Factorization with respect to a divisor-closed multiplicative submonoid of a ring

Ashkan NIKSERESHT[1],[*], Abdulrasool AZIZI[2]
[1]Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran
[2]Department of Mathematics, College of Sciences, Shiraz University, Shiraz, Iran

Abstract: In this paper, we consider factorizations of elements of a divisor-closed multiplicative submonoid of a ring and also factorizations of elements of a module as a product of elements coming from a divisor-closed multiplicative submonoid of the ring and another element of the module. In particular, we study uniqueness and some other properties of such factorizations and investigate the behavior of these factorizations under direct sum and product of rings and modules.

Key words: Unique factorization, bounded factorization, primitive elements, présimplifiability, divisor-closed multiplicative submonoid

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Unless explicitly stated otherwise, we assume that all modules are nonzero. R denotes a ring and M denotes an R-module. Moreover, by $U(R)$, $J(R)$, and $N(R)$ we mean the set of units, Jacobson radical, and nilradical of R, respectively. Furthermore, $Z(N)$, where $N \subseteq M$, means the set of zero divisors of N, that is, $\{r \in R | \exists 0 \neq m \in N : \quad rm = 0\}$. Any other undefined notation is as in [5].

Factorization theory in commutative monoids has gained considerable attention in the last two decades, especially when the considered semigroup is the semigroup of regular elements of a commutative ring; see for example [7, 9, 10, 13–18, 21, 23]. In particular, a result of Facchini was the starting point for an entire new development in factorization theory of monoids. This result states that if $\mathcal{C}$ is a class of R-modules closed under finite direct sums, direct summands, and isomorphisms such that all modules in $\mathcal{C}$ have semilocal endomorphism ring, then the semigroup of isomorphism classes of modules in $\mathcal{C}$ (denoted by $\mathcal{V}(\mathcal{C})$) is a Krull monoid (see [11, Theorem 3.4]). This result could be applied to get interesting results on properties of direct sum decomposition of modules in $\mathcal{C}$, from factorization properties of elements of $\mathcal{V}(\mathcal{C})$ or vice versa; see for example [7, 8, 12, 13].

In [3, 4], Anderson and Valdes-Leon generalized the theory of factorization in integral domains to commutative rings with zero divisors and to modules as well. They called two elements of M, such as m and n, *associates*, denoted by $m \sim n$, when $Rm = Rn$. They also said that m and n are *strong associates*, denoted by $m \approx n$, when $m = un$ for some $u \in U(R)$. They defined m and n to be *very strong associates*, denoted by $m \cong n$, when they are associates and either both are zero or that from $m = rn$ for some $r \in R$, we can deduce $r \in U(R)$.

An element $m \in M$ is called *primitive* (resp. *strongly primitive, very strongly primitive*), when $m = rn$ for some $r \in R$, $n \in M$ implies $m \sim n$ (resp. $m \approx n$, $m \cong n$). A nonunit element $a \in R$ is called irreducible (resp. strongly irreducible, very strongly irreducible) if $a = bc$ for some $b, c \in R$, implies $a \sim b$ or $a \sim c$ (resp. $a \approx b$ or $a \approx c$, $a \cong b$ or $a \cong c$). Note that here by being associates in R, we mean being associates in R as an R-module. Using these concepts they introduced factorization properties such as unique factorization and bounded factorization in rings having zero divisors and in modules over such rings.

Here we investigate these factorization properties in rings and modules but with the restriction that the ring elements appearing in the factorization come from a divisor-closed multiplicative submonoid of the ring. Recall that a *divisor-closed multiplicative submonoid* (abbreviated as *DMS*) of R means a submonoid S of the multiplicative monoid of R, with the property that if $rr' \in S$, then both r and r' are in S (in some commutative algebra texts such an S is called a *saturated multiplicatively closed subset*). In what follows, we assume that S is a DMS of R. Thus $0 \in S$ if and only if $S = R$. If $S \neq R$, we say that S is proper. It is clear that $\mathrm{U}(R) \subseteq S$. If $S = \mathrm{U}(R)$, the concepts that are defined in this article will become trivial. Hence we assume that $S \neq \mathrm{U}(R)$, unless explicitly specified otherwise.

We will see that the concept of regular factorization in a ring (see [4, Section 5]), can be viewed as the special case of our work with $S = R \setminus \mathrm{Z}(R)$. Furthermore, these concepts will generalize the notion of regular bounded factorization modules introduced in [23, Section 3].

To grasp the idea behind this work, let us give an example. Take $\mathbb{Z}$ as a $\mathbb{Z}$-module and let S be the DMS generated by 2, that is, $S = \{\pm\, 2^k | k \in \mathbb{N} \cup \{0\}\}$. In the terminology we will define, the fact that every element of $\mathbb{Z}$ can be written uniquely as $2^n p$, where $n \in \mathbb{N} \cup \{0\}$ and p is an odd integer, will be stated as "$\mathbb{Z}$ is an S-UFM", that is, $\mathbb{Z}$ has unique factorization with respect to S (see Example 2.12). In fact, this is the generic meaning of having unique factorization with respect to a DMS.

In Section 2 of this paper, we state the definitions of the main ideas of this article and give various examples. In Section 3, we study some basic properties of these concepts and finally, in Section 4, we investigate how these notions behave under direct sum and product of rings and modules.

2. Basic concepts

Recall that in this paper, S is a DMS of R and so $\mathrm{U}(R) \subseteq S$. We also assume that $S \neq \mathrm{U}(R)$, unless explicitly stated otherwise.

Definition 2.1 *We say that two elements, m and n of M, are S-associates and write $m \sim^S n$ if there exists $s, s' \in S$, such that $m = sn$ and $n = s'm$. They are called S-strong associates if $m = un$ for some $u \in \mathrm{U}(R) \cap S$ and we denote it by $m \approx^S n$. We call them S-very strong associates, denoted by $m \cong^S n$, when $m \sim^S n$ and either $m = n = 0$ or that $m = sn$ for some $s \in S$ implies $s \in \mathrm{U}(R)$.*

Note that $\mathrm{U}(R) \cap S = \mathrm{U}(R)$ and hence $\approx^S$ in fact does not depend on S. In this definition and other definitions that we give throughout this paper, in the case $S = R$, we drop the S and say primitive element, associates, etc. One can easily verify that this notation is compatible with the definitions in the introduction.

Example 2.2 *Set $M = \mathbb{Z}_{2^\infty}$, $R = \mathbb{Z}$, and $S = \{\pm\, 17^k | k \in \mathbb{N} \cup \{0\}\}$. Then $\frac{1}{16} + \mathbb{Z} \sim \frac{3}{16} + \mathbb{Z}$, but they are not S-associates. Moreover, if $S' = \{\pm\, 2^k | k \in \mathbb{N} \cup \{0\}\}$, then although $\frac{1}{2} + \mathbb{Z} \cong^{S'} \frac{1}{2} + \mathbb{Z}$, they are not very strong associates.*

Definition 2.3 *An R-module M is called S-présimplifiable, when from $sm = m$ for an $s \in S$, $0 \neq m \in M$, we can deduce $s \in \mathrm{U}(R)$. Furthermore, we say R is présimplifiable in S, when $rs = s$ for some $0 \neq s \in S$, $r \in R$ implies $r \in \mathrm{U}(R)$.*

To give an example, we need the following lemma. Here we say that a set $A \subseteq M$ is *finite up to units*, when there is a finite subset $B \subseteq A$ such that for each $a \in A$ there are $b \in B$ and $u \in \mathrm{U}(R)$ such that $a = ub$, it means, $A \subseteq \bigcup_{b \in B}(\mathrm{U}(R)b)$.

Lemma 2.4 *Suppose that for each $s \in S \setminus \mathrm{U}(R)$ there exists $k \in \mathbb{N}$ such that $s^k M$ is finite up to units. Then M is S-présimplifiable if and only if $(S \setminus \mathrm{U}(R)) \subseteq \sqrt{\mathrm{Ann}(M)}$.*

Proof ($\Leftarrow$) If $m = sm$ for an $s \in S \setminus \mathrm{U}(R)$ and $m \in M$, then $s^n M = 0$ for some $n \in \mathbb{N}$ and hence $m = s^n m = 0$, as required (note that the finiteness condition of the statement is not used in the proof of this side).

($\Rightarrow$) Let $s \in S \setminus \mathrm{U}(R)$ and $k \in \mathbb{N}$ be such that $s^k M$ is finite up to units. Then for each $m \in M$ and $k' > k$, $s^{k'} m \in s^k M$ and hence the set $\{m, sm, s^2 m, \ldots\}$ is finite up to units. Therefore, $s^{k_1} m = u s^{k_2} m$ for some $u \in \mathrm{U}(R)$ and $k_1 < k_2 \in \mathbb{N}$. Thus $(u s^{k_2 - k_1})(s^{k_1} m) = (s^{k_1} m)$ and since $u s^{k_2 - k_1} \in S \setminus \mathrm{U}(R)$ and M is S-présimplifiable, we conclude that $s^{k_1} m = 0$. In particular, for each $x \in s^k M$ there is a $n \in \mathbb{N}$ with $s^n x = 0$. However, $s^k M$ is finite up to units and so we can find a $n \in \mathbb{N}$ with $s^n s^k M = 0$, that is $s \in \sqrt{\mathrm{Ann}(M)}$. $\square$

As a simple example, consider $M = (\oplus_{i=1}^{\infty} \mathbb{Z}_8) \oplus \mathbb{Z}_9$ as a $\mathbb{Z}$-module and $S = \{\pm 2^k | k \in \mathbb{N} \cup \{0\}\}$. Then clearly M and S satisfy the condition of the previous lemma and it follows that M is not S-présimplifiable. Note that in this example M itself is not finite up to units.

