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Asymptotic stability of solutions for a certain non-autonomous second-order stochastic delay differential equation

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Abstract: In this paper, sufficient criteria that guarantee the existence of stochastic asymptotic stability of the zero solution of the nonautonomous second-order stochastic delay differential equation (1.1) were established with the aid of a suitable Lyapunov functional. Two examples are given in the last section to illustrate our main result.

Key words: Asymptotic stability, nonautonomous second-order stochastic delay differential equation, Lyapunov functional

1. Introduction

It is well known that random fluctuations are abundant in natural or engineered systems. Therefore stochastic modeling has come to play an important role in various fields such as biology, mechanics, economics, medicine, and engineering (see [6, 20, 21]). Moreover, these systems are sometimes subject to memory effects, when their time evolution depends on their past history with noise disturbance. Stochastic delay differential equations (SDDEs) give a mathematical formulation for such systems. They can be regarded as a natural generalization of stochastic ordinary differential equations by allowing the coefficients to depend on the past values. Lyapunov's direct method has been successfully used to investigate stability problems in deterministic/stochastic differential equations and delay differential equations.

Many papers dealt with the delay differential equations and obtained many good results, for example, [1, 15–19, 22]. Recently, the studies of stochastic differential equations have attracted considerable attention among scholars. Many interesting results have been obtained over the last few years (see, for example, [7, 9, 10, 23] and the references therein). Stability analysis is very important for stochastic delay systems, as we like to know the impact of memory as well as noise. This motivates a lot of recent research; see, for example, [2–5, 8, 11–14] and the references therein. In many references, the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for stability.

Here we consider the second-order stochastic delay differential equation of the following form:

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)f(x(t-r)) + g(t,x)\dot{\omega}(t) = 0, \tag{1.1}$$

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2010 *AMS Mathematics Subject Classification:* 34K20, 34K50, 60H35 where $a(t)$ and $b(t)$ are two positive and continuously differentiable functions on $[0,\infty)$, r is a positive constant, and $f(x)$ and $g(t,x)$ are continuous functions with $f(0)=0$. $\omega(t)\in\mathbb{R}^m$ is a standard Wiener process.

Essentially, our subject is to establish some sufficient conditions for the stochastic asymptotic stability of the zero solution of equation (1.1) by constructing a suitable Lyapunov functional.

2. Stability

Let $(\Omega,\mathcal{F},\{\mathcal{F}\}_{t\geq 0},\mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions. In other words, Ω is a set called the sample space, $\mathcal{F}$ is a σ-field of subsets of Ω (whence $(\Omega,\mathcal{F})$ is a measurable space), and $\mathcal{P}$ is a probability measure on $(\Omega,\mathcal{F})$ (i.e. is closed with respect to the set-theoretic operations executed a countable number of times). $(\Omega,\mathcal{F},\{\mathcal{F}\}_{t\geq 0},\mathcal{P})$ is filtered by a nondecreasing right-continuous family $\{\mathcal{F}\}_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$.

Let $B(t)=(B_1(t),\ldots,B_m(t))$ be an m-dimensional Brownian motion defined on the probability space. Consider an n-dimensional stochastic differential equation

$$dx(t)=f(t,x(t))dt+g(t,x(t))dB(t)\quad \text{on } t\geq 0, \tag{2.1}$$

with initial value $x(0)=x_0\in\mathbb{R}^n$. As a standing condition, we assume that $f:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ satisfy the local Lipschitz condition and the linear growth condition (see, for example, [9, 23]). It is therefore known that equation (2.1) has a unique continuous solution on $t\geq 0$, which is denoted by $x(t;x_0)$ in this paper. Assume furthermore that $f(t,0)=0$ and $g(t,0)=0$, for all $t\geq 0$. Hence the stochastic differential equation admits the zero solution $x(t;0)\equiv 0$.

