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Simultaneous strong proximinality in Banach spaces

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Abstract: Several researchers have discussed the problem of strong proximinality in Banach spaces. In this paper, we generalize the notion of strong proximinality and define simultaneous strong proximinality. It is proved that if W is a simultaneously approximatively compact subset of a Banach space X then W is simultaneously strongly proximinal and the converse holds if the set of all best simultaneous approximations to every bounded subset S of X from W is compact. We show that simultaneously strongly Chebyshev sets are precisely the sets that are simultaneously strongly proximinal and simultaneously Chebyshev. It is also proved that if F and W are subspaces of a Banach space X such that F is simultaneously strongly proximinal, W is finite dimensional and $F+W$ is closed then $F+W$ is simultaneously strongly proximinal in X. How simultaneous strong proximinality is transmitted to and from quotient spaces is discussed in this paper.

Key words: Strongly proximinal, simultaneously strongly proximinal, strongly Chebyshev, simultaneously strongly Chebyshev, approximatively compact, simultaneously approximatively compact

1. Introduction

Let W be a nonempty closed subset of a real Banach space $(X, \|.\|)$ and $x \in X$. An element $w_0 \in W$ is said to be a *best approximation* to x from W if

$$\|x - w_0\| = \inf_{w \in W} \|x - w\| \equiv d(x, W).$$

The set of all best approximations to x from W is denoted by $P_W(x)$. The set W is called *proximinal* if $P_W(x) \neq \phi$ for every $x \in X$. If for each $x \in X$, $P_W(x)$ is a singleton then the set W is called *Chebyshev*.

A proximinal subset W of a Banach space X is said to be *strongly proximinal* (see [1,5,10]) if for each $x \in X$ and for any minimizing sequence $\{y_n\} \subseteq W$ for x, i.e. $\lim_{n\to\infty} \|x - y_n\| = d(x, W)$, there is a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\} \subseteq P_W(x)$ such that $\|y_{n_k} - z_k\| \to 0$.

A subset W of a Banach space X is said to be *approximatively compact* if for any $x \in X$, every minimizing sequence $\{y_n\} \subseteq W$ for x, i.e. $\|x - y_n\| \to d(x, W)$ has a convergent subsequence in W.

A subset W of a Banach space X is said to be *strongly Chebyshev* (see [1]) if for any $x \in X$, every minimizing sequence $\{y_n\} \subseteq W$ for x is convergent in W.

Sometimes, it may happen that an element to be approximated is not known exactly but is known to lie in a bounded set S. In that case, it is reasonable to approximate simultaneously all $s \in S$ by a single element

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of W by solving

$$\inf_{w\in W}\sup_{s\in S}\|s-w\|\equiv d(S,W).$$

An element $w_0\in W$ is said to be a *best simultaneous approximation* (see [6]) to S from W if

$$\sup_{s\in S}\|s-w_0\|=d(S,W).$$

The set of all best simultaneous approximations to S from W is denoted by $L_W(S)$. The set W is called *simultaneously proximinal* if for each bounded subset S of X, $L_W(S)\neq\phi$. If for each bounded subset S of X, $L_W(S)$ is a singleton then the set W is called *simultaneously Chebyshev.* For any $\delta>0$, the set $\{y\in W:\sup_{s\in S}\|s-y\|<\sup_{s\in S}\|s-w\|+\delta \; for\ all\ w\in W\}$ is denoted by $L_W(S,\delta)$.

Motivated by the notions of best simultaneous approximation and strong proximinality, we define the notion of simultaneous strong proximinality.

A simultaneously proximinal subset W of a Banach space X is said to be *simultaneously strongly proximinal* if for each bounded subset S of X and for any minimizing sequence $\{y_n\}\subseteq W$ for S, i.e. $\lim_{n\to\infty}\sup_{s\in S}\|s-y_n\|=d(S,W)$, there is a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\}\subseteq L_W(S)$ such that $\|y_{n_k}-z_k\|\to 0$.

A subset W of a Banach space X is said to be *simultaneously approximatively compact* (see also [7]) if for any bounded subset S of X, every minimizing sequence $\{y_n\}\subseteq W$ for S has a convergent subsequence in W.