Example 2.5 *Let R be a unique factorization domain (UFD), $0 \neq x \in R \setminus \mathrm{U}(R)$, $M = \frac{R}{Rx}$ and assume that M is finite up to units (say $R = \mathbb{Z}$ and $M = \mathbb{Z}_n$, $n \in \mathbb{N}$). Then M is S-présimplifiable if and only if $x \approx p^\alpha$ for some $\alpha \in \mathbb{N}$, a prime element $p \in R$ and $S = \{u p^k | k \in \mathbb{N} \cup \{0\}, u \in \mathrm{U}(R)\}$.*

Proof ($\Leftarrow$) Follows from the above lemma.

($\Rightarrow$) Let $x = v p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ be the prime decomposition of x with $v \in \mathrm{U}(R)$ and $s \in S \setminus \mathrm{U}(R)$. By 2.4, $s \in \sqrt{Rx} = R p_1 p_2 \cdots p_t$. Assume that $s = r p_1 p_2 \cdots p_t$ for some $r \in R$. By S being divisor-closed, $p_1 \in S$ and since p_1 is nonunit, again we have $p_1 \in R p_1 \cdots p_t$. Thus $t = 1$ and $x \approx p_1^{k_1}$. Moreover, every prime divisor of r is in S and hence in $R p_1$, that is, r and hence s are strongly associated to powers of p_1, as required. $\square$

It should be mentioned that in the above example the finiteness condition on M is necessary, as the next example shows.

Example 2.6 *Let $R = D[X]$, where D is a UFD and $M = \frac{R}{RX^2}$. Then $\mathrm{U}(R) = \mathrm{U}(D)$ and the elements of the infinite set $\{x, x+1, x+2, \ldots\}$, where x denotes the image of X in M, are not unit multiples of each other. Consequently, M is not finite up to units. Note that if D is a field, then R is a principal ideal domain. Moreover, if $D = \mathbb{Z}$ and $S = \{\pm 2^\alpha X^\beta | \alpha, \beta \in \mathbb{N} \cup \{0\}\}$, then it is easy to check that M is S-présimplifiable.*

In the sequel, we denote the closure of a multiplicative submonoid T of R by $\widetilde{T}$, which means the smallest DMS of R containing T.

Theorem 2.7 *Assume that S and S' where $S \subseteq S'$ are two DMS's of R.*

(i) *If $m, n \in M$ we have: $m \cong^{S'} n \Rightarrow m \cong^S n \Rightarrow m \approx^{S'} n \Leftrightarrow m \approx^S n \Rightarrow m \sim^S n \Rightarrow m \sim^{S'} n$.*

(ii) *An R-module M is S-présimplifiable if and only if the relations $\sim^S$, $\approx^S$, and $\cong^S$ coincide if and only if $\cong^S$ is reflexive.*

(iii) *Let $S = \widetilde{1 + \mathfrak{I}}$, where $\mathfrak{I}$ is an ideal of R. Then M is S-présimplifiable if and only if $\mathrm{Z}(M) \cap \mathfrak{I} \subseteq \mathrm{J}(R)$.*

(iv) *A ring R is présimplifiable in S if and only if $\sim$, $\approx$ and $\cong$ coincide on S if and only if $\cong$ is reflexive on S if and only if $\mathrm{Z}(S) \subseteq \mathrm{J}(R)$.*

(v) *If M is S-présimplifiable and $s \in S \setminus \mathrm{Ann}(M)$, then all kinds of associativity and irreducibility are equivalent for s.*

Proof   (i) and (ii) are easy to prove.

(iii) First suppose that M is S-présimplifiable and $x \in \mathrm{Z}(M) \cap \mathfrak{I}$. Thus there is an $0 \neq m \in M$ such that $xm = 0$. Now for each $r \in R$ we have $(1 + rx)m = m$ and also $1 + rx \in 1 + \mathfrak{I} \subseteq S$. Thus $1 + rx \in \mathrm{U}(R)$ for all $r \in R$ and hence $x \in \mathrm{J}(R)$.

Now suppose that $\mathrm{Z}(M) \cap \mathfrak{I} \subseteq \mathrm{J}(R)$ and $sm = m$ for some $s \in S$, $0 \neq m \in M$. We must show that $s \in \mathrm{U}(R)$. It suffices to show that for each maximal ideal $\mathfrak{M}$ of R, $s \notin \mathfrak{M}$. Note that $S = R \setminus \bigcup \mathfrak{P}$, where the union is taken over prime ideals $\mathfrak{P}$ of R containing $\mathfrak{I}$ (see [5, p. 44, Exercise 7]). Thus if $\mathfrak{I} \subseteq \mathfrak{M}$, then $s \notin \mathfrak{M}$ as required.

Therefore, suppose that $i \in \mathfrak{I} \setminus \mathfrak{M}$. Now $i(1 - s)m = 0$ and hence $i(1 - s) \in \mathrm{Z}(M) \cap \mathfrak{I} \subseteq \mathrm{J}(R) \subseteq \mathfrak{M}$. However, since $i \notin \mathfrak{M}$, we must have $1 - s \in \mathfrak{M}$ and thus $s \notin \mathfrak{M}$. This completes the proof of (iii).

(iv) If $\mathrm{Z}(S) \subseteq \mathrm{J}(R)$ and $rs = s$ for some $0 \neq s \in S$ and $r \in R$, then $(1 - r)s = 0$ and hence $1 - r \in \mathrm{Z}(S) \subseteq \mathrm{J}(R)$. Thus $r = 1 - (1 - r) \in \mathrm{U}(R)$. The converse is similar and the proof of other parts of (iv) is easy.

(v) Since $s \notin \mathrm{Ann}(M)$, there is an $m \in M$ such that $sm \neq 0$. Suppose that $s = rs$ for some $r \in R$. By S being divisor-closed, $r \in S$. Now $sm = r(sm)$ and since M is S-présimplifiable, $r \in \mathrm{U}(R)$. This shows that $s \cong s$. Now the result follows from [3, Theorem 2.2(2)].   □

Note that the converses of implications in 2.7 (i), are not true as shown in Example 2.2 and [3, Example 2.3 and p. 445].

Corollary 2.8 *The following are equivalent for an R-module M.*

(i) *M is S-présimplifiable for every proper DMS S of R.*

(ii) *M is S-présimplifiable for every DMS S of R of the form $\widetilde{1 + Rx}$ for a nonunit $x \in R$.*

(iii) *M is présimplifiable.*

In particular, every R-module is S-présimplifiable for every proper DMS S of R if and only if R is quasi-local.

Proof   It is clear that (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) If $x \in Z(M)$, then since M is $\widetilde{1+Rx}$-présimplifiable and by Theorem 2.7(iii), $x \in Z(M) \cap Rx \subseteq J(R)$. Thus $Z(M) \subseteq J(R)$ and again by Theorem 2.7(iii) with $\mathfrak{I} = R$, M is présimplifiable. The final assertion follows from [4, p. 203]. $\qquad\qquad\square$

Definition 2.9 *An $m \in M$ is called S-primitive (resp. S-strongly primitive, S-very strongly primitive), when $m = sn$ for some $s \in S$, $n \in M$ implies $n \sim^S m$ (resp. $n \approx^S m$, $n \cong^S m$).*

Example 2.10   *(i) If we take $\mathbb{Z}$ as a $\mathbb{Z}$-module, then its primitive elements are $\{1, -1\}$ but its nonzero S'-primitive elements, where S' is as in Example 2.2, are the elements of $\mathbb{Z}$ coprime to 2.*

  (ii) Suppose that R is a UFD, $x \in R$, $M = \frac{R}{Rx}$ and for each $s \in S \setminus U(R)$, $\mathrm{GCD}(s, x) \neq U(R)$, where $\mathrm{GCD}(s, x)$ is the set of greatest common divisors of s and x. Then $0 \neq \bar{a} \in M$ is S-very strongly primitive if and only if $\mathrm{GCD}(a, p) = U(R)$ for all prime factors p of x with $p \in S$, where a is a preimage of $\bar{a}$ in R.

  (iii) The $\mathbb{Z}$-module $\mathbb{Z}_n$ has no S-very strongly primitive element, if S contains a nonunit number coprime to n.

  (iv) Let $n \in \mathbb{N}$ and $0 \neq a \in \mathbb{Z}$ be such that $(a, p) = 1$ for all prime factors p of n with $p \in S$. Then the image $\bar{a}$ of a in the $\mathbb{Z}$-module $\mathbb{Z}_n$ is S-primitive.

Proof   (i) Clear.

