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CC-normal topological spaces

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Abstract: A topological space X is called *CC-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. We will investigate this property and produce some examples to illustrate the relation between CC-normality and other weaker kinds of normality.

Key words: Normal, L-normal, C-normal, CC-normal, mildly normal, almost normal, epinormal, submetrizable, π-normal

1. Introduction

A. V. Arhangel'skiĭ introduced in 2012, when he was visiting the Department of Mathematics at King Abduaziz University, a new weaker version of normality, called *C-normality* [2]. A topological space X is called *C-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. We use the idea of this definition to introduce another new weaker version of normality that will be called *CC-normality.* The purpose of this paper is to investigate this property. We prove that normality implies CC-normality but the converse is not true in general. We present some examples to show relationships between CC-normality and other weaker versions of normality such as C-normality, L-normality, almost normality, mild normality, epinormality, and π-normality. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space is a T_1 completely regular space. We do not assume T_2 in the definition of compactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinals α such that $\alpha < \gamma$. The first infinite ordinal is ω_0, the first uncountable ordinal is ω_1, and the successor cardinal of ω_1 is ω_2.

2. CC-normality

Definition 2.1 *A topological space X is called CC-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

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We do not assume T_2 in the above definition. Recall that a topological space X is called L*-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$ [9]. Any normal space is CC-normal, just by taking $X = Y$ and f to be the identity function. We will give an example of a CC-normal space that is neither normal nor locally compact, but first we give a theorem that will be used in the example.

Theorem 2.2 *If X is an L-normal space such that each countably compact subspace is contained in a Lindelöf subspace, then X is CC-normal.*

Proof Let X be any L-normal space such that if A is any countably compact subspace of X; then there exists a Lindelöf subspace B such that $A \subseteq B$. Let Y be a normal space and $f : X \longrightarrow Y$ be a bijective function such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each Lindelöf subspace C of X. Now let A be any countably compact subspace of X. Pick a Lindelöf subspace B of X such that $A \subseteq B$. Then $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism, and hence $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism as $(f_{|_B})_{|_A} = f_{|_A}$. □

By similar arguments, we obtain the following corollary.

Corollary 2.3 *(a) If X is C-normal and any Lindelöf subspace of X is contained in a compact subspace of X, then X is L-normal [9].*

(b) If X is C-normal and any countably compact subspace of X is contained in a compact subspace of X, then X is CC-normal.

(c) If X is CC-normal and any Lindelöf subspace of X is contained in a countably compact subspace of X, then X is L-normal.

Example 2.4 *We modify the Dieudonné plank [15] to define a new topological space. Let*

$$X = ((\omega_2 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_2, \omega_0 \rangle\}.$$

Write $X = A \cup B \cup N$, where $A = \{\langle \omega_2, n\rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0\rangle : \alpha < \omega_2\}$, and $N = \{\langle \alpha, n\rangle : \alpha < \omega_2$ and $n < \omega_0\}$. The topology $\mathcal{T}$ on X is generated by the following neighborhood system: for each $\langle \alpha, n\rangle \in N$, let $\mathcal{B}(\langle \alpha, n\rangle) = \{\{\langle \alpha, n\rangle\}\}$. For each $\langle \omega_2, n\rangle \in A$, let $\mathcal{B}(\langle \omega_2, n\rangle) = \{V_\alpha(n) = (\alpha, \omega_2] \times \{n\} : \alpha < \omega_2\}$. For each $\langle \alpha, \omega_0\rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0\rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. Then X is a Tychonoff nonnormal space that is not locally compact as any basic open neighborhood of any element in A is not Lindelöf, and hence not compact. Now a subset $C \subseteq X$ is countably compact if and only if C satisfies all of the following conditions: (1) $C \cap A$ and $C \cap B$ are both finite; (2) the set $\{\langle \alpha, n\rangle \in C \cap N : \langle \omega_2, n\rangle \in C \cap A\}$ is finite; and (3) the set $\{\langle \alpha, n\rangle \in C \cap N : \langle \alpha, \omega_0\rangle \notin C \cap B, \langle \omega_2, n\rangle \notin C \cap A\}$ is finite. This means that any countably compact subspace is countable and hence Lindelöf. Since the modified Dieudonné plank is L-normal, see [9], by Theorem 2.2, it is CC-normal.

Theorem 2.5 *If X is a T_1 space such that the only countably compact subspaces of X are the finite subsets, then X is CC-normal.*

Proof Let X be a T_1 space such that the only countably compact subspaces of X are the finite subsets of X. By T_1, we conclude that any countably compact subspace of X is discrete. Thus, let $Y = X$ and consider Y with the discrete topology. Then the identity function from X onto Y works. □

C-normality is a generalization of CC-normality because any compact space is countably compact; hence, any CC-normal space is C-normal. Obviously, any countably compact CC-normal space must be normal. Thus, $\omega_1 \times I^{\omega_1}$, where $I = [0,1]$ is the closed unit interval with its usual metric topology and I^{ω_1} is an uncountable product of I, is not CC-normal because it is a countably compact nonnormal space [15], but $\omega_1 \times I^{\omega_1}$ is C-normal being locally compact [2]. The space $\omega_1 \times (\omega_1 + 1)$ is an example of an L-normal space, see [9], which is not CC-normal because it is a countably compact nonnormal space. Here is an example of a CC-normal space that is not L-normal.