A subset W of a Banach space X is said to be *simultaneously strongly Chebyshev* if for any bounded subset S of X, every minimizing sequence $\{y_n\}\subseteq W$ for S is convergent in W.

In this paper, we prove that if W is simultaneously approximatively compact then W is simultaneously strongly proximinal and the converse holds if $L_W(S)$ is compact for every bounded subset S of X. We shall also show that simultaneously strongly Chebyshev sets are precisely the sets that are simultaneously strongly proximinal and simultaneously Chebyshev.

The following question was raised by Cheney in [2]:

If F and W are proximinal subspaces of a Banach space X, and $F+W$ is closed, does it follows that $F+W$ is proximinal in X?

Feder [4] gave a negative answer to this problem and proved that if W is reflexive and F is proximinal such that $W\bigcap F$ is finite dimensional then $F+W$ is proximinal. Lin [9] improved this result and proved that W is reflexive if and only if for every Banach space X with $W\subseteq X$, F is proximinal in X and $W+F$ is closed implies $W+F$ is proximinal in X. Sun and Luo [13] proved that if F is strongly proximinal and W is a finite dimensional subspace of X such that $F+W$ is closed then $F+W$ is a strongly proximinal subspace of X. Rawashdeh [11] generalized the problem of sum of proximinal subspaces to sum of simultaneously proximinal subspaces and proved that if F and W are subspaces of a Banach space X such that F is simultaneously proximinal and W is of finite dimension then $F+W$ is simultaneously proximinal. Thus, the question is:

Can we prove an analogous result for the sum of simultaneously strongly proximinal subspaces?

In this paper, we take up this problem and prove that if F and W are subspaces of a Banach space X such that F is simultaneously strongly proximinal, W is finite dimensional, and $F+W$ is closed then $F+W$ is simultaneously strongly proximinal in X. We also prove that if F is a simultaneously proximinal subspace and W a subspace of X such that $W+F$ is closed then $(W+F)/F$ is simultaneously strongly proximinal in X/F if $W+F$ is simultaneously strongly proximinal in X.

The results proved in this paper generalize several results of [1,3,8,10,11,13].

2. Simultaneous strong proximinality and simultaneous approximative compactness

It was shown in [11] that if W is a reflexive subspace of a Banach space X then W is simultaneously proximinal in X. Thus, it is natural to ask whether we can prove a similar result for simultaneous strong proximinality? In this section, we prove that if W is a finite dimensional subspace of a Banach space X then W is simultaneously strongly proximinal. We also give some relationships between simultaneous strong proximinality, simultaneous approximative compactness, and simultaneous strong Chebyshevity.

Proposition 2.1 *Let W be a finite dimensional subspace of a Banach space X; then W is simultaneously approximatively compact.*

Proof Let S be a bounded subset of X and $\{y_n\} \subseteq W$ a minimizing sequence for S, i.e.

$$\lim_{n\to\infty} \sup_{s\in S} \|s-y_n\| = d(S,W).$$

Then the sequence $\{a_n\}$ given by $a_n = \sup_{s\in S}\|s-y_n\|$ is a convergent sequence of real numbers and hence is bounded. Therefore, there exist $M_1>0$ such that $\|a_n\| = \sup_{s\in S}\|s-y_n\| \le M_1$ for every positive integer n. The inequality $\|y_n\| \le \|y_n - s\| + \|s\|$ for any $s\in S$ and for all n gives $\|y_n\| \le \sup_{s\in S}\|y_n-s\| + \sup_{s\in S}\|s\| \le M_1+M_2 = M$ for all n. This implies that $\{y_n\}$ is a bounded sequence in W. Since W is finite dimensional, there exists a subsequence $\{y_{n_k}\} \to y_0 \in W$. Thus W is simultaneously approximatively compact for S. □

The above proposition shows that if $\{y_n\}\subseteq W$ is a minimizing sequence for S then $\{y_n\}$ is bounded. Therefore, if W is a closed subset of a finite dimensional normed linear space X then W is simultaneously approximatively compact.

