

Generalized trial equation method and its applications to Duffing and Poisson–Boltzmann equations

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Abstract: The trial equation method, which was proposed by Cheng-Shi Liu, is a very powerful method for solving nonlinear differential equations. After the original trial method, some modified versions of the trial equation method were introduced and applied to some famous nonlinear differential equations. Although each modified trial equation method provides a different perspective, they have some weaknesses according to the given differential equations. This is the main reason for introducing modified trial equation methods. This study aims to define a general representation of trial methods for solving nonlinear differential equations. The generalized trial equation method consists of the simple trial equation method, irrational trial method, and extended trial equation method as a common coverage. A suitable trial equation can also be structured according to the given nonlinear differential equations. To demonstrate the applicability of the generalized trial equation method, the solutions of the Duffing equation and Poisson–Boltzmann equation are examined and new solutions of these equations are obtained based on some nonlinear functions that have not been considered before within the trial equation methods.

Key words: Trial method, nonlinear differential equations, undamped Duffing equation, Poisson–Boltzmann equation

1. Introduction

The trial method was developed by Cheng-Shi Liu for the solution of nonlinear differential equations. This new method provides considerably different approaches for solutions of nonlinear differential equations. The idea of this method is very simple and it depends on a representation of the desired solution that satisfies a first-order constant coefficient differential equation. In [10], the simple trial method is defined as

$$u' = F(u) = \sum_{i=0}^m a_i u^i \quad (1)$$

for the solution of nonlinear differential equation

$$M(u, u', u'', \dots) = 0. \quad (2)$$

The author in [10] then showed that method (1) is not applicable for some nonlinear differential equations. Consequently, a modified trial equation method was introduced in [11] as

$$u = \sum_{i=0}^s d_i \varphi^i \quad (3)$$

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where φ' is a rational function of u and u is the solution of nonlinear differential equation (2). On the other hand, Du in his paper [5] indicated that trial equation methods (1) and (3) are not applicable for some nonlinear differential equations. He suggested an irrational trial equation method in [5] as

$$u' = \sum_{i=0}^{k_1} a_i u^i + \left(\sum_{i=0}^{k_2} b_i u^i \right) \sqrt{\sum_{i=0}^{k_3} c_i u^i}. \tag{4}$$

Finally, an extended trial method was introduced in [6] as as

$$u = \sum_{i=0}^{\delta} \tau_i \gamma^i \tag{5}$$

where $(\gamma')^2$ is a rational function of γ and u is the solution of nonlinear differential equation (2).

Each introduced trial method has been applied and many exact solutions of some important nonlinear equations were obtained. Therefore, selection of a proper trial equation method depends on the characteristics of the nonlinear differential equation in the area of study.

For example, the irrational trial equation method can be applicable for the sine-Gordon equation considered in [5]:

$$\alpha u'' + \beta u' = a \sin u + b \sin(2u).$$

On the other hand, the extended trial method can be more suitable for the equation of the fractional KdV,

$$\alpha u'' u + \beta (u')^2 = au^2 + bu^3, \tag{6}$$

given in [12]. Therefore, these works show that trial equations should be written according to the nonlinear differential equation in the study. Finally, papers [3] and [4] are referred to for important applications of trial equation methods to partial differential equations.

2. Generalized trial equation method

In this section, the generalized trial method will be defined and then the relation between it and other trial methods will be discussed. Consider the following nonlinear ordinary differential equation:

$$L(u, u', u'', \dots) = 0. \tag{7}$$

The *generalized trial equation* for the solution of nonlinear differential equation (7) can be defined as

$$F(u) = \frac{\alpha(u)}{\beta(u)} \tag{8}$$

where $F(u)$ is the solution of nonlinear equation (7) and

$$u' = D(u) = \frac{\gamma(u)}{\rho(u)}. \tag{9}$$

It is important to note that the functions $\alpha(u), \beta(u), \gamma(u)$, and $\rho(u)$ in the generalized trial equation can be nonlinear according to the given differential equation. Consequently, from (8),