Definition 2.1 *The zero solution of the stochastic differential equation is said to be stochastically stable or stable in probability, if for every pair of $\varepsilon\in(0,1)$ and $r>0$, there exists a $\delta=\delta(\varepsilon,r)>0$ such that*

$$P\{|x(t;x_0)|<r \text{ for all } t\geq 0\}\geq 1-\varepsilon,$$

whenever $|x_0|<\delta$. Otherwise it is said to be stochastically unstable.

Definition 2.2 *The zero solution of the stochastic differential equation is said to be stochastically asymptotically stable, if it is stochastically stable, and moreover for every $\varepsilon\in(0,1)$, there exists a $\delta_0=\delta_0(\varepsilon)>0$ such that*

$$P\{\lim_{t\to\infty}x(t;x_0)=0\}\geq 1-\varepsilon,$$

whenever $|x_0|<\delta_0$.

Let $C^{1,2}(\mathbb{R}^+\times\mathbb{R}^n;\mathbb{R}^+)$ denote the family of nonnegative functions $V(t,x)$ defined on $\mathbb{R}^+\times\mathbb{R}^n$, which are once continuously differentiable in t and twice continuously differentiable in x.

Define an operator $\mathcal{L}$ acting on $C^{1,2}(\mathbb{R}^+\times\mathbb{R}^n;\mathbb{R}^+)$ functions by

$$\mathcal{L}V(t,x)=V_t(t,x)+V_x(t,x).f(t,x)+\frac{1}{2}trace[g^T(t,x)V_{xx}(t,x)g(t,x)], \tag{2.2}$$

where $V_x=(V_{x_1},\ldots,V_{x_n})$ and $V_{xx}=(V_{x_ix_j})_{n\times n}$. Moreover, let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\rho:\mathbb{R}^+\to\mathbb{R}^+$ such that $\rho(0)=0$ and $\rho(r)>0$, if $r>0$.

Theorem 2.1 [10] *Assume that there exist* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ *and* $\rho \in \mathcal{K}$ *such that*

$$V(t,0) = 0, \qquad \rho(|x|) \leq V(t,x),$$

and

$$\mathcal{L}V(t,x) \leq 0, \quad \text{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then the zero solution of the stochastic differential equation is stochastically stable.

Theorem 2.2 [10] *Assume that there exist* $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ *and* $\rho_1, \rho_2,\ \rho_3 \in \mathcal{K}$ *such that*

$$\rho_1(|x|) \leq V(t,x) \leq \rho_2(|x|),$$

and

$$\mathcal{L}V(t,x) \leq -\rho_3(|x|), \quad \text{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then the zero solution of the stochastic differential equation is stochastically asymptotically stable.

Now we present the main stability result of (1.1).

Theorem 2.3 *Suppose that* $a(t)$ *and* $b(t)$ *are two continuously differentiable functions on* $[0,\infty)$ *and the following conditions are satisfied:*

(i) $A \geq a(t) \geq a_0 > \frac{1}{2}$ *and* $B \geq b(t) \geq b_0 > 0$, *for* $t \in [0,\infty)$.

(ii) $f(0) = 0$, $\frac{f(x)}{x} \geq f_0 > 0$ $(x \neq 0)$ *and* $f'(x) \leq f_1$, *for all* x.

(iii) $g(t,x) \leq Cx$ *for positive constant* C.

(iv) $a'(t) \leq \alpha$ *and* $b'(t) \leq \beta$ *for positive constants* α, β.

(v) $b_0 f_0 \geq \frac{3}{4}$ *and* $2\beta f_1 + \alpha + 2C^2 < \frac{3}{2}$.

Then the zero solution of (1.1) *is stochastically asymptotically stable, provided that*

$$r < \min\left\{\frac{2b_0 f_0 - 2\beta f_1 - \alpha - 2C^2}{2Bf_1}, \frac{2a_0 - 1}{5Bf_1}\right\}.$$

3. Proof of Theorem 2.3

We can write equation (1.1) in the following equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a(t)y - b(t)f(x) + b(t)\int_{t-r}^{t} f'(x(s))y(s)ds - g(t,x)\dot{\omega}(t). \end{aligned} \tag{3.1}$$

We define the Lyapunov functional $V(t, x_t, y_t)$ as

$$V(t,x_t,y_t) = 2b(t)\int_0^x f(\xi)d\xi + \frac{1}{2}a(t)x^2 + xy + y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds, \tag{3.2}$$

where $x_t = x(t+s),\ s \le 0$ and λ is a positive constant, which will be determined later.