Concerning simultaneous approximative compactness and simultaneous strong proximinality, we have

Theorem 2.2 *Let W be a closed subset of a Banach space $(X, \|.\|)$. Then W is simultaneously approximatively compact if and only if W is simultaneously strongly proximinal and $L_W(S)$ is compact for every bounded subset S of X.*

Proof Suppose W is simultaneously approximatively compact and S a bounded subset of X. Let $\{y_n\}\subseteq W$ be a minimizing sequence for S, i.e.

$$\lim_{n\to\infty} \sup_{s\in S} \|s-y_n\| = d(S,W). \tag{2.1}$$

Since W is simultaneously approximatively compact, $\{y_n\}$ has a subsequence $\{y_{n_k}\}\to y_0\in W$. From (2.1), we have $\sup_{s\in S}\|s-y_0\| = d(S,W)$, i.e. $y_0\in L_W(S)$ and so W is simultaneously proximinal for S. Then for the constant sequence $\{y_0\}\subseteq L_W(S)$, we have $\|y_{n_k}-y_0\|\to 0$. Hence W is simultaneously strongly proximinal for S.

Now suppose that $\{z_n\}$ is any sequence in $L_W(S)$, i.e. $\sup_{s\in S}\|s-z_n\| = d(S,W)$ for every $n\in\mathbb{N}$. Then $\lim_{n\to\infty}\sup_{s\in S}\|s-z_n\| = d(S,W)$ and so $\{z_n\}$ is a minimizing sequence for S. Since W is simultaneously approximatively compact, $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$. Hence $L_W(S)$ is compact.

Conversely, suppose that W is simultaneously strongly proximinal and $L_W(S)$ is compact for every bounded subset S of X. Let $\{y_n\} \subseteq W$ be a minimizing sequence for S. Since W is simultaneously strongly proximinal, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and sequence $\{z_k\} \subseteq L_W(S)$ such that $\|y_{n_k} - z_k\| \to 0$. Since $L_W(S)$ is compact, $\{z_k\}$ has a subsequence $\{z_{k_l}\} \to z_0 \in W$. This gives $\{y_{n_k}\} \to z_0$ and so W is simultaneously approximatively compact. □

Using Proposition 2.1, we obtain

Corollary 2.3 *Let W be a finite dimensional subspace of a Banach space $(X, \|.\|)$. Then W is simultaneously strongly proximinal and $L_W(S)$ is compact for every bounded subset S of X.*

Since a closed subset of a finite dimensional normed linear space is simultaneously approximatively compact, we have

Corollary 2.4 *Let W be a closed subset of a finite dimensional normed linear space $(X, \|.\|)$. Then W is simultaneously strongly proximinal and $L_W(S)$ is compact for every bounded subset S of X.*

A closed subset W of a Banach space X is called *simultaneously quasi-Chebyshev* (see [8]) if $L_W(S)$ is nonempty and compact for every bounded subset S of X. Therefore, we have

Corollary 2.5 *Let W be a closed subset of a Banach space $(X, \|.\|)$; then W is simultaneously approximatively compact if and only if W is simultaneously strongly proximinal and simultaneously quasi-Chebyshev.*

The following theorem gives relationships between simultaneous approximative compactness, simultaneous strong Chebyshevity, and simultaneous strong proximinality.

Theorem 2.6 *Let W be a closed subset of a Banach space X. Then the following are equivalent:*
(i) W is simultaneously strongly Chebyshev.
(ii) W is simultaneously strongly proximinal and simultaneously Chebyshev.
(iii) W is simultaneously approximatively compact and simultaneously Chebyshev.

Proof (i) ⇒ (ii). Since W is simultaneously strongly Chebyshev, it is simultaneously approximatively compact and so by Theorem 2.2, W is simultaneously strongly proximinal. Now suppose S is any bounded subset of X and $w_1, w_2 \in L_W(S)$, $w_1 \neq w_2$. Then