  (ii) Let $\mathcal{P}$ be the set of all prime factors of x and $\mathcal{P}' = \mathcal{P} \cap S$. Let $0 \neq a \in R$ such that $\mathrm{GCD}(a, p) = U(R)$ for all $p \in \mathcal{P}'$ and suppose that $\bar{a} = s\bar{b}$ for some $s \in S$, $b \in R$. Thus for some $r \in R$, we have $a = sb + rx$. If s is a nonunit, then by assumption $\mathrm{GCD}(s, x) \neq U(R)$ and hence there is a $p \in \mathcal{P}$ such that $p|s$. However, since S is divisor-closed, $p \in \mathcal{P}'$. Now $p|x$ and $p|s$, and so $p|sb + rx = a$, which is a contradiction. Thus s must be a unit and therefore $\bar{a}$ is S-very strongly primitive.

  Conversely, suppose that $\bar{a}$ is S-very strongly primitive. If $\mathrm{GCD}(a, p) \neq U(R)$ for some $p \in \mathcal{P}'$, then $\bar{a} = p\bar{b}$, where $b = \frac{a}{p}$ and $p \notin U(R)$, in contradiction with S-very strongly primitivity of $\bar{a}$. Therefore, $\mathrm{GCD}(a, p) = U(R)$ for all $p \in \mathcal{P}'$.

  (iii) Suppose that there is an $s \in S \setminus U(R)$ with $(s, n) = 1$. Thus for some $k \in \mathbb{N}$, we have $s^k \equiv 1 \pmod{n}$. Therefore, for every $\bar{m} \in \mathbb{Z}_n$, $\bar{m} = s^k \bar{m}$ and hence $\mathbb{Z}_n$ has no S-very strongly primitive element.

  (iv) Note that if $\mathcal{P}'$, b, and s are as in (ii) with x replaced by n, then by a similar argument, we have $(s, n) = 1$ and hence for some $k \in \mathbb{N}$, $s^k \equiv 1 \pmod{n}$. Now $\bar{a} = s\bar{b}$, $\bar{b} = s^{k-1}\bar{a}$ and so $\bar{a} \sim^S \bar{b}$. Thus $\bar{a}$ is S-primitive. $\qquad\qquad\square$

  From now on, we set $A = \{\text{irreducible, strongly irreducible, very strongly irreducible}\}$ and $B = \{\text{primitive, strongly primitive, very strongly primitive}\}$.

Definition 2.11 *By an S-factorization of $m \in M$ with length k, we mean an equation $m = s_1 \cdots s_k n$ where s_i's are nonunits in S, $k \in \mathbb{N} \cup \{0\}$, and $n \in M$. If, moreover, $\alpha \in A, \beta \in B$ and s_i's are α and n is S-β, we call this an (α, β)-S-factorization. If every nonzero element of M has an (α, β)-S-factorization, we say that M is (α, β)-S-atomic.*

We could similarly define the concepts of "S-associativity" and "S-irreducibility" for elements of R and use them, instead of those defined in the introduction, in our work. However, if we take R as an R-module and $s, s' \in S$, then S-associativity for s and s' turn to be equivalent with the usual associativity, because we have assumed S to be divisor-closed. Consequently, since here we focus on elements of S, we use the usual notions of associativity and irreducibility for ring elements.

Example 2.12 *(i) If we take $\mathbb{Z}$ as a $\mathbb{Z}$-module and S' is as in Example 2.2, then an (irreducible, primitive)-S'-factorization of $0 \neq k \in \mathbb{Z}$ means an equation $k = \pm\, 2^n k'$, where $n \in \mathbb{N} \cup \{0\}$ and k' is an odd integer. It is obvious that this kind of factorization is unique up to the signs of k' and 2^n.*

(ii) It follows easily from (ii) and (iii) of Example 2.10 that the $\mathbb{Z}$-module $\mathbb{Z}_n$ is (irreducible, very strongly primitive)-S-atomic if and only if S does not contain any nonunit number coprime to n. However, we can deduce from (iv) of Example 2.10 that $\mathbb{Z}_n$ is an (irreducible, primitive)-S-atomic $\mathbb{Z}$-module, for every DMS, S of $\mathbb{Z}$.

By an S-*atomic factorization* we mean an (irreducible, primitive)-S-factorization and by an S-*atomic module* we mean a module that is (irreducible, primitive)-S-atomic. Moreover, we say two S-atomic factorizations $m = s_1 \cdots s_k n = t_1 \cdots t_{k'} n'$ are *isomorphic*, if $k = k'$, $n \sim^S n'$ and for a permutation σ of $\{1, \ldots, k\}$, we have $s_i \sim t_{\sigma(i)}$ for all $1 \leq i \leq k$. (One can use '$\cong$' or '$\approx$' instead of '$\sim$' to get different isomorphisms of S-factorizations, but as we will see in the next section, here we mainly focus on S-présimplifiable modules and so we just work with the weakest form of isomorphism.)

Definition 2.13 *We call a module M an S-unique factorization module (S-UFM) when every nonzero element of M has exactly one S-atomic factorization up to isomorphism. Furthermore, we say that M is an S-bounded factorization module (S-BFM) if for every $0 \neq m \in M$ there is an $N_m \in \mathbb{N}$ such that the length of every S-factorization of m is at most N_m.*

We say that R is *atomic in S* when every nonzero nonunit element of S has a factorization into irreducible elements of R and if these factorizations are unique up to order and associates, we say that R *has unique factorization (UF) inside S*. Similarly we say that R *has BF in S* if the lengths of factorizations of every nonzero nonunit element of S are bounded above.

One can easily check that if $S = R \setminus Z(R)$, then R is atomic (resp. has UF, has BF) in S if and only if in the terminology of [4, Section 5], R is r-atomic (resp. factorial, an r-BFR). Moreover, if $S = R \setminus Z(M)$ and M is an S-BFM, then in [23, Section 3] we have called M an r-BFM and have presented a necessary and sufficient condition for $M[x]$ to be an r-BFM over $R[x]$.

It is easy to see that if R has UF inside S, then it has BF inside S. Furthermore, if R has BF inside S and for some $s \in S$, $r \in R$ we have $s = rs$, then $s = r^2 s = r^3 s = \cdots$ and hence either $s = 0$ or $r \in U(R)$. By a similar argument one can easily see that every S-BFM is S-présimplifiable, but the following example shows that there are S-UFMs that are not S-BFMs, even in the case $S = R$. The idea of this example is stated in [4, p.206].

In this example, we use the concept of semigroup rings. If R is a ring and T is a semigroup, then the semigroup ring of T over R denoted by $R[x; T]$ is the set of formal sums of the form $\sum_{t \in F} r_t x^t$, where F is a finite subset of T and $r_t \in R$, which equipped with an addition and a product similar to that of polynomials

forms a ring. Of course, for this ring to be commutative with identity, we in fact assume that T is a commutative monoid. A nice text on semigroup rings is [19].

Example 2.14 *Let $\mathbb{Q}^{\geq 0}$ be the additive semigroup of nonnegative rational numbers and $D = \mathbb{C}\left[x; \mathbb{Q}^{\geq 0}\right]$. Set $\mathfrak{M} = \{f \in D | f(0) = 0\}$ and $M = \frac{D}{\mathfrak{M}}$. Then M is a UFM that is not présimplifiable.*

Proof   As noted (without proof) in [1, p. 3], D has no irreducible elements. To see this, let $f(x) = a_0 + \sum_{i=1}^{n} a_i x^{\frac{p_i}{q}}$ be a nonunit element of D with a_i's in $\mathbb{C}$ and p_i's and q and n in $\mathbb{N}$. Set $y = x^{\frac{1}{2q}}$. Then $f(x) = a_0 + \sum_{i=1}^{n} a_i y^{2p_i}$ is a polynomial in y with complex coefficients and degree at least 2. Consequently, there are nonunit polynomials $f_1, f_2 \in \mathbb{C}[y]$ such that $f(x) = f_1(y)f_2(y) = f_1\left(x^{\frac{1}{2q}}\right) f_2\left(x^{\frac{1}{2q}}\right)$, whence $f(x)$ is not irreducible in D.

Now it is easy to check that $\mathfrak{M}$ is a non-unique maximal ideal of D. Thus M is a simple D-module and hence every nonzero element of M is primitive. Therefore, the only S-atomic factorization of $0 \neq m \in M$ is $m = m$ and M is a UFM. On the other hand, since $\mathrm{Z}(M) = \mathfrak{M} \not\subseteq \mathrm{J}(D)$, by 2.7(iii) with $\mathfrak{I} = D$, we see that M is not présimplifiable. □

The module in the previous example has the property that each of its nonzero elements is primitive. We end this section by characterizing modules with this property.

Theorem 2.15 *Every nonzero element of M is primitive if and only if $M \cong \bigoplus_{a \in A} \frac{R}{\mathfrak{M}}$, for some maximal ideal $\mathfrak{M}$ of R and an index set A if and only if $\mathrm{Ann}(M)$ is a maximal ideal of R.*

Proof The last two conditions are clearly equivalent. Suppose that $M = \bigoplus_{a \in A} \frac{R}{\mathfrak{M}}$. If for some $0 \neq m, m' \in M$ we have $m = rm'$; then since $\frac{R}{\mathfrak{M}}$ is a field and $r \notin \mathrm{Ann}(M) = \mathfrak{M}$, there is an $r' \in R$ such that $rr' + \mathfrak{M} = 1 + \mathfrak{M}$. Now $r'm = r'rm' = m'$, that is, $m \sim m'$. Hence every nonzero element of M is primitive.

