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The most important inequalities of m-convex functions

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Abstract: The intention of this article is to investigate the most important inequalities of m-convex functions without using their derivatives. The article also provides a brief survey of general properties of m-convex functions.

Key words: m-Convex function, Jensen inequality, Fejér inequality, Hermite–Hadamard inequality

1. Introduction

Let $[a,b]\subset\mathbb{R}$ be an interval where $a<b$. Each point $c\in[a,b]$ can be represented by the binomial convex combination

$$c=\frac{b-c}{b-a}a+\frac{c-a}{b-a}b.$$

Using the coefficient $t=(c-a)/(b-a)$, which belongs to $[0,1]$, the above combination takes the form $c=(1-t)a+tb$. The above representations also apply to $a>b$. It is suitable to use the convex hull of points a and b as the set $\operatorname{conv}\{a,b\}=\{(1-t)a+tb : t\in[0,1]\}$, including the case $a=b$.

If the interval $[a,b]$ contains the zero, then the product mc belongs to $[a,b]$ for every $c\in[a,b]$ and $m\in[0,1]$. Thus, if points $x,y\in[a,b]$ and coefficients $t,m\in[0,1]$, then the convex combination $(1-t)x+tmy$ of points x and my belongs to the convex hull $\operatorname{conv}\{x,my\}\subseteq[a,b]$. Any interval $I\subseteq\mathbb{R}$ containing the zero possesses the above properties. Throughout the paper, we will use proper intervals (with the nonempty interior) of real numbers containing the zero.

Definition 1 *Let $I\subseteq\mathbb{R}$ be an interval containing the zero, and let $m\in(0,1]$ be a number. A function $f:I\to\mathbb{R}$ is said to be m-convex if the inequality*

$$f\big((1-t)x+tmy\big)\le(1-t)f(x)+tmf(y) \tag{1.1}$$

holds for every pair of points $x,y\in I$ and every coefficient $t\in[0,1]$.

We point out the following note relating to the above definition. Using the point $x=my$ or the coefficient $t=1$ in formula (1.1), we get the same, $f(my)\le mf(y)$. Thus, the case $x=my$ can be omitted in Definition 1.

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Geometric presentation of the inequality of m-convexity in formula (1.1) indicates that the line segment connecting the graph point $F(x,f(x))$ and the point $M(my,mf(y))$ is above the graph of the restriction $f/\mathrm{conv}\{x,my\}$. According to the definition, a 1-convex function represents a usually convex function. The notion of m-convexity was introduced in [14] by employing a function f defined on the bounded closed interval $I=[0,b]$, where $b>0$.

Convex functions are continuous in the interior of their domains. Contrary to this, m-convex functions may be discontinuous at the interior points. The following example illustrates just such an m-convex function.

Example 1.1 *The function*

$$f(x)=\begin{cases}\frac{1}{2}x & \text{for } 0\le x<1\\ \frac{3}{2}x-\frac{1}{2} & \text{for } 1\le x\le 2\end{cases}$$

is m-convex for every $m\in(0,1/2]$, and it is discontinuous at $x=1$.

Let $f:I\to\mathbb{R}$ be a function, where I contains the zero. If f is m-convex for some $m\in(0,1)$, then $f(0)\le 0$. If f is convex and $f(0)\le 0$, then f is m-convex for every $m\in(0,1]$.

In [15], it was shown that for each $m\in(0,1)$ there are differentiable functions that are m-convex but are not usually convex, that is, 1-convex. Such an interesting example was given in [11]; the polynomial f defined by

$$f(x)=\frac{1}{12}\big(x^4-5x^3+9x^2-5x\big)$$

on the interval $[0,\infty)$ is 16/17-convex, but it is not m-convex for any $m\in(16/17,1]$.

The type of generalization of convex functions that relies on the secant line was considered in [12].

2. General properties of m-convex functions

We consider the primary properties of m-convex functions defined on the interval containing the zero.

Theorem 2.1 *Let $I\subseteq\mathbb{R}$ be an interval containing the zero, let $m\in(0,1]$ be a number, and let $f:I\to\mathbb{R}$ be a function.*

Then the following statements are equivalent.

(i) The function f satisfies the inequality

$$f\big((1-t)x+tmz\big)\le(1-t)f(x)+tmf(z) \tag{2.1}$$

for every pair of points $x,z\in I$ such that $x\ne mz$, and every coefficient $t\in[0,1]$.