Thus from (3.2), (3.1) and by using It$\hat{o}$ formula, we get

$$\begin{aligned}\mathcal{L}V(t,x_t,y_t) =& 2b'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}a'(t)x^2 + \lambda r y^2 - \lambda\int_{t-r}^t y^2(s)ds + y^2 \\ & - b(t)f(x)x - 2a(t)y^2 + (x+2y)b(t)\int_{t-r}^t f'(x(s))y(s)ds \\ & + g^2(t,x).\end{aligned} \tag{3.3}$$

Since $b(t) \le B$, $f'(x) \le f_1$ and by using the inequality $2uv \le u^2 + v^2$, we have

$$b(t)x\int_{t-r}^t f'(x(s))y(s)ds \le Bf_1\int_{t-r}^t x(t)y(s)ds \le \frac{1}{2}Bf_1 r x^2 + \frac{1}{2}Bf_1\int_{t-r}^t y^2(s)ds,$$

$$2b(t)y\int_{t-r}^t f'(x(s))y(s)ds \le 2Bf_1\int_{t-r}^t y(t)y(s)ds \le Bf_1 r y^2 + Bf_1\int_{t-r}^t y^2(s)ds.$$

Then by substituting in (3.3) we obtain

$$\begin{aligned}\mathcal{L}V \le& 2b'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}a'(t)x^2 + \lambda r y^2 - \lambda\int_{t-r}^t y^2(s)ds + y^2 - b(t)f(x)x \\ & - 2a(t)y^2 + \frac{1}{2}Bf_1 r x^2 + Bf_1 r y^2 + \frac{3}{2}Bf_1\int_{t-r}^t y^2(s)ds + g^2(t,x).\end{aligned}$$

Since $f'(x) \le f_1$ and $f(0) = 0$, then by using the mean-value theorem, we obtain $f(x) \le f_1 x$. From this and conditions $(i)-(iv)$ of Theorem 2.3 we get

$$\begin{aligned}\mathcal{L}V \le& 2\beta\int_0^x f_1\xi d\xi + \frac{1}{2}\alpha x^2 + \lambda r y^2 - \lambda\int_{t-r}^t y^2(s)ds + y^2 - b_0 f_0 x^2 \\ & - 2a_0 y^2 + \frac{1}{2}Bf_1 r x^2 + Bf_1 r y^2 + \frac{3}{2}Bf_1\int_{t-r}^t y^2(s)ds + C^2x^2 \\ \le& -(b_0 f_0 - \beta f_1 - \frac{1}{2}\alpha - \frac{1}{2}Bf_1 r - C^2)x^2 - (2a_0 - 1 - Bf_1 r - \lambda r)y^2 \\ & + (\frac{3}{2}Bf_1 - \lambda)\int_{t-r}^t y^2(s)ds.\end{aligned}$$

If we take $\lambda = \frac{3}{2}Bf_1$, then we find

$$\mathcal{L}V \le -(b_0 f_0 - \beta f_1 - \frac{1}{2}\alpha - \frac{1}{2}Bf_1 r - C^2)x^2 - (2a_0 - 1 - \frac{5}{2}Bf_1 r)y^2.$$

Therefore, if

$$r < \min\left\{\frac{2b_0 f_0 - 2\beta f_1 - \alpha - 2C^2}{2Bf_1}, \frac{2a_0 - 1}{5Bf_1}\right\},$$

we have

$$\mathcal{L}V(t,x_t,y_t) \le -D_1(x^2+y^2), \quad \text{for some } D_1 > 0. \tag{3.4}$$