Conversely, suppose that every element of M is primitive. Assume that $a \in A$ and $0 \neq a' \in Ra$. Then by primitivity of a' we see that $Ra' = Ra$. Consequently, Ra is a simple module and for some maximal ideal $\mathfrak{M}_a$ of R, $Ra \cong \frac{R}{\mathfrak{M}_a}$.

Assume that there exist $a, b \in A$ such that $\mathfrak{M}_a \neq \mathfrak{M}_b$. If $Ra \cap Rb \neq 0$, since Ra and Rb are simple, we must have $Ra = Rb$ and $\mathfrak{M}_a = \mathfrak{M}_b$, a contradiction. Therefore, $Ra + Rb = Ra \oplus Rb \cong \frac{R}{\mathfrak{M}_a} \oplus \frac{R}{\mathfrak{M}_b} \cong \frac{R}{\mathfrak{M}_a \cap \mathfrak{M}_b}$ is a cyclic submodule of M, say $Ra \oplus Rb = Rc$. Now we have $a \in Rc$ and $Ra \neq Rc$, which is in contradiction with the primitivity of a. Thus there is a maximal ideal $\mathfrak{M}$ of R such that for all $a \in A$, $Ra \cong \frac{R}{\mathfrak{M}}$ and hence $\mathrm{Ann}(M) = \mathfrak{M}$. □

3. Basic properties of S-factorizations

In this section, we state and prove some basic results on the properties of S-factorizations in modules.

Theorem 3.1   *(i) An S-UFM is an S-BFM if and only if it is S-présimplifiable. In particular, if R is atomic in S, then every S-UFM is an S-BFM.*

(ii) A module M (resp. a ring R) is an S-BFM (resp. has BF in S) if and only if it is S-atomic (resp. is atomic in S) and the length of S-atomic factorizations of each of its nonzero elements (resp. atomic factorizations of nonzero nonunit elements of S) is bounded.

Proof (i) As noted in the previous section, every S-BFM is S-présimplifiable, whence one side of the assertion is obvious. Suppose that M is an S-présimplifiable S-UFM and $m = s_1 s_2 \cdots s_k m'$ is an S-factorization of $0 \neq m \in M$. Also assume that N is the length of the unique S-atomic factorization of m.

Let $s_1 m' = s_1' s_2' \cdots s_{t_1}' m_1'$ be the S-atomic factorization of $s_1 m'$. If $t_1 = 0$, then since m_1' is S-primitive and M is S-présimplifiable, s_1 must be a unit, which is a contradiction. Thus $t_1 \geq 1$. Now by writing down an S-atomic factorization for $s_2 m_1'$ and continuing this way, we get an S-atomic factorization of m with length $t_1 + t_2 + \cdots + t_k$ (where t_i is the length of S-atomic factorization of $s_i m_{i-1}'$). A similar argument shows that each t_i is at least 1 and hence $k \leq t_1 + t_2 + \cdots + t_k$, which by the uniqueness of S-atomic factorizations must equal N. Therefore, M is an S-BFM.

If R is atomic in S and M is an S-UFM and for some nonunit $s \in S$ and nonzero $m \in M$, we have $sm = m$, then by writing down the S-atomic factorization of m and an atomic factorization for s, we get two different S-atomic factorizations for m, a contradiction. Hence M is S-présimplifiable and, by the above argument, an S-BFM.

(ii) Suppose that M is an S-BFM. If $m = s_1 s_2 \cdots s_n m'$ is an S-factorization of $0 \neq m \in M$ with the maximum length, then every s_i must be irreducible and m' must be S-primitive, else we get a longer S-factorization. Thus M is S-atomic.

For the converse note that by an argument similar to the proof of (i), from each S-factorization of m, we can get an S-atomic factorization of m without reducing its length, from which the result follows. The ring case is similar and even easier. □

Example 3.2 *The $\mathbb{Z}$-module $\mathbb{Z}_n$ is an S-UFM if and only if either $n = p$ is prime and $S = \{\pm p^k | k \in \mathbb{N} \cup \{0\}\}$ or $n = 4$ and $S = \{\pm 2^k | k \in \mathbb{N} \cup \{0\}\}$.*

Proof One can readily verify that in these cases $M = \mathbb{Z}_n$ is an S-UFM. Conversely, if M is an S-UFM, then by 3.1, since $\mathbb{Z}$ is an atomic ring, M is S-présimplifiable and by 2.5, $n = p^\alpha$ for a prime number p and $S = \{\pm p^k | k \in \mathbb{N} \cup \{0\}\}$. However, we have $p^{\alpha-1}\bar{1} = p^{\alpha-1}(\bar{1} + \bar{p})$. Thus by the uniqueness of S-factorizations, we get $\bar{1} \cong \bar{1} + \bar{p}$ or equivalently $\pm 1 = 1 + p \pmod{n}$. Hence either $p = 0 \pmod{n}$, that is, $\alpha = 1$ and n is prime, or $p = -2 \pmod{n}$, which means $n = 4$. □

Proposition 3.3 *The following are equivalent for a ring R.*

(i) *Every R-module is an S-UFM for every proper DMS, S of R.*

(ii) *Every R-module is an S-BFM for every proper DMS, S of R.*

(iii) *R is a zero dimensional quasi-local ring.*

Proof (i) $\Rightarrow$ (ii) Let S be a proper DMS of R. Then R is an S-UFM as an R-module. Thus each $s \in S$ has an S-atomic factorization such as $s = s_1 s_2 \cdots s_n r$, where each s_i is irreducible and r is primitive. However, $r \in S$ and it is easy to see that every S-primitive element of R that is in S is a unit. Therefore, R is atomic inside S and by Theorem 3.1 every S-UFM is S-BFM, as required.

(ii) $\Rightarrow$ (iii) Suppose that (iii) is not true. Then there exists an $r \in R \setminus (\mathrm{N}(R) \cup \mathrm{U}(R))$. Consider $M = \frac{\prod_{i \in \mathbb{N}} R}{\bigoplus_{i \in \mathbb{N}} R}$ and set S to be the closure of $\{r^k | k \in \mathbb{N} \cup \{0\}\}$. Then we have $(1, r, r^2, \ldots) + \oplus R =$

$r(0, 1, r, r^2, \ldots) + \oplus R = r^2(0, 0, 1, r, \ldots) + \oplus R = \cdots$, and M is not an S-BFM. However, S does not contain 0 and hence is a proper DMS, which is in contradiction with (ii), and the result follows.

(iii) $\Rightarrow$ (i) Just note that the only proper DMS of R is $\mathrm{U}(R)$. $\hfill \square$

Example 3.4 *Let K be a field, $D = K\left[\{x_i\}_{i \in \mathbb{N}}, y\right]$, $\mathfrak{I} = Dy^2 + \sum_{i \in \mathbb{N}} Dx_i^{i+1} + \sum_{i \in \mathbb{N}} Dy(1 - x_i^i)$ and $R = \frac{D}{\mathfrak{I}}$. Then it is easy to see that R has exactly one prime ideal and hence by the above result every R module is a UFM with respect to every proper DMS of R. However, $\bar{y} = \bar{x_i}^i \bar{y}$ for all $i \in \mathbb{N}$. Consequently, R is not a BFR, that is, there are R-modules that are not S-BFM when $S = R$.*

In the sequel, we focus on S-présimplifiable modules, so that we can use Theorems 3.1 and 2.7.

Notation 3.5 *In the rest of this paper, when we talk about an S-UFM such as M, we assume that M is S-présimplifiabletoo.*

Proposition 3.6 *Let M be an R-module.*

(i) If $\mathrm{Ann}(M) \cap S \subseteq \{0\}$ and M is S-présimplifiable (resp. an S-BFM, an S-UFM), then inside S, R is présimplifiable (resp. has BF, has UF).

(ii) If M is faithful and S-présimplifiable (resp an S-BFM), then R as an R-module is S-présimplifiable (resp. an S-BFM).

Proof (i) Suppose that $0 \neq s \in S$ and $m \in M$ is such that $sm \neq 0$. If $s = rs$ for some $r \in R$, then because S is divisor-closed $r \in S$ and also $(sm) = r(sm)$, whence if M is S-présimplifiable, then $r \in \mathrm{U}(R)$. Similarly every factorization of s leads to an S-factorization of sm with the same length. Thus if M is an S-BFM, then R has BF in S.

Moreover, if M is an S-UFM, then it is an S-BFM (note that we are using Notation 3.5). Thus R has BF and hence is atomic inside S. Now by writing down an atomic factorization of s and appending it with an S-atomic factorization of m, we get the unique S-atomic factorization of sm, whence up to order and associates there exists exactly one atomic factorization for s.

