)\Psi_{\mathbf{x}_0}\right\|_{\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)}$$

$$= \left|e^{-p\,\bar{t}_0}\right| \left(\left|\big\langle \varphi_{\overline{\mathbf{x}}_0,\mathbf{x}_0}(\mathbf{x}), \Psi_{\overline{\mathbf{x}}_0} - \Psi_{\mathbf{x}_0}\big\rangle\right| + \kappa\right) + A_{\mathbf{x}_0}\left|e^{-p\,\bar{t}_0} - e^{-p\,t_0}\right| =$$

$$= \left|\big\langle e^{-Re(p)t}\varphi_{\overline{\mathbf{x}}_0,\mathbf{x}_0}, \Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} - \Psi_{\mathbf{x}_0,\bar{t}_0}\big\rangle\right| + \kappa\, e^{-Re(p)(\bar{t}_0-t_0)}e^{-Re(p)t_0} + A_{\mathbf{x}_0}\left|e^{-p\bar{t}_0} - e^{-pt_0}\right|$$

and by (6) there is $B_\varepsilon(Re(p), T) > 0$ such that $\kappa > 0$ being arbitrary we obtain

$$\left\|\mathfrak{L}\big(\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} - \Psi_{\mathbf{x}_0,t_0}\big)(p,\mathbf{x})\right\|_{\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)} \leq B_\varepsilon(Re(p), T)\left\|\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0} - \Psi_{\mathbf{x}_0,\bar{t}_0}\right\|_{\Xi^{-\left(\frac{n}{2}+\varepsilon\right)}(\Omega)}$$

$$+A_{\mathbf{x}_0}\left(\left|e^{-Re(p)\bar{t}_0}-e^{-Re(p)t_0}\right|+e^{-Re(p)t_0}\left|e^{-Im(p)\bar{t}_0\, i}-e^{-Im(p)t_0\, i}\right|\right) \tag{42}$$

which approaches 0 if $(\overline{\mathbf{x}}_0,\bar{t}_0)\to(\mathbf{x}_0,t_0)$ by part a). □

Proposition 16 *Given* $(\mathbf{x}_0,t_0)\in\Omega\times\mathbb{R}$, *let* $G_{\mathbf{x}_0,t_0}(\mathbf{x},t)$ *be the unique function verifying* $\mathcal{A}(G_{\mathbf{x}_0,t_0}(\mathbf{x},t))=\Psi_{\mathbf{x}_0,t_0}$ *in* $\Omega\times\mathbb{R}$, $\mathfrak{B}(G_{\mathbf{x}_0,t_0}(\mathbf{x},t)=0$ *in* $\partial\Omega\times\mathbb{R}$ *and* $G_{\mathbf{x}_0,t_0}(\mathbf{x},t)=0$ *in* $\Omega\times]-\infty,V]$ *for some* $V\in\mathbb{R}$. *Given* $0<\eta<\xi<\frac{1}{2}$, $\varepsilon>0$ *and* $T>0$ *one has*

$$\lim_{(\overline{\mathbf{x}}_0,\bar{t}_0)\to(\mathbf{x}_0,t_0)} G_{\overline{\mathbf{x}}_0,\bar{t}_0}(\mathbf{x},t)=G_{\mathbf{x}_0,t_0}(\mathbf{x},t)\quad in\ H^{\eta}\big(]-T,T[,H^{-\left(\frac{n}{2}-1+\varepsilon+2\xi\right)}(\Omega)\big). \tag{43}$$

Proof By corollary 14 we have $G(\mathbf{x},t,\mathbf{x}_0,t_0)\in H^{\eta}\big(]-T,T[,H^{-\left(\frac{n}{2}-1+\varepsilon+2\xi\right)}(\Omega)\big)$ for every $(\mathbf{x}_0,t_0)\in\Omega\times\mathbb{R}$ such that $|t_0|<T$. Arguing as in theorem 13, formula (31), and using its notations as well as those of corollary 14, part b, and writing $R_{\overline{\mathbf{x}}_0,\bar{t}_0}(\mathbf{x},t):=\big(G_{\overline{\mathbf{x}}_0,\bar{t}_0}(\mathbf{x},t)-G_{\mathbf{x}_0,t_0}((\mathbf{x},t)\big)$, by Hölder's inequality we have

$$\int_{-\infty}^{\infty}\Big(1+|\psi+i\,\nu|^2\Big)^{\eta}\left\|\mathfrak{L}\Big[R_{\overline{\mathbf{x}}_0,\bar{t}_0}(\mathbf{x},t)\Big](\psi+i\,\nu)\right\|^2_{H^{-\left(\frac{n}{2}-1+\xi\right)}(\Omega)}\,d\nu\le$$

$$\le C\int_{-\infty}^{\infty}\frac{\Big(1+|\psi+i\,\nu|^2\Big)^{\eta}}{\left|\psi+\frac{1}{2\,\tau}\right|^{1+2\xi}}\frac{\left\|\mathfrak{L}\big[\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0}-\Psi_{\mathbf{x}_0,t_0}\big](\psi+i\,\nu)\right\|^2}{\left|\psi+i\,\nu+\frac{1}{2\,\tau}\right|^{1+2\xi}}\,d\nu$$

and by (42) and Hölder's inequality

$$\le 3\,C\,B_{\varepsilon}(\psi,T)^2\int_{-\infty}^{\infty}\frac{\Big(1+|\psi+i\,\nu|^2\Big)^{\eta}}{\left|\psi+\frac{1}{2\,\tau}\right|^{1+2\xi}}\frac{\left\|\big(\Psi_{\overline{\mathbf{x}}_0,\bar{t}_0}-\Psi_{\mathbf{x}_0,t_0}\big)(\psi+i\,\nu,t)\right\|^2}{\left|\psi+i\,\nu+\frac{1}{2\,\tau}\right|^{1+2\xi}}\,d\nu+ \tag{44}$$

$$+3CA^2_{\mathbf{x}_0}\left|e^{-\psi\bar{t}_0}-e^{-\psi)t_0}\right|^2\int_{-\infty}^{\infty}\frac{\Big(1+|\psi+i\,\nu|^2\Big)^{\eta}}{\left|\psi+\frac{1}{2\,\tau}\right|^{1+2\xi}}\frac{d\nu}{\left|\psi+i\,\nu+\frac{1}{2\,\tau}\right|^{1+2\xi}}+ \tag{45}$$

$$+3\,Ce^{-2\psi t_0}\int_{-\infty}^{\infty}\frac{\Big(1+|\psi+i\,\nu|^2\Big)^{\eta}}{\left|\psi+\frac{1}{2\,\tau}\right|^{1+2\xi}}\frac{4}{\left|\psi+i\,\nu+\frac{1}{2\,\tau}\right|^{1+2\xi}}\,d\nu. \tag{46}$$

Since $1+2\xi-2\eta>1$, by lemma 15, we can choose $(\overline{\mathbf{x}}_0,\bar{t}_0)$ close enough to $(\mathbf{x}_0,t_0)$ in order that (44), (45), and (46) be arbitrarily small, finishing the proof. □

It is important to note that the results of theorem 13 cannot essentially be improved. In fact, in [9] there is an example for $n=2$ such that for every t in a set of positive measure of $]-T,T[,T>0$, the function $G(\mathbf{x},t)\notin L^2(\Omega)$ and hence $G\notin L^2(\Omega_T)$.

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