(ii) The function f satisfies the inequality

$$f(y)\le\frac{mz-y}{mz-x}f(x)+\frac{y-x}{mz-x}mf(z) \tag{2.2}$$

for every triplet of points $x,y,z\in I$ such that $x\ne mz$ and $y\in\mathrm{conv}\{x,mz\}$.

(iii) The function f satisfies the following: the inequality

$$\begin{vmatrix} x & f(x) & 1 \\ y & f(y) & 1 \\ mz & mf(z) & 1 \end{vmatrix} \geq 0 \tag{2.3}$$

for a triplet of points $x, y, z \in I$ in the case that $x < mz$ and $x \leq y \leq mz$, and the inequality with replacement in the first and third row in the determinant of formula (2.3) in the case that $x > mz$ and $mz \leq y \leq x$.

Proof From (i) follows (ii) by applying formula (2.1) to a pair of points x and z such that $x \neq mz$, and the coefficient $t = (y - x)/(mz - x)$. Herein we utilize the convex combination of points x and mz in the form of

$$y = \frac{mz - y}{mz - x}x + \frac{y - x}{mz - x}mz.$$

From (ii) follows (iii) because formula (2.2) can be arranged into formula

$$x[f(y) - mf(z)] - y[f(x) - mf(z)] + mz[f(x) - f(y)] \geq 0.$$

From (iii) follows (i) having discussed two cases for a pair of points $x, z \in I$ such that $x \neq mz$, a coefficient $t \in [0, 1]$, and the convex combination of x and mz in the form of

$$y = (1 - t)x + tmz.$$

In the case that $x < mz$, we apply formula (2.3) to the order $x \leq y \leq mz$ with $(1 - t)x + tmz$ instead of y, and we obtain the inequality

$$(mz - x)f(y) \leq (1 - t)(mz - x)f(x) + t(mz - x)mf(z),$$

which, divided by $mz - x$, gives the inequality in formula (2.1).

In the case that $x > mz$, we apply formula (2.3) to the order $mz \leq y \leq x$ with $(1 - t)x + tmz$ instead of y, and we obtain an inequality that divided by $x - mz$ also yields the inequality in formula (2.1). □

With respect to the note related to Definition 1, each of the statements $(i) - (iii)$ in Theorem 2.1 can be used as a definition of m-convexity of the function f.

Corollary 2.1 *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $m \in (0, 1]$ be a number.*

Then each m-convex function $f : I \to \mathbb{R}$ satisfies the inequality

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(y) - mf(z)}{y - mz} \tag{2.4}$$

for every triplet of points $x, y, z \in I$ in the order of $x < y < mz$. If $mz < y < x$, then the reverse inequality is valid in formula (2.4).

The inequality in formula (2.4) is useful to study derivatives of m-convex functions. Derivatives of m-convex functions on the interval $[0, b]$ were considered in [2].

Corollary 2.2 *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $m \in (0,1)$ be a number.*

Then each m-convex function $f : I \to \mathbb{R}$ satisfies the following inequalities.

If $x \le y < 0$ or $0 < x \le y$, then

$$\frac{f(x)}{x} \le \frac{f(y)}{y}. \tag{2.5}$$

If $x < 0 < y$, then

$$\frac{f(x)}{x} - \frac{f(y)}{y} \le \left(\frac{1}{x} - \frac{1}{y}\right) \frac{(m+1)f(0)}{2m}. \tag{2.6}$$

Proof Formula (2.5) can be obtained by arranging formula (2.2) in the orders of $x \le y < my$ and $mx < x \le y$.

Formula (2.6) can be obtained by summing the inequalities arising from formula (2.2) in the orders of $x < 0 < my$ and $mx < 0 < y$. □

Respecting the above corollary, the function $f(x)/x$ is nondecreasing on the intervals $I \cap (-\infty, 0)$ and $I \cap (0, +\infty)$. Thus, the function $f(x)/x$ is almost everywhere continuous on $I \setminus \{0\}$. We therefore reach the conclusion that an m-convex function $f : I \to \mathbb{R}$ is almost everywhere continuous for every $m \in (0,1]$.

Now we consider m-convex functions defined on the interval $[a,b]$ such that $a < 0 < b$. The following lemma shows that these functions are bounded.

Lemma 2.1 *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a < 0 < b$, and let $m \in (0,1]$ be a number.*

Then each m-convex function $f : [a,b] \to \mathbb{R}$ is bounded by the affine functions as follows.