$$\sup_{s \in S} \|s - w_1\| = d(S, W) = \sup_{s \in S} \|s - w_2\|.$$

Consider the sequence $\{y_n\}$ in W such that $y_{2n} = w_1$ and $y_{2n+1} = w_2$. Then $\{y_n\}$ is a minimizing sequence for S in W. Since $w_1 \neq w_2$, $\{y_n\}$ is not convergent, a contradiction to simultaneous strong Chebyshevity of W. Thus $w_1 = w_2$ and hence W is simultaneously Chebyshev.
(ii) ⇒ (iii) follows from Theorem 2.2.
(iii) ⇒ (i). Let $\{y_n\} \subseteq W$ be a minimizing sequence for a bounded subset S of X, i.e. $\lim_{n\to\infty} \sup_{s \in S} \|s - y_n\| = d(S, W)$. Since W is simultaneously approximatively compact, $\{y_n\}$ has a subsequence $\{y_{n_k}\} \to y_0$. Then $\sup_{s \in S} \|s - y_0\| = d(S, W)$, i.e. $y_0 \in L_W(S)$. We claim that every subsequence of $\{y_n\}$ also converges to y_0. Suppose $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ such that $\{y_{n_i}\} \to z_0$, $z_0 \neq y_0$. Then $\sup_{s \in S} \|s - z_0\| = d(S, W)$,

i.e. z_0 is also a best simultaneous approximation to S from W. However, W is simultaneously Chebyshev and so $y_0 = z_0$, a contradiction. Therefore, every subsequence of $\{y_n\}$ converges to y_0 and hence $\{y_n\} \to y_0$. □

The following example shows that a simultaneously approximatively compact set need not be simultaneously strongly Chebyshev.

Example 2.7 *Let $X = (\mathbb{R}^2, \|.\|)$, where $\|(x,y)\| = max(|x|,|y|)$, $W = \{(x,0) : x \in \mathbb{R}\}$ and $S = \{(1,1),(1,2),(2,1),(2,2)\}$. Then W being finite dimensional subspace of X, is simultaneously approximatively compact. But W is not simultaneously strongly Chebyshev for S. Consider the sequence $\{y_n\}$ such that $y_{2n} = (2.5, 0)$ and $y_{2n+1} = (1.5, 0)$. Then $\{y_n\}$ is a minimizing sequence for S which is not convergent.*

Remarks

1. While an approximatively compact subset of a Banach space is strongly proximinal (see [1]), a strongly proximinal subset of a Banach space need not be approximatively compact.

   Let $X = l_\infty$, $W = c_0$. Then W, being an M-ideal, is strongly proximinal in X. However, for $x = (1,1,1,...) \in l_\infty$, the sequence $y_n = (1,1,...,1,\ 0,0,...) \in W$ is a minimizing sequence for x but $\{y_n\}$ has no convergent subsequence (see [1]).

2. A proximinal subset of a Banach space need not be strongly proximinal. Even a proximinal convex subset of a Banach space need not be strongly proximinal.

   Let $X = (l_1, \|.\|_H)$, as constructed by Smith (see [12], Example 5). Then the unit ball $B(X_H)$ is proximinal in X but not strongly proximinal in X (see [14]).

3. The results proved in this section generalize the corresponding results proved in [1,10] for strong proximinality.

3. Sum and quotient of simultaneously strongly proximinal subspaces

In this section, we discuss simultaneous strong proximinality for the sum and quotient subspaces of X and see how simultaneous strong proximinaliy is transmitted to and from quotient spaces and to sum of subspaces. The results proved in this section are motivated by the corresponding results proved for proximinality in [3], simultaneous proximinality in [8,11], and strong proximinality in [13].

The following two results of [11] will be used in the sequel:

Lemma 3.1 *Let F and W be subspaces of a Banach space X such that F is simultaneously proximinal, W is finite dimensional, and $F+W$ is closed; then $F+W$ is simultaneously proximinal in X.*

Lemma 3.2 *Let F be a simultaneously proximinal subspace of a Banach space X and W a subspace of X such that $W+F$ is closed. If $W+F$ is simultaneously proximinal in X then $(W+F)/F$ is simultaneously proximinal in X/F.*

Concerning the sum of simultaneously strongly proximinal subspaces, we have

Theorem 3.3 *Let F and W be subspaces of a Banach space X such that F is simultaneously strongly proximinal, W is finite dimensional, and $F+W$ is closed; then $F+W$ is simultaneously strongly proximinal in X.*