Since $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ is nonnegative, then we obtain

$$V(t,x_t,y_t) \geq 2b(t)\int_0^x f(\xi)d\xi + \frac{1}{2}a(t)x^2 + xy + y^2.$$

Since $a(t) \geq a_0, b(t) \geq b_0$, and $\frac{f(x)}{x} \geq f_0$, therefore we have

$$\begin{aligned} V &\geq b_0 f_0 x^2 + \frac{1}{2}a_0 x^2 + xy + y^2 \\ &= \left(b_0 f_0 + \frac{a_0}{2}\right)x^2 + \left(x + \frac{1}{2}y\right)^2 - x^2 - \frac{1}{4}y^2 + y^2 \\ &\geq \left(b_0 f_0 + \frac{a_0}{2} - 1\right)x^2 + \frac{3}{4}y^2. \end{aligned}$$

However, $b_0 f_0 + \frac{a_0}{2} > 1$; thus we can get

$$V(t,x_t,y_t) \geq D_2(x^2+y^2), \qquad \text{for some } D_2 > 0. \tag{3.5}$$

Now since $f(x) \leq f_1 x$ and from the condition (i) of Theorem 2.3, we find

$$V(t,x_t,y_t) \leq Bf_1 x^2 + \frac{1}{2}Ax^2 + xy + y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds. \tag{3.6}$$

However,

$$\begin{aligned} \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds &= \int_{t-r}^{t}(\theta - t + r)y^2(\theta)d\theta \\ &\leq \|y\|^2\int_{t-r}^{t}(\theta - t + r)d\theta = \frac{r^2}{2}\|y\|^2, \end{aligned}$$

then by substituting in (3.6) and by using the inequality $uv \leq \frac{1}{2}(u^2+v^2)$, we have

$$\begin{aligned} V &\leq \left(Bf_1 + \frac{1}{2}A\right)x^2 + \frac{1}{2}(x^2+y^2) + y^2 + \lambda\frac{r^2}{2}\|y\|^2 \\ &\leq \left\{Bf_1 + \frac{1}{2}(A+1)\right\}\|x\|^2 + \frac{\lambda r^2 + 3}{2}\|y\|^2. \end{aligned}$$

Hence we can get

$$V(t,x_t,y_t) \leq D_3(x^2+y^2), \qquad \text{for some } D_3 > 0. \tag{3.7}$$

Therefore from (3.4), (3.5), and (3.7) all the assumptions of Theorem 2.2 are satisfied and so the zero solution of (1.1) is stochastically asymptotically stable. Thus the proof of Theorem 2.3 is now complete.

4. Examples

In this section we provide two examples to illustrate the application of the result we obtained in the previous section.

Example 1 let

$$a(t) = e^{-\frac{1}{2}t} + \frac{11}{20},\ b(t) = \frac{1}{t+1} + \frac{3}{4},\ f(x) = \frac{x}{x^2+1} + x \quad \text{and} \quad g(t,x) = x\ \frac{t}{t^2+1}.$$

Since

$$\frac{31}{20} \geq a(t) = e^{-\frac{1}{2}t} + \frac{11}{20} \geq \frac{11}{20} > \frac{1}{2}, \quad \text{for} \quad t \in [0,\infty),$$

we can take $A = \frac{31}{20}$ and $a_0 = \frac{11}{20}$. As a result, we have

$$a'(t) = -\frac{1}{2}e^{-\frac{1}{2}t} \leq 0, \quad \text{for} \quad t \in [0,\infty).$$

Thus we can take $\alpha = 0.001 \times 10^{-3}$. Moreover, since

$$\frac{7}{4} \geq b(t) = \frac{1}{t+1} + \frac{3}{4} \geq \frac{3}{4}, \quad \text{for} \quad t \in [0,\infty),$$

then we can take $B = \frac{7}{4}$ and $b_0 = \frac{3}{4}$. It follows that

$$b'(t) = -\frac{1}{(t+1)^2} \leq 0, \quad \text{for} \quad t \in [0,\infty),$$

and hence we can take $\beta = 0.001$. Next we can note that

$$\frac{f(x)}{x} = \frac{1}{x^2+1} + 1 \geq 1, \quad \text{for all} \ \ x,$$

then we can take $f_0 = 1$. As a result, we have

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} + 1 \leq 2, \quad \text{for all} \ \ x;$$

thus we can take $f_1 = 2$. We also have

$$g^2(t,x) = x^2\ \frac{t^2}{(t^2+1)^2} \leq \frac{1}{4}x^2, \quad \text{for} \quad t \in [0,\infty);$$

then we can take $C = \frac{1}{2}$.