(ii) Again if $0 \neq r \in R$ and $m \in M$ with $rm \neq 0$, then an S-factorization of r, when we take R as an R-module, leads to an S-factorization of rm with at least the same length. $\hfill \square$

Remark 3.7 *Indeed, the proof of Proposition 3.6 shows that if $s \in S \setminus \mathrm{Ann}(M)$ is a nonunit and M is an S-UFM (S-BFM), then s has a unique factorization into irreducibles (s has BF).*

If R as an R-module is S-présimplifiable (resp. an S-BFM, an S-UFM), then according to Proposition 3.6(i) R is présimplifiable (resp. has BF, has UF) inside S. However, Example 4.14 shows that the converse is not true.

Theorem 3.8 *Let $S \subseteq S'$ be two DMS's of R.*

(i) If M is S'-présimplifiable (resp. an S'-BFM), then it is S-présimplifiable (resp. an S-BFM). Moreover, in this case if $0 \neq m, m' \in M$, then $m \sim^{S'} m'$ if and only if $m \sim^{S} m'$ if and only if $m = um'$ for some $u \in \mathrm{U}(R)$ and also if $0 \neq s, s' \in S' \setminus \mathrm{Ann}(M)$, then $s \sim s'$ if and only if $s = vs'$ for some $v \in \mathrm{U}(R)$.

(ii) Set $T = \{ut_1t_2 \cdots t_k | k \in \mathbb{N} \cup \{0\}$, each t_i is an irreducible element in $S' \setminus S$, $u \in \mathrm{U}(R)\}$. If M is an S'-UFM, then it is an S-UFM and nonzero S-primitive elements of M are exactly nonzero elements of the form tm, for some $t \in T$ and some S'-primitive $m \in M$. If moreover $T \cap \mathrm{Ann}(M) = \varnothing$, then T is a DMS of R.

Proof (i) The first assertion is obvious and the second follows from Theorem 2.7.

(ii) Assume that M is an S'-UFM. Suppose that $0 \neq m \in M$ is S-primitive. If $m = t_1t_2 \cdots t_km'$ is the S'-atomic factorization of m, then no t_i can be in S, since m is S-primitive. Thus $m = tm'$ for some $t \in T$ and an S'-primitive $m' \in M$.

Conversely, assume that $0 \neq ut_1t_2 \cdots t_km = sm'$, where $u \in U(R)$, t_i's are irreducibles in $S' \setminus S$, m is an S'-primitive element of M, $m' \in M$, and $s \in S \setminus \mathrm{U}(R)$. By Remark 3.7, since $s \notin \mathrm{Ann}(M)$, it has an atomic factorization such as $s = s_1s_2 \cdots s_t$. Let $m' = s_1's_2' \cdots s_{t'}'m''$ be the S'-atomic factorization of m'.

Now $ut_1t_2 \cdots t_km = s_1s_2 \cdots s_ts_1's_2' \cdots s_{t'}'m''$ and by uniqueness of S'-atomic factorizations, for some i we must have $t_i \sim s_1$. Thus t_i is a unit multiple of s_1 by (i), and hence is in S, which is a contradiction. From this contradiction we conclude that $ut_1t_2 \cdots t_km$ is S-primitive.

To see that M is an S-UFM, first note that it is an S'-BFM and by (i), an S-BFM. Thus M is S-atomic. Now let $m = s_1 \cdots s_kn$ and $m = s_1' \cdots s_{k'}'n'$ be two S-atomic factorizations of $0 \neq m \in M$. Suppose that $n = t_1 \cdots t_pa$ and $n' = t_1' \cdots t_{p'}'a'$ are the S'-atomic factorizations of n and n', respectively.

Since n and n' are S-primitive, t_i's and t_i''s are in $S' \setminus S$. However, by uniqueness of S'-atomic factorizations of m, each s_i must be an associate of some s_i' or t_i', but $s_i \sim t_j'$ results in $t_j' \in S$, which is a contradiction. Therefore, each s_i is an associate of some s_i'. Similarly each s_i' is an associate of some s_i. Thus in fact we can assume that $k = k'$ and for a reordering of s_i''s, we have $s_i \sim s_i'$ for each $1 \leq i \leq k$.

By a similar reasoning, we see that $p = p'$ and we can assume $t_i = v_it_i'$ for $v_i \in \mathrm{U}(R)$ and $1 \leq i \leq p$. Furthermore, $a = ua'$ for some $u \in \mathrm{U}(R)$ and hence if we set $v = v_1 \cdots v_pu$, then $n = vn'$, that is, $n \sim^S n'$. Thus the two S-atomic factorizations of m are isomorphic. Hence M is an S-UFM.

Now suppose that $T \cap \mathrm{Ann}(M) = \varnothing$. It is obvious that T is closed under multiplication and $1 \in T$. Suppose that $xy \in T$, say $xy = ut_1t_2 \cdots t_k$ for irreducible elements $t_i \in S' \setminus S$ and $u \in \mathrm{U}(R)$. Since $xy \in S'$ and S' is divisor-closed, we have $x \in S'$ and $y \in S'$.

By 3.7, x, y, and xy have unique atomic factorizations. Let $x = s_1 \cdots s_p$ and $y = s_1' \cdots s_{p'}'$ be the atomic factorizations of x and y, respectively. By the uniqueness of atomic factorizations of xy, each s_i must be a unit multiple of one of the t_i's and hence cannot be in S. Thus s_i's are in $S' \setminus S$ and hence $x \in T$. Similarly $y \in T$. $\qquad\square$

Theorem 3.9 *Assume that $S \cap \mathrm{Z}(M) = \varnothing$ and $S \subseteq S'$ be two DMS's of R. Let $T = S^{-1}S' = \left\{\frac{r}{s} \in S^{-1}R | r \in S'\right\}$. If M is S-présimplifiable (an S-BFM) and $S^{-1}M$ as an $S^{-1}R$-module is T-présimplifiable (a T-BFM), then M is S'-présimplifiable (an S'-BFM).*

Proof First note that if $m \in M$, $r \in R$, and $s \in S$, then since $S \cap \mathrm{Z}(M) = \varnothing$, we have $\frac{m}{1} = 0$ if and only if $m = 0$ and since S is divisor-closed we have $\frac{r}{s} \in \mathrm{U}\left(S^{-1}R\right)$ if and only if $r \in S$.

Suppose that M is an S-BFM, $S^{-1}M$ is a T-BFM, and let $m = s_1s_2 \cdots s_ks_1's_2' \cdots s_{k'}'m'$ be an S'-factorization of $0 \neq m \in M$, where s_i's are in S and s_i''s are in $S' \setminus S$. Since M is an S-BFM, we have $k \leq N$, where N is the bound on the lengths of S-factorizations of m.

Now $\frac{m}{1} = \frac{s_1}{1} \cdots \frac{s_k}{1} \frac{s'_1}{1} \cdots \frac{s'_{k'}}{1} \frac{m'}{1}$. Since $S^{-1}M$ is a T-BFM, there is an upper bound such as N' on the number of nonunit elements in T-factorizations of $0 \neq \frac{m}{1}$. Thus $k' \leq N'$ and $N + N'$ is a bound on the length of S'-factorizations of m. The présimplifiable case is similar. $\qquad\square$

The following example shows that the condition $S \cap \mathrm{Z}(M) = \varnothing$ is necessary in Theorem 3.9.

Example 3.10 *If $M = \mathbb{Z}_{2^\infty} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module, $S = \{\pm\, 2^k | k \in \mathbb{N} \cup \{0\}\}$ and $S' = \mathbb{Z}$, then M is S-présimplifiable and $S^{-1}M$ is an $(S^{-1}S')$-UFM, but M is not S'-présimplifiable.*

If $S \cap \mathrm{Z}(M) \neq \varnothing$, then we can apply Theorem 3.9 with $S_0 = S \setminus \mathrm{Z}(M) \subseteq S'$. However, in some cases like the above example, $S_0 = \mathrm{U}(R)$ and applying Theorem 3.9 with S_0 is of no use.

The converse of Theorem 3.9 is not true. For example, if R is the ring of integer-valued polynomials over $\mathbb{Q}$, that is, $\{f \in \mathbb{Q}[x] | f(\mathbb{Z}) \subseteq \mathbb{Z}\}$, then R is a BFR (that is, R as an R-module is a BFM), but for some prime ideal $\mathfrak{P}$ of R, $R_\mathfrak{P}$ is not a BFR (see [1, Example 2.7(b)]). Thus if $S = R \setminus \mathfrak{P}$, $S' = R = M$, then although M is an S'-BFM, but $S^{-1}M$ is not an $(S^{-1}S')$-BFM as an $S^{-1}R$-module.

Question 3.11 *Does the UFM version of Theorem 3.9 hold?*

Consider the UFM version of Theorem 3.9 in the very special case that R is a domain and $M = S' = R$. In this case, the question is "can we say that R is a UFD, assuming that $S^{-1}R$ is a UFD and R is an S-UFM as an R-module?" This is an example of what some authors call 'Nagata-type' questions. This type of question, which is well-studied, asks "under what conditions can we deduce that R is a UFD, assuming that $S^{-1}R$ is a UFD?" For example, if R is a Krull domain, S is generated by a set of primes and $S^{-1}R$ is a UFD, then R is a UFD (see [20, Corollary 8.32]). To see some other 'Nagata-type' theorems and a brief literature review of this subject see [2, Section 3].