If $a \le x \le 0$, then

$$\frac{f(b) - f(0)}{b} x + \frac{f(0)}{m} \le f(x) \le \frac{f(a)}{a} x. \tag{2.7}$$

If $0 \le x \le b$, then

$$\frac{f(a) - f(0)}{a} x + \frac{f(0)}{m} \le f(x) \le \frac{f(b)}{b} x. \tag{2.8}$$

Proof Formula (2.7) can be derived by applying the m-convexity of f to the ordered triplets $mx \le 0 < b$ and $a \le x \le 0$. Formula (2.8) can be derived by applying the m-convexity of f to the ordered triplets $a < 0 \le mx$ and $0 \le x \le b$. □

Each m-convex function $f : [a,b] \to \mathbb{R}$ is bounded and almost everywhere continuous, and therefore it is the Riemann integrable. Concise notations on Riemann integrals and integrability are given in [10].

We finish the section by estimating the integral of an m-convex function $f : [a,b] \to \mathbb{R}$ satisfying $f(0) = 0$. Integrating the inequalities in formulae (2.7) and (2.8), and then summing, we obtain the bounds as follows.

Corollary 2.3 *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a < 0 < b$, and let $m \in (0,1]$ be a number.*

Then each m-convex function $f : [a,b] \to \mathbb{R}$ such that $f(0) = 0$ satisfies the double inequality

$$\frac{b^3 f(a) - a^3 f(b)}{2ab} \le \int_a^b f(x)\, dx \le \frac{bf(b) - af(a)}{2}. \tag{2.9}$$

3. Main results

The main results primarily include the next two independent theorems. Further considerations are trying to correlate these results. In studying the discrete inequalities, we rely on mathematical induction. In studying the integral inequalities, we apply the integral method including convex combinations.

Now we are looking for an upper bound of the Riemann integral of an m-convex function f on the interval $[a,b]$ by using an intermediate point $mc \in [a,b]$.

Theorem 3.1 *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $m \in (0,1]$ be a number. Let $a,b,c \in I$ be points such that $a \le mc \le b$.*

Then each m-convex function $f : I \to \mathbb{R}$ satisfies the inequality

$$\int_a^b f(x)\,dx \le \frac{mc-a}{2}f(a) + \frac{b-mc}{2}f(b) + \frac{b-a}{2}mf(c). \tag{3.1}$$

Proof The inequality in formula (3.1) trivially holds if $a = b$, so let $a < b$.

Suppose that $a \le x \le mc$, where $a < mc$. Applying the m-convexity of the function f to the convex combination

$$x = \frac{mc-x}{mc-a}a + \frac{x-a}{mc-a}mc,$$

we get

$$f(x) \le \frac{mc-x}{mc-a}f(a) + \frac{x-a}{mc-a}mf(c). \tag{3.2}$$

Integrating the above function inequality over the interval $[a,mc]$, the integral estimation follows:

$$\int_a^{mc} f(x)\,dx \le \frac{mc-a}{2}\big(f(a) + mf(c)\big), \tag{3.3}$$

which also applies to $mc = a$. As regards the interval $[mc,b]$, in a similar way we obtain the estimation

$$\int_{mc}^b f(x)\,dx \le \frac{b-mc}{2}\big(f(b) + mf(c)\big). \tag{3.4}$$

Applying the inequalities in formulae (3.3) and (3.4) to the corresponding terms of the right-hand side of the decomposition,

$$\int_a^b f(x)\,dx = \int_a^{mc} f(x)\,dx + \int_{mc}^b f(x)\,dx,$$

and arranging, we achieve the inequality in formula (3.1). □

If $I = [a,b]$ is an interval containing the zero, then we can use any point $c \in [a,b]$ in formula (3.1) because $mc \in [a,b]$. Using $c = a$ or $c = b$ in this case, we get the following appropriate inequality.

Corollary 3.1 *Let $[a,b] \subset \mathbb{R}$ be an interval containing the zero, and let $m \in (0,1]$ be a number.*

Then each m-convex function $f : [a,b] \to \mathbb{R}$ satisfies the inequality

$$\int_a^b f(x)\,dx \le \frac{mb-a}{2}f(a) + \frac{b-ma}{2}f(b). \tag{3.5}$$

Now we consider the discrete form of Jensen's inequality (see [6]) for m-convex functions.

Theorem 3.2 *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $\sum_{i=1}^{n} t_i x_i$ be a convex combination of points $x_i \in I$ with coefficients $t_i \in [0,1]$. Let $m \in (0,1]$ be a number.*

Then each m-convex function $f : I \to \mathbb{R}$ satisfies the inequality

$$f\left(m\sum_{i=1}^{n} t_i x_i\right) \leq m\sum_{i=1}^{n} t_i f(x_i). \tag{3.6}$$

Proof The proof can be done by applying mathematical induction on the number of points x_i.