Proof Since F is simultaneously proximinal and W is finite dimensional, it follows from Lemma 3.1 that $F+W$ is simultaneously proximinal. Let S be a bounded subset of X and $\{h_n+w_n\} \subseteq F+W$ is such that $\sup_{s\in S}\|s-(h_n+w_n)\| \to d(S,W+F)$, where $\{h_n\}$ is a sequence in F and $\{w_n\}$ is a sequence in W. Therefore, the sequence $\{a_n\}$ given by $a_n = \sup_{s\in S}\|s-(h_n+w_n)\|$ is a convergent sequence of real numbers and hence bounded. Thus, there exist $M_1 > 0$ such that $|a_n| \leq M_1$ for every n. The inequality $\|h_n+w_n\| \leq \|h_n+w_n-s\|+\|s\|$ for any $s\in S$ and for any n implies that $\|h_n+w_n\| \leq \sup_{s\in S}\|h_n+w_n-s\|+\sup_{s\in S}\|s\| \leq M_1+M_2 = M$ for all n and so $\{h_n+w_n\}$ is a bounded sequence in $F+W$.

Since the projection $P: F+W \to W,\ P(h+w)=w$ is closed and $P(F)=0$, by the closed graph theorem there exists a positive constant C such that $\|w_n\| = \|P(h_n+w_n)\| \leq C\|h_n+w_n\| \leq CM$. This implies that $\{w_n\}$ is a bounded sequence and so is $\{h_n\}$. Since W is finite dimensional, by passing to a subsequence if necessary, we may assume that $w_n \to w_0 \in W$. We claim $d(S-w_0,F) = d(S,F+W) = \lim_{n\to\infty}\sup_{s\in S}\|s-(w_0+h_n)\|$.

Since $|\sup_{s\in S}\|s-(h_n+w_n)\| - \sup_{s\in S}\|s-(w_0+h_n)\|| \leq \sup_{s\in S}|\|s-(h_n+w_n)\| - \|s-(w_0+h_n)\|| \leq \|w_n-w_0\|$, we have $\lim_{n\to\infty}\sup_{s\in S}\|s-(h_n+w_n)\| = \lim_{n\to\infty}\sup_{s\in S}\|s-(h_n+w_0)\|$. Therefore, $\lim_{n\to\infty}\sup_{s\in S}\|s-(h_n+w_n)\| = d(S,F+W) = \lim_{n\to\infty}\sup_{s\in S}\|s-(h_n+w_0)\| \geq d(S-w_0,F)$. We also have $d(S,F+W) = \inf_{h+w\in F+W}\sup_{s\in S}\|s-(h+w)\| \leq d(S-w_0,F)$. Hence the claim holds. Therefore, $\{h_n\} \subseteq F$ is a minimizing sequence for $S-w_0$. Since F is simultaneously strongly proximinal, there exist a subsequence $\{h_{n_k}\}$ and a sequence $\{z_k\} \subseteq L_F(S-w_0)$ such that $\|h_{n_k}-z_k\| \to 0$. As $z_k \in L_F(S-w_0)$, $\sup_{s\in S}\|(s-w_0)-z_k\| = d(S-w_0,F) = d(S,F+W)$, i.e. $w_0+z_k \in L_{F+W}(S)$. Now $\{w_0+z_k\} \subseteq L_{F+W}(S)$ and $\|h_{n_k}+w_{n_k}-(w_0+z_k)\| \to 0$. Thus $F+W$ is a simultaneously strongly proximinal subspace of X. □

If we take $L_F(S)$ to be compact for every bounded subset S of X then from the proof of the above theorem, we obtain

Corollary 3.4 *Let F and W be subspaces of a Banach space X such that F is simultaneously strongly proximinal, W is finite dimensional, and $F+W$ is closed; then $F+W$ is simultaneously approximatively compact.*

We require the following lemma in the proof of the next theorem, which shows that if $W+F$ is simultaneously strongly proximinal then so is its quotient space.