$$F'(u) = \frac{\partial F}{\partial u} \cdot D(u).$$

Using (8) and (9), the second- and third-order derivatives of solution function (8) can be obtained as

$$\begin{aligned} F''(u) &= \frac{\partial^2 F}{\partial u^2} \cdot D^2(u) + \frac{\partial F}{\partial u} \cdot \frac{\partial D}{\partial u} \cdot D(u), \\ F'''(u) &= \frac{\partial^3 F}{\partial u^3} \cdot D^3(u) + 2 \frac{\partial^2 F}{\partial u^2} \cdot \frac{\partial D}{\partial u} \cdot D^2(u) + \frac{\partial^2 F}{\partial u^2} \cdot \frac{\partial D}{\partial u} \cdot D^2(u) \\ &\quad + \frac{\partial F}{\partial u} \left(\frac{\partial^2 D}{\partial u^2} \cdot D^2(u) + \left(\frac{\partial D}{\partial u} \right)^2 \cdot D(u) \right). \end{aligned}$$

By selection of appropriate functions $F(u), \alpha(u), \beta(u), \gamma(u)$, and $\rho(u)$, the introduced trial methods can be obtained as follows:

Case 1 If $F(u) = u'$, $\alpha(u) = \sum_{i=0}^{k_1} a_i u^i$, $\beta(u) = 1$, and $F(u) = D(u)$, then the generalized trial method returns to the simple trial method.

Case 2 If $F(u) = u$, $\alpha(u) = \sum_{i=0}^{k_1} a_i u^i$, $\beta(u) = 1$, $\gamma(u) = \frac{a_m u^m + \dots + a_0}{b_n u^n + \dots + b_0}$; $m, n \in \mathbb{N}$ and $\rho(u) = u'$, then the generalized trial method returns to the extended trial method.

Case 3 If $F(u) = u'$, $\alpha(u) = \sum_{i=0}^{k_1} a_i u^i + \left(\sum_{i=0}^{k_2} b_i u^i \right) \sqrt{\sum_{i=0}^{k_3} c_i u^i}$, $\beta(u) = 1$, and $F(u) = D(u)$, then the generalized trial method returns to the irrational trial method.

Unfortunately, there are many nonlinear differential equations that cannot be solved by simple trial method, extended trial method, or irrational trial method. For instance, the nonlinear differential equation

$$u''(x) - u'(x) = e^{u(x)} + e^{2u(x)} \tag{10}$$

cannot directly be solved by one of the introduced trial methods. By logarithmic transformation, equation (10) can be rewritten as

$$u''(x) - \frac{(u'(x))^2}{u(x)} = (u(x))^2 + (u(x))^3. \tag{11}$$

Equation (11) is almost the same as the equation of the fractional KdV given in [12].

The original nonlinear equation (10) can be easily solved via the generalized trial method. By using the generalized trial method, the necessary trial equations for nonlinear equation (10) can be written as

$$F(u) = a_1 \ln(u) + \ln a_2 \tag{12}$$

and

$$u' = D(u) = b_1 + b_2 u + b_3 u^2, \tag{13}$$

where a_1, a_2, b_1, b_2, b_3 are constants to be determined. When $F(u)$ and $D(u)$ are substituted into equation (10), the nonlinear system of equations

$$\begin{aligned} a_1 b_1^2 &= 0; a_1 b_1 u + a_1 b_1 b_2 u = 0, \\ a_1 b_1 u^2 &= 0; +a_1 b_3 u^3 - a_1 b_2 b_3 u^3 = 0, \\ -a_1 b_3^2 u^4 &= 0; a_2 u^{2+a_1} + a_2^2 u^{2+2a_1} = 0 \end{aligned}$$

will be obtained. Consequently, the unknown parameters a_1, a_2, b_1, b_2, b_3 should be found as