The base of induction for $n = 1$ holds by the inequality $f(mx_1) \leq mf(x_1)$.

To prove the step of induction for $n \geq 2$, we assume that the inequality in formula (3.6) applies to all convex combinations having less than or equal to $n-1$ members. Assuming that $t_1 < 1$, and applying the inductive assumption to the point

$$y_1 = m\sum_{i=2}^{n} \frac{t_i}{1-t_1} x_i, \tag{3.7}$$

where the sum $\sum_{i=2}^{n}(t_i/(1-t_1))x_i$ is a convex combination belonging to I, we get

$$f(y_1) \leq m\sum_{i=2}^{n} \frac{t_i}{1-t_1} f(x_i). \tag{3.8}$$

Using the inequalities in formulae (1.1) and (3.8), we obtain

$$\begin{aligned} f\left(m\sum_{i=1}^{n} t_i x_i\right) &= f\big((1-t_1)y_1 + mt_1x_1\big) \\ &\leq (1-t_1)f(y_1) + mt_1 f(x_1) \\ &\leq (1-t_1)m\sum_{i=2}^{n} \frac{t_i}{1-t_1} f(x_i) + mt_1 f(x_1) \\ &= m\sum_{i=1}^{n} t_i f(x_i), \end{aligned} \tag{3.9}$$

achieving the inequality in formula (3.6). □

Corollary 3.2 *Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $\sum_{i=1}^{n} t_i x_i$ be a convex combination of points $x_i \in I$ with coefficients $t_i \in [0,1]$. Let $m \in (0,1]$ be a number.*

Then each m-convex function $f : I \to \mathbb{R}$ satisfies the inequality

$$f\left(t_1x_1 + m\sum_{i=2}^{n} t_i x_i\right) \leq t_1 f(x_1) + m\sum_{i=2}^{n} t_i f(x_i). \tag{3.10}$$

Proof We will once again employ the mathematical induction on the number of points x_i.

The base of induction for $n = 2$ holds by the definition in formula (1.1).

The step of induction for $n \geq 3$, assuming that $t_n < 1$, relies on the presentation

$$t_1 x_1 + m \sum_{i=2}^{n} t_i x_i = (1 - t_n) \left(\frac{t_1}{1 - t_n} x_1 + m \sum_{i=2}^{n-1} \frac{t_i}{1 - t_n} x_i \right) + t_n m x_n, \tag{3.11}$$

which enables us to apply the inductive premise. □

Relying on Theorem 3.2, we demonstrate the integral form of Jensen's inequality (see [7]) for m-convex functions as follows.

Corollary 3.3 *Let $[a, b] \subset \mathbb{R}$ be an interval containing the zero, and let $g : [a, b] \to \mathbb{R}$ be an integrable function with the image in $[a, b]$. Let $m \in (0, 1]$ be a number.*

Then each continuous m-convex function $f : [a, b] \to \mathbb{R}$ satisfies the inequality

$$f \left(\frac{m}{b - a} \int_a^b g(x)\, dx \right) \leq \frac{m}{b - a} \int_a^b f(g(x))\, dx. \tag{3.12}$$

Proof Let $I = [a, b]$, let n be a positive integer, and let $I = \cup_{i=1}^{n} I_{ni}$ be a partition of disjoint subintervals I_{ni} so that each of them contracts to the point as n approaches infinity. We take a point x_{ni} from each interval I_{ni} and form the convex combination

$$\sum_{i=1}^{n} \frac{|I_{ni}|}{|I|} g(x_{ni})$$

of points $y_{ni} = g(x_{ni})$ with coefficients $t_{ni} = |I_{ni}|/|I|$, where $|\ |$ denotes the length. Applying the inequality in formula (3.6) to the above convex combination, we produce the inequality

$$f \left(\frac{m}{|I|} \sum_{i=1}^{n} |I_{ni}| g(x_{ni}) \right) \leq \frac{m}{|I|} \sum_{i=1}^{n} |I_{ni}| f(g(x_{ni})),$$

and sending n to infinity, we realize the inequality in formula (3.12). To provide the limit on the left side, we have assumed the continuity of f. The composition $f(g)$ is integrable because it is bounded and almost everywhere continuous. □