Therefore we can prove that condition (v) of Theorem 2.3 is satisfied. Note that

$$\frac{2b_0f_0 - 2\beta f_1 - \alpha - 2C^2}{2Bf_1} \simeq 0.1423,$$

and

$$\frac{2a_0 - 1}{5Bf_1} \simeq 0.0057.$$

Hence the zero solution of the following equation

$$\ddot{x}(t) + \big(e^{-\frac{1}{2}t} + \frac{11}{20}\big)\dot{x}(t) + \big(\frac{1}{t+1} + \frac{3}{4}\big)\big\{\frac{x(t-r)}{x^2(t-r)+1} + x(t-r)\big\} + \big(\frac{xt}{t^2+1}\big)\ \dot{\omega}(t) = 0,$$

is stochastically asymptotically stable, provided that $r = 0.0057$.

Example 2 let

$$a(t) = \frac{2}{\sqrt{t+1}} + \frac{3}{5},\ b(t) = \frac{1}{t^2+1} + 4,\ f(x) = \sin x + \frac{3}{5}x \quad \text{and} \quad g(t,x) = \frac{1}{4}x\ e^{-\frac{1}{2}t}.$$

Since

$$\frac{13}{5} \geq a(t) = \frac{2}{\sqrt{t+1}} + \frac{3}{5} \geq \frac{3}{5} > \frac{1}{2}, \quad \text{for} \quad t \in [0, \infty),$$

we can take $A = \frac{13}{5}$ and $a_0 = \frac{3}{5}$. It follows that

$$a'(t) = -\frac{1}{(t+1)^{\frac{3}{2}}} \leq 0, \quad \text{for} \quad t \in [0, \infty);$$

thus we can take $\alpha = 0.1$. Furthermore, since

$$5 \geq b(t) = \frac{1}{t^2+1} + 4 \geq 4, \quad \text{for} \quad t \in [0, \infty),$$

then we can take $B = 5$ and $b_0 = 4$. Therefore

$$b'(t) = -\frac{2t}{(t^2+1)^2} \leq 0, \quad \text{for} \quad t \in [0, \infty);$$

hence we can take $\beta = 0.01$. Next we can see that

$$\frac{f(x)}{x} = \frac{\sin x}{x} + \frac{3}{5} \geq \frac{1}{5}, \quad \text{for all} \quad x;$$

then we can take $f_0 = \frac{1}{5}$. As a result, we obtain

$$f'(x) = \cos x + \frac{3}{5} \leq \frac{8}{5}, \quad \text{for all} \quad x;$$

thus we can take $f_1 = \frac{8}{5}$. We also have

$$g^2(t,x) = \frac{1}{16}x^2 e^{-t} \leq \frac{1}{16}x^2, \quad \text{for} \quad t \in [0, \infty);$$

then we can take $C = \frac{1}{4}$.

Then we can show that condition (v) of Theorem 2.3 is satisfied. Note that

$$\frac{2b_0 f_0 - 2\beta f_1 - \alpha - 2C^2}{2Bf_1} \simeq 0.084,$$

and

$$\frac{2a_0 - 1}{5Bf_1} = 0.005.$$

Hence the zero solution of the following equation

$$\ddot{x}(t) + \big(\frac{2}{\sqrt{t+1}} + \frac{3}{5}\big)\dot{x}(t) + \big(\frac{1}{t^2+1} + 4\big)\big\{\sin x(t-r) + \frac{3}{5}x(t-r)\big\} + \frac{1}{4}x\ e^{-\frac{1}{2}t}\ \dot{\omega}(t) = 0,$$

is stochastically asymptotically stable, provided that $r = 0.005$.

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