$$a_1 = 1, a_2 = a_2, b_1 = b_2 = 0 \text{ and } b_3 = -a_2.$$

Hence, the solution of nonlinear equation (10) is

$$F(u(x)) = u(x) = \ln\left[\frac{a_2}{a_2 x - c_1}\right]$$

where c_1 is the integral constant. It is necessary to mention that Mathematica does not give any output for the solution of equation (10). It is important to mention a study from paper [1, pp. 616] about the solutions of (2+1)-dimensional Burgers partial differential equations. In this study, the (2+1)-dimensional Burgers equation has been converted into the following ordinary differential equation:

$$(c^2 - 1) u''(x) = -e^{u(x)} - e^{-2u(x)}. \tag{14}$$

Consequently, transformation based on a natural logarithm was used before applying the generalized Kudryashov method (similar to a trial equation). From the derivation of the solution of equation (10), the proposed generalized trial equation method can directly be applied for differential equation (14) given in [1, pp. 616] with no requirement of using logarithmic transformation. Hence, the generalized trial equation method can alternatively be applicable in the process of the generalized Kudryashov method. For more details about this important method, see [2].

Moreover, the original equation (10) is also similar to the Poisson–Boltzmann type differential equation

$$u''(x) = -e^{u(x)} + e^{-u(x)} \tag{15}$$

discussed in [8, pp. 142]. The solutions of this Poisson–Boltzmann type differential equation will be discussed in the next section.

3. Applications

In this section, the new solutions of the Poisson–Boltzmann type differential equation and the Duffing type differential equation will be derived respectively via a generalized trial equation method.

3.1. Poisson–Boltzmann differential equation

The Poisson–Boltzmann equation (16) can be widely used as a modeling tool in a variety of fields. Poisson distribution is particularly used in predicting the failures of components in electrical power and communication systems. It enables a solution to detect the exact failed components within the system. The outcome of such

application reduces search time that may be costly in some applications. In this subsection, the convenient trial equations for the solution of differential equation (16) will be derived. In light of the solution of equation (10), the necessary nonlinear trial equations for the Poisson–Boltzmann type differential equation

$$u''(x) = -e^{u(x)} + e^{-u(x)} \tag{16}$$

can be derived as

$$F(u) = \ln\left(a_1 + a_2u^{\frac{1}{4}} + a_3u^{\frac{1}{2}}\right) \tag{17}$$

and

$$u' = D(u) = b_1u + b_2u^{\frac{5}{4}}, \tag{18}$$

where a_1, a_2, a_3, b_1, b_2 are constants to be determined. After substituting (17) and (18) into the nonlinear equation (16), the complete nonlinear trial equation satisfying nonlinear equation (16) is derived as

$$F(u) = \ln\left(-1 + a_2u^{\frac{1}{4}} - \left(\frac{a_2^2}{4}\right)u^{\frac{1}{2}}\right) \tag{19}$$

and

$$u' = D(u) = -\sqrt{2}a_2u^{\frac{5}{4}} + 4\sqrt{2}u. \tag{20}$$

Consequently, solutions of the Poisson–Boltzmann type differential equation (16) can be derived from the simple set of trial equations (19) and (20). First, the solutions of the trial equation (20) should be derived. By using Mathematica, the solutions of equation (20) have been derived as

$$u_1(x) = \frac{256e^{4\sqrt{2}x}}{\left(-\left(a_2e^{4\sqrt{2}x}\right) + e^{c_1}\right)^4}, \tag{21}$$

$$u_2(x) = \frac{256e^{4\sqrt{2}x}}{\left(a_2e^{4\sqrt{2}x} + e^{c_1}\right)^4}, \tag{22}$$

$$u_3(x) = \frac{\left(256e^{8\sqrt{2}x}\right)}{a_2^4e^{8\sqrt{2}x} + e^{4(\sqrt{2}x+c_1)} - 6a_2^2e^{6\sqrt{2}x+2c_1} + 4\sqrt{-\left(a_2^2e^{2(5\sqrt{2}x+c_1)}\left(-\left(a_2^2e^{2\sqrt{2}x}\right) + e^{2c_1}\right)^2\right)}, \tag{23}$$

and

$$u_4(x) = \frac{256\left(\frac{4}{2}e^{8\sqrt{2}x} + e^{4(\sqrt{2}x+c_1)} - 6a_2^2e^{6\sqrt{2}x+2c_1} + 4\sqrt{-\left(a_2^2e^{2(5\sqrt{2}x+c_1)}\left(-\left(a_2^2e^{2\sqrt{2}x}\right) + e^{2c_1}\right)^2\right)}\right)}{\left(a_2^2e^{2\sqrt{2}x} + e^{2c_1}\right)^4}. \tag{24}$$