Lemma 3.5 *[8] Let W be a proximinal subspace of a Banach space X; then for any bounded subset S of X we have*

$$d(S,W) = \sup_{s\in S}\inf_{w\in W}\|s-w\|.$$

Theorem 3.6 *Let F be a simultaneously proximinal subspace of a Banach space X and W a subspace of X such that $W+F$ is closed. If $W+F$ is simultaneously strongly proximinal in X then $(W+F)/F$ is simultaneously strongly proximinal in X/F.*

Proof Suppose $W+F$ is simultaneously strongly proximinal in X; then $W+F$ is simultaneously proximinal and so, using Lemma 3.2, $(W+F)/F$ is simultaneously proximinal in X/F.

Let A be any bounded subset of X/F; then $A = S/F$ for some bounded subset S of X (see [11]). Let $\{y_n+F\} \subseteq (W+F)/F$ be any minimizing sequence for S/F, i.e. $\lim_{n\to\infty}\sup_{s\in S}\|(s+F)-(y_n+$

$F)\| = d(S/F, (W+F)/F)$. Then $y_n + F \in L_{(W+F)/F}(S/F, \delta_n)$ for any $\delta_n > 0$ after some stage. Since $y_n + F \in L_{(W+F)/F}(S/F, \delta_n)$

$$\sup_{s \in S} \|(s+F) - (y_n+F)\| < \sup_{s \in S} \|(s+F) - (g+F)\| + \delta_n \ for\ all\ g+F \in (W+F)/F$$

$$\Rightarrow \sup_{s \in S} \inf_{f \in F} \|(s-y_n) - f\| < \sup_{s \in S} \inf_{f \in F} \|(s-g) - f)\| + \delta_n \ for\ all\ g \in (W+F)$$

Using Lemma 3.5 and the proximinality of F, we can find $f_n \in F$ such that

$$\sup_{s \in S} \|(s-y_n) - f_n\| < \inf_{f \in F} \sup_{s \in S} \|(s-g) - f)\| + \delta_n \ for\ all\ g \in (W+F)$$

Letting $\delta_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n \to \infty} \sup_{s \in S} \|(s-y_n) - f_n\| \leq d(S, W+F).$$

Moreover, $d(S, W+F) \leq \lim_{n\to\infty} \sup_{s \in S} \|(s-y_n) - f_n\|$ and so $\lim_{n\to\infty} \sup_{s\in S} \|(s-y_n) - f_n\| = d(S, W+F)$, i.e. $\{y_n + f_n\}$ is a minimizing sequence for S in $W+F$. Since $W+F$ is simultaneously strongly proximinal for S, there exist a subsequence $\{y_{n_k} + f_{n_k}\}$ and $\{z_k\} \subseteq L_{W+F}(S)$ such that $\|(y_{n_k} + f_{n_k}) - z_k\| \to 0$. Now $\|(y_{n_k} + F) - (z_k + F)\| = \inf_{f \in F} \|y_{n_k} - z_k + f\| \leq \|y_{n_k} - z_k + f_{n_k}\| \to 0$. Hence $(W+F)/F$ is simultaneously strongly proximinal for S/F. □

Using Theorem 3.3, we have

Corollary 3.7 *Let F and W be subspaces of a Banach space X such that F is simultaneously strongly proximinal, W is finite dimensional, and $F+W$ is closed; then $(F+W)/F$ is simultaneously strongly proximinal in X/F.*

Concerning strong proximinality in quotient spaces, we have

Theorem 3.8 *Let W and F be subspaces of a Banach space X and $F \subseteq W$ is proximinal in X. If W is simultaneously strongly proximinal in X then W/F is simultaneously strongly proximinal in X/F.*

Proof Since W is simultaneously proximinal, W/F is simultaneously proximinal (see [8]). Let S/F be any bounded subset of X/F and $\{y_n + F\} \subseteq W/F$ be any minimizing sequence for S/F, i.e.