Corollary 3.4 *Let $[a, b] \subset \mathbb{R}$ be an interval containing the zero, let $g : [a, b] \to \mathbb{R}$ be an integrable function with the image in $[a, b]$, and let $h : [a, b] \to \mathbb{R}$ be a positive integrable function. Let $m \in (0, 1]$ be a number.*

Then each continuous m-convex function $f : [a, b] \to \mathbb{R}$ satisfies the inequality

$$f \left(m \frac{\int_a^b g(x) h(x)\, dx}{\int_a^b h(x)\, dx} \right) \leq m \frac{\int_a^b f(g(x)) h(x)\, dx}{\int_a^b h(x)\, dx}. \tag{3.13}$$

Proof The procedure used in the proof of Corollary 3.3 should be applied to the convex combination

$$\sum_{i=1}^{n} \frac{|I_{ni}| h(x_{ni})}{\sum_{i=1}^{n} |I_{ni}| h(x_{ni})} g(x_{ni}) = \frac{\sum_{i=1}^{n} |I_{ni}| g(x_{ni}) h(x_{ni})}{\sum_{i=1}^{n} |I_{ni}| h(x_{ni})}$$

of points $y_{ni} = g(x_{ni})$ with coefficients $t_{ni} = |I_{ni}|h(x_{ni}) / \sum_{i=1}^{n} |I_{ni}|h(x_{ni})$. □

Inequalities for m-convex functions on the bounded interval of nonnegative real numbers were considered in [2]. Research on inequalities via s-convexity and log-convexity can be found in [1]. Usage of functionals in studying inequalities can be seen in [13].

4. Applications and generalizations

We begin the section by recalling the important Fejér inequality (see [3]) and the famous Hermite–Hadamard inequality (see [4, 5]). For this purpose, we use a convex function $f : [a, b] \to \mathbb{R}$ and a positive integrable function $h : [a, b] \to \mathbb{R}$ that is symmetric respecting the midpoint $(a + b)/2$ of the interval $[a, b]$. As a consequence of the convexity of f, and the symmetry of h, which can be represented by the equation $h(a+b-x) = h(x)$, the Fejér inequality follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)h(x)\,dx}{\int_a^b h(x)\,dx} \leq \frac{f(a)+f(b)}{2}, \tag{4.1}$$

and the substitution $h(x) = 1$ yields the Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{4.2}$$

Our aim is to establish the Fejér and Hermite–Hadamard inequality for m-convex functions.

Lemma 4.1 *Let $[a, b] \subset \mathbb{R}$ be an interval, and let $h : [a, b] \to \mathbb{R}$ be an integrable function that is symmetric respecting the midpoint of $[a, b]$.*

Then each affine function $u : \mathbb{R} \to \mathbb{R}$ satisfies the equality

$$\int_a^b u(x)h(x)\,dx = u\left(\frac{a+b}{2}\right)\int_a^b h(x)\,dx. \tag{4.3}$$

Proof Since the function h satisfies the equation $h(a+b-x) = h(x)$, the function $\overline{h} : [a, b] \to \mathbb{R}$ defined by

$$\overline{h}(x) = \left(x - \frac{a+b}{2}\right)h(x)$$

satisfies the equation $\overline{h}(a+b-x) = -\overline{h}(x)$. It in fact ensures that $\overline{h}$ is antisymmetric respecting the midpoint $(a+b)/2$, so we have

$$\int_a^b \overline{h}(x)\,dx = \int_a^b xh(x)\,dx - \frac{a+b}{2}\int_a^b h(x)\,dx = 0,$$

and consequently

$$\int_a^b xh(x)\,dx = \frac{a+b}{2}\int_a^b h(x)\,dx. \tag{4.4}$$

The equality in formula (4.3) can be derived by using the above equality and the affine equation of u taken as $u(x) = kx + l$ for some real constants k and l. □

Let $[a,b]$ be an interval containing the zero, let $c \in [a,b]$ be a point, and let $f : [a,b] \to \mathbb{R}$ be an m-convex function. Then applying the m-convexity of f, we can obtain the affine estimations

$$f(x) \le \frac{mf(c) - f(a)}{mc - a}x + m\frac{cf(a) - af(c)}{mc - a} \quad \text{for } x \in [a, mc] \tag{4.5}$$

and

$$f(x) \le \frac{f(b) - mf(c)}{b - mc}x + m\frac{bf(c) - cf(b)}{b - mc} \quad \text{for } x \in [mc, b]. \tag{4.6}$$

Suppose that a positive integrable function h is still symmetric respecting the midpoints of the intervals $[a, mc]$ and $[mc, b]$, that is, $h(a + mc - x) = h(x)$ for $x \in [a, mc]$ and $h(mc + b - x) = h(x)$ for $x \in [mc, b]$. Then multiplying the inequalities in formulae (4.5) and (4.6) with $h(x)$, and integrating by using formula (4.3) for the intervals $[a, mc]$ and $[mc, b]$, we obtain the following inequality, which also applies to $mc = a$ and $mc = b$.