Hence, the solutions of nonlinear equation (16) according to (21) – (24) with suitable coefficients can be derived as

$$F(u(x)) = u(x) = \ln\left(-1 + a_2(u_i(x))^{\frac{1}{4}} - \frac{a_2^2}{2}(u_i(x))^{\frac{1}{2}}\right), \forall i = 1, 2, 3, 4. \tag{25}$$

If we consider $u_1(x)$ with $a_2 = 1$ and $c_1 = 0$, for simplicity, we obtain the solution of nonlinear equation (16) as follows:

$$u(x) = \ln \left(-1 + \left(\frac{256e^{4\sqrt{2}x}}{\left(-\left(e^{4\sqrt{2}x}\right) + 1\right)^4} \right)^{\frac{1}{4}} - \frac{1}{2} \left(\frac{256e^{4\sqrt{2}x}}{\left(-\left(e^{4\sqrt{2}x}\right) + 1\right)^4} \right)^{\frac{1}{2}} \right). \tag{26}$$

It is important to mention that the solution (26) of equation (16) is satisfied by using Mathematica.

Remark 4 *The solution (26) according to the selections of coefficients involves functions of complex variables. The Poisson–Boltzmann type differential equation is particularly used in quantum mechanics. For interesting applications, see [7]. Since complex numbers play a central role in representing the wave function of a quantum system in quantum mechanics, solution (26) is meaningful. It is also important to note that the real valued solutions of equation (16) can be derived. All solutions obtained from trial equations (19) and (20) are natural logarithms of exponential functions.*

3.2. Undamped Duffing differential equation

The nonlinear differential equation

$$u''(x) - \alpha u(x)^3 - \beta u(x) = 0 \tag{27}$$

is called the undamped Duffing equation. The Duffing equation is one of the famous nonlinear equations commonly used in science and engineering. Therefore, researchers pay it remarkable attention due to the variety of its applications. For more details about the Duffing equation, see [9].

As shown in the previous section, a suitable nonlinear trial equation via the generalized trial equation will be considered for the undamped Duffing equation. In this subsection, a new solution of the undamped Duffing equation will be derived via the generalized trial equation method. By using the generalized trial method, the necessary trial equations for nonlinear equation (27) can be written as

$$F(u) = a_1 u^{\frac{1}{2}} + a_2 \tag{28}$$

and

$$u' = D(u) = b_1 u^{\frac{1}{2}} + b_2 u + b_3 u^{\frac{3}{2}} \tag{29}$$

where a_1, a_2, b_1, b_2, b_3 are constants to be determined. When $F(u)$ and $D(u)$ are substituted in equation (27), the nonlinear system of equation

$$\begin{aligned} -4\alpha a_2^3 + a_1 b_1 b_2 - 4a_2 \beta &= 0, (-12\alpha a_1^2 a_2 + 3a_1 b_2 b_3)u = 0, \\ (-12\alpha a_1 a_2^2 + a_1 b_2^2 + 2a_1 b_1 b_3 - 4a_1 \beta)u^{\frac{1}{2}} &= 0, (-4\alpha a_1^3 + 2a_1 b_3^2)u^{\frac{3}{2}} = 0 \end{aligned}$$

will be obtained. Then the unknown parameters a_1, a_2, b_1, b_2, b_3 can be considered as

$$a_2 = 0, a_1 = a_1, b_1 = -\frac{\sqrt{2}\beta}{\sqrt{\alpha a_1}}, b_2 = 0 \text{ and } b_3 = -\sqrt{2\alpha} a_1.$$

Hence, the solution of nonlinear equation (27) by means of coefficients is

$$F(u(x)) = u(x) = a_1 \sqrt{\frac{\beta \tan \left[\frac{1}{2} \sqrt{\beta} (\sqrt{2}x - \sqrt{\alpha a_1} c_1) \right]^2}{\alpha a_1^2}} \tag{30}$$

where c_1 is an integral constant. It is important to mention that solution (30) of equation (27) is satisfied by using Mathematica.