$$\lim_{n \to \infty} \sup_{s+F \in S/F} \|(s+F) - (y_n+F)\| = d(S/F, W/F).$$

Thus $y_n + F \in L_{W/F}(S/F, \delta_n)$ for any $\delta_n > 0$ after some stage. Then $\sup_{s+F \in S/F} \|(s+F) - (y_n + F)\| < \sup_{s+F\in S/F} \|(s+F) - (w+F)\| + \delta_n$ for all $w + F \in W/F$, i.e. $\sup_{s\in S} \inf_{f \in F} \|(s-y_n) - f\| < \sup_{s \in S} \inf_{f\in F} \|(s-w) - f\| + \delta_n$ for all $w \in W$. Using Lemma 3.5 and proximinality of F, we can find $f_n \in F$ such that $\sup_{s\in S} \|(s-y_n) - f_n\| < \inf_{f\in F} \sup_{s\in S} \|(s-w) - f\| + \delta_n < \sup_{s \in S} \|(s-w)\| + \delta_n$ for all $w \in W$. Therefore, letting $\delta_n \to 0$ as $n \to \infty$, we get $\lim_{n\to\infty} \sup_{s\in S} \|s - (y_n + f_n)\| \leq d(S, W)$. We also have $d(S, W) \leq \lim_{n\to\infty} \sup_{s\in S} \|s - (y_n + f_n)\|$. Therefore, $\lim_{n\to\infty} \sup_{s\in S} \|s - (y_n + f_n)\| = d(S, W)$. This implies that $\{y_n + f_n\} \subseteq W$ is a minimizing sequence for S. Since W is simultaneously strongly proximinal for S there exist a subsequence $\{y_{n_k} + f_{n_k}\}$ and $\{z_k\} \subseteq L_W(S)$ such that $\|(y_{n_k} + f_{n_k}) - z_k\| \to 0$. Now $\|(y_{n_k} + F) - (z_k + F)\| = \inf_{f \in F} \|(y_{n_k} - z_k) - f\| \leq \|y_{n_k} - z_k + f_{n_k}\| \to 0$. Hence W/F is simultaneously strongly proximinal for S/F. □

It was proved in [8] that for subspaces W and F of X such that $F \subseteq W$ is simultaneously Chebyshev then W is simultaneously Chebyshev if and only if W/F is simultaneously Chebyshev. Therefore, we have

Corollary 3.9 *Let W and F be subspaces of a Banach space X and $F \subseteq W$ is simultaneously Chebyshev in X. If W is simultaneously strongly Chebyshev then W/F is simultaneously strongly Chebyshev.*

The authors do not know whether the converse parts of Theorems 3.6 and 3.8 hold.

Remarks Taking S to be a singleton set, we obtain several results of strong proximinality proved in [1,10,13].

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References

[1] Bandyopadhyay P, Li Y, Lin BL, Narayana D. Proximinality in Banach Spaces. J Math Anal Appl 2008; 341: 309-317.

[2] Cheney EW. Five problems on best approximation. In: Canad Math Soc Conf Proc 1983; 3: 390-391.

[3] Cheney EW, Wulbert DE. The existence and uniqueness of best approximation. Math Scand 1969; 24: 113-140.

[4] Feder M. On the sum of proximinal subspaces. J Approx Theory 1987; 49: 144-148.

[5] Godefroy G, Indumathi V. Strong proximinality and polyhedral spaces. Rev Mat Complut 2001; 14: 105-125.

[6] Goel DS, Holland ASB, Nasim C, Sahney BN. On best simultaneous approximation in normed linear spaces. Can Math Bulletin 1974; 17: 523-527.

[7] Govindarajulu P. On best simultaneous approximation. J Math Phy Sci 1984; 18: 345-351.

[8] Iranmanesh M, Mohebi H. On best simultaneous approximation in quotient spaces. Anal Theory Appl 2007; 23: 35-49.

[9] Lin PK. A remark on the sum of proximinal subspaces. J Approx Theory 1989; 58: 55-57.

[10] Narayana D. Strong proximinality and renorming. P Am Math Soc 2005; 134: 1167-1172.

[11] Rawashdeh M, Al-Sharif Sh, Domi WB. On the sum of best simultaneously proximinal subsapces. Hacet J Math Stat 2014; 43: 595-602.

[12] Smith MA. Some examples concerning rotundity in Banach spaces. Math Ann 1978; 233: 155-161.

[13] Sun FL, Luo HZ. The sum of strongly proximinal sets. Xiamen Daxue Xuebao Ziran Kexue Ban 2013; 52: 312-315.

[14] Zhang ZH, Liu CY, Zhou Z. Some examples concerning proximinality in Banach spaces. J Approx Theory 2015; 200: 136-143.