4. S-factorizations in decomposable rings and modules

The proof of the following result is easy and we leave it to the reader.

Proposition 4.1 *Assume that for each $i \in I$, M_i is an R-module, $M = \bigoplus_{i \in I} M_i$ and $N = \prod_{i \in I} M_i$. Then M is S-présimplifiable (an S-BFM) if and only if N is S-présimplifiable (an S-BFM) if and only if each M_i is S-présimplifiable (an S-BFM).*

To obtain a similar result on S-UFMs, we first need a lemma.

Lemma 4.2 *Suppose that each M_i ($i \in I$) is an R-module and $M = \prod_{i \in I} M_i$. Also assume that M_i's are S-UFMs and $0 \neq m_i \in M_i$. If $(m_i)_{i \in I} = a(x_i)_{i \in I} = b(y_i)_{i \in I}$ for some $a, b \in S$, $x_i, y_i \in M_i$ such that $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ are S-primitive elements of M, then there exist unit elements u and v_i's in R such that $a = ub$, $x_i = v_i y_i$.*

Proof Assume that $s_1, \ldots, s_k \in S$ are the irreducible factors that are common (up to associates) in the S-atomic factorizations of all m_i's, and let t_{ij}'s ($1 \leq j \leq n_i$) be the other irreducible factors of m_i. More concretely, let

$$m_i = s_1^{\alpha_{i1}} s_2^{\alpha_{i2}} \cdots s_k^{\alpha_{ik}} t_{i1}^{\beta_{i1}} \cdots t_{in_i}^{\beta_{in_i}} z \tag{1}$$

be the S-atomic factorization of m_i. By Remark 3.7, since $a, b \in S \setminus \mathrm{Ann}(M)$ and M is an S-UFM, a and b have atomic factorizations. By appending an atomic factorization of a to an S-atomic factorization of x_i, we get an S-atomic factorization of m_i. However, by M_i's being S-UFMs, these factorizations must be isomorphic to (1).

Because (x_i) is S-primitive, there is no common irreducible in the S-atomic factorizations of x_i's. In addition, every irreducible element in the atomic factorization of a is a common factor of all m_i's and hence must be one of the s_i's. Using these notes, it is easy to see that in fact

$$a = t s_1^{\gamma_1} \cdots s_k^{\gamma_k},$$

$$x_i = l_i s_1^{\alpha_{i1} - \gamma_1} \cdots s_k^{\alpha_{ik} - \gamma_k} t_{i1}^{\beta_{i1}} \cdots t_{in_i}^{\beta_{in_i}} z,$$

where $\gamma_j = \min_{i \in I}(\alpha_{ij})$ for $1 \le j \le k$ and $t, l_i \in \mathrm{U}(R)$ for all $i \in I$.

A similar argument shows that b and y_i's must have similar formulations as a and x_i's, respectively, and the only difference can happen in the unit multiples. Thus b and y_i's are unit multiples of a and x_i's, respectively, as required. $\qquad \square$

Theorem 4.3 *Assume that for each $i \in I$, M_i is an R-module, $M = \bigoplus_{i \in I} M_i$, and $N = \prod_{i \in I} M_i$. Then M is an S-UFM if and only if N is an S-UFM if and only if each M_i is an S-UFM and $S \cap \mathrm{Z}(M_i) = S \cap \mathrm{Ann}(M_j)$ for each $i, j \in I$.*

Proof Suppose that N is an S-UFM. We will show that the conditions on M_i's hold. First assume that $I = \{1, 2\}$. Because N is an S-BFM and by 4.1, each M_i is an S-BFM and S-atomic. One can easily check that for each $0 \neq m \in M_1$, $(m, 0)$ is S-primitive in N if and only if m is so in M_1. Thus, if $m = s_1 s_2 \cdots s_k x = s_1' s_2' \cdots s_{k'}' x'$ are two nonisomorphic S-atomic factorizations of m, then $(m, 0) = s_1 s_2 \cdots s_k (x, 0) = s_1' s_2' \cdots s_{k'}' (x', 0)$ are two nonisomorphic S-atomic factorizations of $(m, 0)$, a contradiction. From this contradiction we deduce that M_1 (similarly M_2) is an S-UFM.

Now suppose that $s \in S \cap \mathrm{Z}(M_2)$. Thus $sm' = 0$ for some $0 \neq m' \in M_2$. Also suppose that $s \notin \mathrm{Ann}(M_1)$ and there is an $m \in M_1$ such that $sm \neq 0$. We can assume that m is S-primitive (else replace m with the S-primitive element in the S-atomic factorization of m).

According to Remark 3.7, s has an atomic factorization say $s = s_1 \cdots s_k$. Now $(sm, 0) = s_1 \cdots s_k(m, 0) = s_1 \cdots s_k(m, m')$ are two S-factorizations of $(sm, 0)$ and the former is S-atomic. By replacing (m, m') with its S-atomic factorization and using the fact that N is an S-UFM, we see that in fact (m, m') is S-primitive and $(m, 0) \sim^S (m, m')$. Since N is S-présimplifiable, this means $(m, m') = u(m, 0)$ for some $u \in \mathrm{U}(R)$, which is impossible. Therefore, $s \in \mathrm{Ann}(M_1)$ and whence $S \cap \mathrm{Z}(M_2) \subseteq S \cap \mathrm{Ann}(M_1)$. By a similar reasoning, $S \cap \mathrm{Z}(M_1) \subseteq S \cap \mathrm{Ann}(M_2)$ and so

$$S \cap \mathrm{Z}(M_1) \subseteq S \cap \mathrm{Ann}(M_2) \subseteq S \cap \mathrm{Z}(M_2) \subseteq S \cap \mathrm{Ann}(M_1) \subseteq S \cap \mathrm{Z}(M_1)$$

and hence all the inequalities must be equalities.

The result for the case that $|I| < \infty$ follows by an easy induction. Now assume that $|I| = \infty$ and $i \neq j \in I$. Since $N = M_i \oplus M_j \oplus \left(\prod_{i, j \neq k \in I} M_k \right)$, it follows from the finite case that $S \cap \mathrm{Z}(M_i) = S \cap \mathrm{Ann}(M_j)$,

as required. Assuming that M (instead of N) is an S-UFM and by a similar argument, one can deduce that each M_i is an S-UFM and $S \cap Z(M_i) = S \cap \mathrm{Ann}(M_j)$ for each $i, j \in I$.

Conversely, suppose that M_i's are S-UFMs and $S \cap Z(M_i) = S \cap \mathrm{Ann}(M_j)$ for each $i, j \in I$. By Proposition 4.1, N is an S-BFM and hence S-atomic. Let $0 \neq m = (m_i) \in N$. First assume that all m_i's are nonzero and $(m_i) = s_1 \cdots s_k(x_i) = t_1 \cdots t_l(y_i)$ are two S-atomic factorizations of m. By Lemma 4.2, there are $u, v_i \in \mathrm{U}(R)$ $(i \in I)$ such that $x_i = v_i y_i$ and $s = s_1 \cdots s_k = u t_1 \cdots t_l$. Because $s \in S \setminus \mathrm{Ann}(M)$, by Remark 3.7, s has just one unique atomic factorization. Hence $l = k$ and after reordering t_i's, we have $s_i \sim t_i$ for each $1 \leq i \leq k$.

In addition, if we set $t = t_1 \cdots t_k$, we have $t y_i = m_i = s x_i = u t v_i y_i$ and hence $t(1 - u v_i) y_i = 0$. However, $t \in S \setminus \mathrm{Ann}(M_i) = S \setminus Z(M_i)$, whence $(1 - u v_i) y_i = 0$ and $u x_i = u v_i y_i = y_i$. Thus $(x_i) \sim^S (y_i)$, that is, m has a unique S-atomic factorization up to isomorphism.

Now assume that $m_i = 0$ for $i \in J \subseteq I$ and $m_i \neq 0$ for $i \in I \setminus J$. If $(m_i) = s_1 \cdots s_k(x_i)$ is an S-atomic factorization of m and $x_j \neq 0$ for a $j \in J$, then $s_1 \cdots s_k \in S \cap Z(M_j) \subseteq \mathrm{Ann}(M_i)$ for all i and hence $m = 0$, a contradiction. Thus $x_j = 0$ for all $j \in J$. Using this, one can see that nonisomorphic S-atomic factorizations of m in N lead to nonisomorphic S-atomic factorizations of $m' = (m_i)_{i \in I \setminus J}$ in $N' = \prod_{i \in I \setminus J} M_i$. However, by the above case in which all m_i's are nonzero, m' has a unique S-atomic factorization in N'. Therefore, m has just a unique S-atomic factorization in N. Consequently, N is an S-UFM.