Alternatively, the unknown parameters a_1, a_2, b_1, b_2, b_3 can be considered as

$$a_1 = a_1, a_2 = a_2, b_1 = \frac{\sqrt{2}(\alpha a_2^2 + \beta)}{\sqrt{\alpha} a_1}, b_2 = 2\sqrt{2\alpha} a_2, \text{ and } b_3 = \sqrt{2\alpha} a_1.$$

Hence, the other solution of nonlinear equation (27) with suitable coefficients is

$$F(u(x)) = u(x) = a_2 + a_1 \sqrt{\frac{(\sqrt{\alpha} a_2 - \sqrt{\beta} \tan [\frac{1}{2} \sqrt{\beta} (\sqrt{2} x + \sqrt{\alpha} a_1 C_1)])^2}{\alpha a_1^2}} \tag{31}$$

where C_1 is an integral constant.

Remark 5 *The same class of trigonometric solutions for differential equation (27) has been introduced from the nonlinear trial equations (28) and (29).*

4. Conclusion

The generalized trial equation method is defined and used for the solutions of some differential equations. The generalized trial method is a very simple and powerful method for solutions of nonlinear ordinary differential equations. The appropriate trial equation via the generalized trial equation can easily be written for many nonlinear differential equations. Besides, this paper shows that nonlinear trial equations consisting of nonlinear functions should be considered for many nonlinear differential equations. Since some effective trial equations have been introduced in different studies, this paper also provides a general representation for trial equation methods. Finally, new solutions of the undamped Duffing equation and the Poisson–Boltzmann equation are obtained in this paper.

References

- [1] Baskonus HM, Bulut H. On some new analytical solutions for the (2+1)-dimensional Burgers equation and the special type of Dodd-Bullough-Mikhailov equation. *J Appl Anal Comput* 2015; 5: 613-625.
- [2] Baskonus M, Bulut H. New hyperbolic function solutions for some nonlinear partial differential equations arising in mathematical physics. *Entropy* 2015; 17: 4255-4270.
- [3] Bulut H. Classification of exact solutions for generalized form of K(m,n) equation. *Abst Appl Anal* 2013; 2013: 742643.
- [4] Bulut H, Pandir Y, Baskonus HM. On the exact solutions of high order wave equations of KdV type (I). *AIP Conference Proceedings* 2014; 1637: 114-126.
- [5] Du XH. An irrational trial equation method and its applications. *PRAMANA J Phys* 2010; 75: 415-422.
- [6] Gurefe Y, Misirli E, Sonmezoglu A, Ekici M. Extended trial equation method to generalized nonlinear partial differential equations. *Appl Math Comp* 2013; 219: 5253-5260.
- [7] Gogonea V, Merz KM. A quantum mechanical-Poisson–Boltzmann equation approach for studying charge flow between ions and a dielectric continuum. *J Chem Phys* 2000; 112: 3227-3235.
- [8] Jordan DW, Smith P. *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers*. 4th ed. Oxford, UK: Oxford University Press, 2007.

- [9] Kovacic I, Brennan MJ. [The Duffing Equation: Nonlinear Oscillators and Their Behaviour](#). New York, NY, USA: Wiley, 2011.
- [10] Liu C. Trial equation method to nonlinear evolution equations with rank inhomogeneous: mathematical discussions and its applications. [Commun Theor Phys](#) 2006; 45: 219-223.
- [11] Liu C. Trial equation method based on symmetry and applications to nonlinear equations arising in mathematical physics. [Found Phys](#) 2011; 41: 793-804.
- [12] Pandir Y, Gurefe Y, Misirli E. The extended trial equation method for some time fractional differential equations. [Disc Dynam Nature Society](#) 2013; 2013: 491359.