Lemma 4.2 *Let $[a,b] \subset \mathbb{R}$ be an interval containing the zero, and let $c \in [a,b]$ be a point. Let $m \in (0,1]$ be a number. Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function that is symmetric respecting the midpoints of $[a, mc]$ and $[mc, b]$.*

Then each m-convex function $f : [a,b] \to \mathbb{R}$ satisfies the inequality

$$\int_a^b f(x)h(x)\,dx \le \frac{f(a)+mf(c)}{2}\int_a^{mc} h(x)\,dx + \frac{mf(c)+f(b)}{2}\int_{mc}^b h(x)\,dx. \tag{4.7}$$

To obtain the following version of the Fejér inequality for m-convex functions, we must combine Corollary 3.4 with the identity function $g(x) = x$, Lemma 4.1, and Lemma 4.2.

Theorem 4.1 *Let $[a,b] \subset \mathbb{R}$ be an interval containing the zero, and let $c \in [a,b]$ be a point. Let $m \in (0,1]$ be a number. Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function that is symmetric respecting the midpoints of $[a, mc]$ and $[mc, b]$.*

Then each continuous m-convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} \frac{1}{m}f\left(m\frac{(a+mc)\int_a^{mc}h(x)\,dx + (mc+b)\int_{mc}^b h(x)\,dx}{2\int_a^b h(x)\,dx}\right) &\le \frac{\int_a^b f(x)h(x)\,dx}{\int_a^b h(x)\,dx} \\ &\le \frac{\big(f(a)+mf(c)\big)\int_a^{mc} h(x)\,dx + \big(mf(c)+f(b)\big)\int_{mc}^b h(x)\,dx}{2\int_a^b h(x)\,dx}. \end{aligned} \tag{4.8}$$

Substituting $h(x) = 1$ and $c = a$ or $c = b$ in the above inequality, we obtain the Hermite–Hadamard inequality for m-convex functions as follows.

Corollary 4.1 *Let $[a,b] \subset \mathbb{R}$ be an interval containing the zero, and let $m \in (0,1]$ be a number.*

Then each continuous m-convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$\frac{1}{m}f\left(m\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{mb - a}{2(b-a)}f(a) + \frac{b - ma}{2(b-a)}f(b). \tag{4.9}$$

Putting $m = 1$, the inequalities in formulae (4.8) and (4.9) are reduced to the classical inequalities in formulae (4.1) and (4.2), respectively.

The Jensen inequality for m-convex functions in formula (3.12) can also be presented by using measurable sets.

Corollary 4.2 *Let X be a nonempty set, and let μ be a positive measure on X such that $\mu(X)$ is a positive number. Let $g : X \to \mathbb{R}$ be a μ-integrable function with the image in the interval $[a, b]$ containing the zero. Let $m \in (0, 1]$ be a number.*

Then each continuous m-convex function $f : [a, b] \to \mathbb{R}$ satisfies the inequality

$$f\left(\frac{m}{\mu(X)}\int_X g(x)\,d\mu(x)\right) \le \frac{m}{\mu(X)}\int_X f(g(x))\,d\mu(x). \tag{4.10}$$

Proof Let $I = [a, b]$, let n be a positive integer, and let $I = \cup_{i=1}^{n} I_{ni}$ be a partition of disjoint intervals I_{ni}, each of which contracts to the point as n approaches infinity. Putting $X_{ni} = g^{-1}(I_{ni})$, we have the partition $X = \cup_{i=1}^{n} X_{ni}$ of disjoint μ-measurable sets X_{ni}. Taking a point x_{ni} from each set X_{ni}, we form the convex combination

$$\sum_{i=1}^{n} \frac{\mu(X_{ni})}{\mu(X)} g(x_{ni}),$$

which belongs to the interval $[a, b]$. This combination can be inserted into the proof of Corollary 3.3. □

Inequalities for differentiable m-convex functions were considered in [9] and [8], particularly the Jensen inequality, the Slater inequality, and the Hermite–Hadamard inequality.

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