To see that M is also an S-UFM, note that any S-atomic factorizations of an element of M is indeed an S-atomic factorization in $\bigoplus_{i \in F} M_i$ for a finite subset F of I. However, for the finite index sets the result follows from the product case proved above. $\square$

Proposition 4.4 *For an R-module M, $S \cap Z(M) = S \cap \mathrm{Ann}(M)$ for every proper DMS S of R if and only if $Z(M) = \mathrm{N}(R) \cup \mathrm{Ann}(M)$.*

Proof ($\Rightarrow$) Clearly $\mathrm{N}(R) \cup \mathrm{Ann}(M) \subseteq Z(M)$. Let $x \in Z(M) \setminus \mathrm{N}(R)$. Then the closure S of $\{x^n | n \in \mathbb{N} \cup \{0\}\}$ is a proper DMS of R and $x \in S \cap Z(M) \subseteq \mathrm{Ann}(M)$.

($\Leftarrow$) Let $s \in S \cap Z(M)$. If $s \notin \mathrm{Ann}(M)$, then $s \in \mathrm{N}(R)$, whence $0 \in S$ and $S = R$, a contradiction. $\square$

Note that $S \cap Z(M) = S \cap \mathrm{Ann}(M)$ for every DMS S of R (including $S = R$) if and only if $Z(M) = \mathrm{Ann}(M)$ if and only if the zero submodule of M is prime. (A submodule N of M is called *prime* when from $rm \in N$ for $r \in R$, $m \in M$, we can deduce that either $m \in N$ or $rM \subseteq N$. This notion has been studied extensively; see for example [6, 22].)

In what follows, we assume that $R = \prod_{\alpha \in A} R_\alpha$ and that π_α is the canonical projection from R onto R_α. If for each $\alpha \in A$, S_α is a DMS of R_α, then obviously $\prod_{\alpha \in A} S_\alpha$ is a DMS of R. Set

$$S = \left\{ (s_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} S_\alpha \,\middle|\, s_\alpha \in \mathrm{U}(R_\alpha) \text{ for all but finitely many } \alpha \in A \right\}.$$

It can easily be verified that S is a DMS of R.

Notation 4.5 *We call the above set S, the weak product of S_α's, and denote it by $\prod_{\alpha \in A}^w S_\alpha$.*

Notation 4.6 *By $(r_\alpha; r)\underset{\substack{\alpha \in A \\ \alpha \neq \alpha_0}}{}$, we mean the element $(a_\alpha) \in R$, with $a_\alpha = r_\alpha$ for all $\alpha \in A$, except for $\alpha = \alpha_0$ and $a_{\alpha_0} = r$.*

Thus, for example, if $\alpha \neq \alpha_0$, then $\pi_\alpha \left((1; 0)\underset{\substack{\alpha \in A \\ \alpha \neq \alpha_0}}{} \right) = 1$ and $\pi_{\alpha_0} \left((1; 0)\underset{\substack{\alpha \in A \\ \alpha \neq \alpha_0}}{} \right) = 0$.

Proposition 4.7 *Suppose that S is a DMS of R and $S_\alpha = \pi_\alpha(S)$. Then each S_α is a DMS of R_α and $\prod_{\alpha \in A}^{w} S_\alpha \subseteq S \subseteq \prod_{\alpha \in A} S_\alpha$.*

Proof Obviously each S_α is closed under multiplication. If $xy \in S_\beta$ for a $\beta \in A$, then there is an $s = (s_\alpha) \in S$ with $s_\beta = xy$. Now $(1; x)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{} (s_\alpha; y)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{} = s \in S$. Since S is divisor-closed, $(1; x)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{}$ and $(s_\alpha; y)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{}$ must be in S and hence $x, y \in S_\beta$.

It is clear that $S \subseteq \prod_{\alpha \in A} S_\alpha$. Suppose that $s = (s_\alpha) \in S' = \prod_{\alpha \in A}^{w} S_\alpha$. Thus for each $\alpha \in A$, we have $s_\alpha \in U(R_\alpha)$ except for a finite number of α's, say $\alpha_1, \ldots, \alpha_n$. For each $1 \le i \le n$ there exists a $t_i \in S$ such that $\pi_{\alpha_i}(t_i) = s_{\alpha_i}$. However, $(\pi_\alpha(t_i); 1)\underset{\substack{\alpha \in A \\ \alpha \neq \alpha_i}}{} (1; \pi_{\alpha_i}(t_i))\underset{\substack{\alpha \in A \\ \alpha \neq \alpha_i}}{} = t_i \in S$. Because S is divisor-closed, we must have $t_i' = (1; \pi_{\alpha_i}(t_i))\underset{\substack{\alpha \in A \\ \alpha \neq \alpha_i}}{} \in S$.

Choose $u \in R$ such that for each $\alpha \in A \setminus \{\alpha_1, \ldots, \alpha_n\}$, we have $\pi_\alpha(u) = s_\alpha$ and for $1 \le i \le n$, $\pi_{\alpha_i}(u) = 1$. Thus every component of u is a unit and hence $u \in U(R) \subseteq S$. Now $s = ut_1' t_2' \cdots t_n' \in S$, as required. $\qquad \square$

Corollary 4.8 *If $|A| < \infty$, then every DMS of R is a product of DMS's of R_α's.*

Example 4.9 *Set $A = \mathbb{N}$ and for each $i \in \mathbb{N}$, let $R_i = \mathbb{Z}$ and $S_i = \left\{ \pm 2^k | k \in \mathbb{N} \cup \{0\} \right\}$. Moreover, let S be the subset of $\prod_{i \in \mathbb{N}} S_i$, for which almost all odd components of its elements are units. Then $(2, 2, \ldots) \in \prod_{i \in \mathbb{N}} S_i \setminus S$ and $(1, 2, 1, 2, \ldots) \in S \setminus \prod_{i \in \mathbb{N}}^{w} S_i$. Thus for this S the two inequalities of Proposition 4.7 are strict.*

Lemma 4.10 *Let $r = (r_\alpha)_{\alpha \in A}$ and $s = (s_\alpha)_{\alpha \in A}$ be two elements of R.*

(i) $r \sim s$ if and only if for each $\alpha \in A$, we have $r_\alpha \sim s_\alpha$.

(ii) r is irreducible if and only if each $r_\alpha \in U(R_\alpha)$ except for one α_0 and that r_{α_0} is irreducible in R_{α_0}.

Proof See [3, Theorem 2.15]. $\qquad \square$

Theorem 4.11 *Assume that $|A| > 1$, S is a DMS of R, and $S_\alpha = \pi_\alpha(S)$.*

(i) R is présimplifiable in S, if and only if for each $\alpha \in A$, R_α is présimplifiable in S_α and $S_\alpha \neq R_\alpha$.

(ii) R has BF (UF) in S, if and only if for each $\alpha \in A$, R_α has BF (UF) in S_α, $S_\alpha \neq R_\alpha$, and $S = \prod_{\alpha \in A}^{w} S_\alpha$.

Proof (i) Assume that R is présimplifiable inside S. If for some $\beta \in A$, $r_\beta \in R_\beta$, and $0 \neq s_\beta \in S_\beta$ we have $r_\beta s_\beta = s_\beta$, then $(1; r_\beta)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{} (1; s_\beta)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{} = (1; s_\beta)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{}$. However, by Proposition 4.7, $(1; s_\beta)\underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{} \in \prod_{\alpha \in A}^{w} S_\alpha \subseteq S$. Thus from R being présimplifiable in S, we deduce that $r_\beta \in U(R_\beta)$ and hence R_β is présimplifiable in S_β.

If for some $\beta \in A$, we have $0 \in S_\beta$, then $s = (1;0)_{\substack{\alpha \in A \\ \alpha \neq \beta}} \in S$ and $s^2 = s$, but $0 \neq s \notin \mathrm{U}(R)$, which is impossible. Thus for all $\alpha \in A$, we have $R_\alpha \neq S_\alpha$.

Conversely, suppose that for all $\alpha \in A$, R_α is présimplifiable in S_α and $R_\alpha \neq S_\alpha$. If for an $r = (r_\alpha) \in R$ and an $s = (s_\alpha) \in S$, we have $rs = s$, then for each $\alpha \in A$, $r_\alpha s_\alpha = s_\alpha$ and since $s_\alpha \neq 0$ (because $R_\alpha \neq S_\alpha$), we have $r_\alpha \in \mathrm{U}(R_\alpha)$. Therefore, $r \in \mathrm{U}(R)$.

(ii) First suppose that R has BF in S. Then by (i), we see that for each $\alpha \in A$, we have $R_\alpha \neq S_\alpha$. Moreover, by mapping $s \in S_\beta$ to $(1;s)_{\substack{\alpha \in A \\ \alpha \neq \beta}} \in S$, one can easily see that lengths of factorizations of s are bounded.

Now suppose that there is an $s = (s_\alpha) \in S$ with an infinite number of nonunit components. We can assume that $\mathbb{N} \subseteq A$ and for each $i \in \mathbb{N}$, the i'th component of s is a nonunit. For each $n \in \mathbb{N}$ set $a_n = (1;s_n)_{\substack{\alpha \in A \\ \alpha \neq n}}$ and let b_n be the element of R with $\pi_i(b_n) = 1$ for $1 \leq i \leq n$ and with other components as s.

However, now we have $s = a_1 b_1 = a_1 a_2 b_2 = \cdots$ and since each a_i and b_i has at least one nonunit component, these are a family of factorizations of s with arbitrary large lengths, a contradiction. Therefore, no $s \in S$ has an infinite number of nonunit components and $S \subseteq \prod_{\alpha \in A}^w S_\alpha$. The converse inclusion holds by Proposition 4.7.

Conversely, assume that $S = \prod_{\alpha \in A}^w S_\alpha$ and for all $\alpha \in A$, R_α has BF in S_α and $S_\alpha \neq R_\alpha$. Let $s = (s_\alpha) \in S$ and suppose that $s_{\alpha_1}, \ldots, s_{\alpha_k}$ are the nonunit components of s. Assume that $s = t_1 \cdots t_n$, where for each $1 \leq i \leq n$, $t_i \in S \setminus \mathrm{U}(R)$. For each $1 \leq j \leq k$, set p_j to be the number of t_i's with a nonunit α_j'th component. Since the α_j'th components of t_i's form a factorization for s_{α_j}, we get $p_j \leq N_j$, where N_j is a bound on the lengths of factorizations of s_{α_j}.

Furthermore, let q_i be the number of nonunit components of t_i. Then $1 \leq q_i$ for each $1 \leq i \leq n$. Hence $n \leq \sum_{i=1}^n q_i =$ the total number of nonunit components of t_i's $= \sum_{j=1}^k p_j \leq \sum_{j=1}^k N_j$. Thus $\sum_{j=1}^k N_j$ is a bound on the lengths of factorizations of s and R has BF inside S.

Finally suppose that R has UF in S. Then by the BF case, we see that $S = \prod_{\alpha \in A}^w S_\alpha$ and for each $\alpha \in A$, $R_\alpha \neq S_\alpha$. Moreover, by mapping $s \in S_\beta$ to $(1;s)_{\substack{\alpha \in A \\ \alpha \neq \beta}} \in S$ and using Lemma 4.10, one can easily see that nonisomorphic atomic factorizations of s lead to nonisomorphic atomic factorizations of $(1;s)_{\substack{\alpha \in A \\ \alpha \neq \beta}}$. Hence every R_α must have UF inside S_α.

Conversely, assume that each R_α has UF in S_α and $s = t_1 \cdots t_n$ is an atomic factorization of $0 \neq s \in S = \prod_{\alpha \in A}^w S_\alpha$. Suppose that $s_{\alpha_1}, \ldots, s_{\alpha_k}$ are the nonunit components of s. By Lemma 4.10(ii), for each $1 \leq i \leq n$, t_i has exactly one nonunit component and that component is irreducible. Suppose that the nonunit components of $t_1, \ldots, t_{n_1}$ occur at the place of α_1, the nonunit components of $t_{n_1+1}, \ldots, t_{n_2}$ occur at the place of α_2, and so on.

Hence $s_{\alpha_i} = \pi_{\alpha_i}(t_{n_{i-1}+1}) \cdots \pi_{\alpha_i}(t_{n_i})$ is an atomic factorization of s_{α_i}. Thus using Lemma 4.10(i) and the fact that each R_α has UF in S_α, it can easily be verified that every atomic factorization of s is isomorphic to $s = t_1 \cdots t_n$. This concludes the proof. $\qquad \square$

At the end of this article, we consider modules over decomposable rings. Note that if $R = R_1 \times R_2$ for two nontrivial rings R_1 and R_2 and M is an R-module, then $M = (1,0)M \oplus (0,1)M$ and also $M_1 = (1,0)M$ and $M_2 = (0,1)M$ are modules over R_1 and R_2, respectively.

Proposition 4.12 *Let R_1 and R_2 be nontrivial rings and M_1 and M_2 be modules, not both zero, over R_1 and R_2, respectively. In addition, suppose that S_1 and S_2 are DMS's of R_1 and R_2, respectively. Set $R = R_1 \times R_2$, $M = M_1 \oplus M_2$, and $S = S_1 \times S_2$.*

Then M as an R-module is S-présimplifiable (resp. an S-BFM, an S-UFM), if and only if either $S = \mathrm{U}(R)$ or one of the M_i's, say M_1, is zero, $S_1 = \mathrm{U}(R_1)$, and M_2 is S_2-présimplifiable (an S_2-BFM, an S_2-UFM).

Proof Assume that M is S-présimplifiable. If $0 \neq m \in M_1$ and $s \in S_2 \setminus \mathrm{U}(R_2)$, then $(1, s)(m, 0) = (m, 0)$, a contradiction with M being S-présimplifiable. Thus, if $M_1 \neq 0$, then $S_2 = \mathrm{U}(R_2)$. Similarly, if $M_2 \neq 0$, then $S_1 = \mathrm{U}(R_1)$. Thus, if neither of M_1 and M_2 is zero, then $S = \mathrm{U}(R)$. Moreover, it is easy to see that each M_i must be an S_i-présimplifiable R_i-module. The cases of S-BFM and S-UFM are similar and the proof of the converse is easy. $\qquad\square$

Corollary 4.13 *Assume that $R = \prod_{i \in I} R_i$ and $\pi_i(S) = S_i$, where each R_i is a ring and S_i is a DMS of R_i, and let M be an R-module. Then M is S-présimplifiable (resp. an S-BFM, an S-UFM) if and only if either $S = \mathrm{U}(R)$ or there is an $i_0 \in I$ such that $\prod_{i_0 \neq i \in I} R_i \subseteq \mathrm{Ann}(M)$, $S_i = \mathrm{U}(R_i)$ for each $i_0 \neq i \in I$ and M is S_{i_0}-présimplifiable (resp. an S_{i_0}-BFM, an S_{i_0}-UFM) as an R_{i_0}-module.*

Example 4.14 *Let $R = \mathbb{Z} \times \mathbb{Z}$ and $S = R \setminus \mathrm{Z}(R) = (\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \setminus \{0\})$. Then by Theorem 4.11, R has UF inside S, but if we take R as an R-module, Proposition 4.12 says that it is not even S-présimplifiable.*

Acknowledgments

References

[1] Anderson DD, Anderson DF, Zafrullah M. Factorization in integral domains. J Pure Appl Algebra 1990; 69: 1-19.

[2] Anderson DD, Anderson DF, Zafrullah M. Factorization in integral domains II. J Algebra 1992; 152: 78-93.

[3] Anderson DD, Valdes-Leon S. Factorization in commutative rings with zero divisors. Rocky Mt J Math 1996; 26: 439-480.

[4] Anderson DD, Valdes-Leon S. Factorization in commutative rings with zero divisors, II. In: Anderson DD, editor. Factorization in Integral Domains. New York, NY, USA: Marcel Dekker, 1997, pp. 197-219.

[5] Atiyah MF, Macdonald IG. Introduction to Commutative Algebra. London, UK: Addison-Wesley, 1969.

[6] Azizi A, Sharif H. On prime submodules. Honam Math J 1999; 21: 1-12.

[7] Baeth NR, Geroldinger A. Monoids of modules and arithmetic of direct-sum decompositions. Pac J Math 2014; 271: 257-319.

[8] Baeth NR, Wiegand RA. Factorization theory and decompositions of modules. Am Math Mon 2013; 120: 3-34.

[9] Chang GW, Smertnig D. Factorization in the self-idealization of a PID. B Unione Mat Ital 2013; 6: 363-377.

[10] Chun S, Anderson DD, Valdez-Leon S. Reduced factorizations in commutative rings with zero divisors. Commun Algebra 2011; 39: 1583-1594.

[11] Facchini A. Direct sum decompositions of modules, semilocal endomorphism rings, and Krull monoids. J Algebra 2002; 256: 280-307.

[12] Facchini A. Geometric regularity of direct-sum decompositions in some classes of modules. J Math Sci 2006; 139: 6814-6822. (Translated from: Fundam Prikl Mat 2004; 10: 231-244.)

[13] Facchini A. Direct-sum decompositions of modules with semilocal endomorphism rings. B Math Sci 2012; 2: 225-279.

[14] Foroutan A, Hassler W. Factorization of powers in C-monoids. J Algebra 2006; 304: 755-781.

[15] Frei C, Frisch S. Non-unique factorization of polynomials over residue class rings of the integers. Commun Algebra 2011; 39: 1482-1490.

[16] Frisch S. A construction of integer-valued polynomials with prescribed sets of lengths of factorizations. Monatsh Math 2013; 171: 341-350.

[17] Geroklinger A, Halter-Koch F. Non-Unique Factorizations: Algebraic, Combinatorial and Analytic Theory. Boca Raton, FL, USA: Chapman & Hall/CRC, 2006.

[18] Geroldinger A, Ramacher S, Reinhart A. On v-marot mori rings and c-rings. J Korean Math Soc 2015; 52: 1-21.

[19] Gilmer R. Commutative Semigroup Rings. Chicago, IL, USA: The University of Chicago Press, 1984.

[20] Larsen MD, McCarthy PJ. Multiplicative Theory of Ideals. New York, NY, USA: Academic Press, 1971.

[21] Mooney CP. Generalized factorization in commutative rings with zero-divisors. PhD, University of Iowa, Iowa City, IA, USA, 2013.

[22] Nikseresht A, Azizi A. Envelope dimension of modules and the simplified radical formula. Can Math Bull 2103; 56: 683-694.

[23] Nikseresht A, Azizi A. On factorization in modules. Commun Algebra 2011; 39: 292-311.