Example 2.6 *Consider* $(\mathbb{R}, \mathcal{CC})$*, where* $\mathcal{CC}$ *is the countable complement topology [15]. The space* $(\mathbb{R}, \mathcal{CC})$ *is* T_1 *as any singleton is closed. Let* C *be any countably infinite subset of* $\mathbb{R}$*. For each* $c \in C$*, define* $V_c = (\mathbb{R} \setminus C) \cup \{c\}$*. Then the family* $\{V_c : c \in C\}$ *is a countable open cover for* C *that has no finite subcover, and hence* C *is not countably compact. Thus, the only countably compact subspaces are the finite sets. Therefore, by Theorem 1.5,* $(\mathbb{R}, \mathcal{CC})$ *is* CC*-normal. It is not* L*-normal because it is a Lindelöf nonnormal space.*

In Example 2.10, we give a Tychonoff separable first countable locally compact CC-normal space that is not L-normal.

Theorem 2.7 *CC-normality is a topological property.*

Proof Let X be a CC-normal space and $X \cong Z$. Let Y be a normal space and $f : X \longrightarrow Y$ be a bijective function such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each countably compact subspace C of X. Let $g : Z \longrightarrow X$ be a homeomorphism. Then $f \circ g : Z \longrightarrow Y$ satisfies all requirements. □

Theorem 2.8 *CC-normality is an additive property.*

Proof Let X_α be a CC-normal space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is CC-normal. For each $\alpha \in \Lambda$, pick a normal space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_{\alpha_{|C_\alpha}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each countably compact subspace C_α of X_α. Since Y_α is normal for each $\alpha \in \Lambda$, the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is normal ([4], 2.2.7). Consider the function sum ([4], 2.2.E), $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is countably compact if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is countably compact in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is countably compact, then $(\oplus_{\alpha\in\Lambda} f_\alpha)_{|C}$ is a homeomorphism because $f_{\alpha_{|C\cap X_\alpha}}$ is a homeomorphism for each $\alpha \in \Lambda_0$. □

CC-normality is not a multiplicative property. For example, the normal spaces ω_1 and $\omega_1 + 1$ are both CC-normal, but $\omega_1 \times (\omega_1 + 1)$ is not CC-normal. CC-normality is not hereditary. For example, the space $\omega_1 \times (\omega_1 + 1)$ is not CC-normal while it is a subspace of its Stone–Čech compactification $(\omega_1 + 1)^2$.

It is clear that a function $f : X \longrightarrow Y$ bearing the CC-normality of X need not be continuous. For example, consider the modified Dieudonné plank X, see Example 1.4. Let $Y = X$ with the topology generated by the following neighborhood system: points of B and N have the same local base as in X and each point of A is isolated. Then the identity function from X onto Y is not continuous, but it bears the CC-normality of X. A function $f : X \longrightarrow Y$ bearing the CC-normality of X will be continuous if X is Fréchet. Recall that a space X is called *Fréchet* if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of points of A such that $a_n \longrightarrow x$, see [4].

Theorem 2.9 *If X is CC-normal and Fréchet and $f : X \longrightarrow Y$ bears the CC-normality of X, then f is continuous.*

Proof Assume that X is CC-normal and Fréchet. Let $f : X \longrightarrow Y$ bear of the CC-normality of X. Let $A \subseteq X$ and pick $y \in f(\overline{A})$. Pick the unique $x \in X$ such that $f(x) = y$. Thus, $x \in \overline{A}$. Since X is Fréchet, there exists a sequence $(a_n) \subseteq A$ such that $a_n \longrightarrow x$. The subspace $B = \{x, a_n : n \in \mathbb{N}\}$ of X is countably compact being compact, and thus $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now let $W \subseteq Y$ be any open neighborhood of y. Then $W \cap f(B)$ is open in the subspace $f(B)$ containing y. Since $f(\{a_n : n \in \mathbb{N}\}) \subseteq f(B) \cap f(A)$ and $W \cap f(B) \neq \emptyset$, $W \cap f(A) \neq \emptyset$, hence $y \in \overline{f(A)}$, and thus $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is continuous. □

We conclude from the above proof that if X is C-normal and Fréchet and $f : X \longrightarrow Y$ bears the C-normality of X, then f is continuous. Since any first countable space is Fréchet, the statements are true if X is first countable. In fact, for C-normality the statement is true if X is a k-space, see [9]. For a function that bears L-normality, the following is true: "If X is L-normal and of countable tightness and $f : X \longrightarrow Y$ bears the L-normality of X, then f is continuous.", see [9].

Any L-normal regular separable space of countable tightness is normal, see [9]. This is not true for CC-normality. Here is an example of a CC-normal Tychonoff separable first countable space that is not normal. For simplicity, we will denote the first infinite ordinal just by ω.

Example 2.10 *We choose a suitable Mrówka space. Recall that two countably infinite sets are said to be almost disjoint [16] if their intersection is finite. Call a subfamily of $[\omega]^\omega = \{ A \subset \omega : A$ is infinite$\}$ a mad family [16] on ω if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Let $\mathcal{A}$ be a pairwise almost disjoint subfamily of $[\omega]^\omega$. The Mrówka space $\Psi(\mathcal{A})$* is defined as follows: the underlying set is $\omega \cup \mathcal{A}$, each point of ω is isolated, and a basic open neighborhood of $W \in \mathcal{A}$ has the form $\{W\} \cup (W \setminus F)$, with $F \in [\omega]^{<\omega} = \{B \subseteq \omega : B$ is finite$\}$. It is well known that there exists an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ such that $|\mathcal{A}| > \omega$ and the Mrówka space $\Psi(\mathcal{A})$ is a Tychonoff, separable, first countable, and locally compact space that is neither countably compact nor normal. $\mathcal{A}$ is a *mad* family if and only if $\Psi(\mathcal{A})$ is pseudocompact [11]. Since $\Psi(\mathcal{A})$ is locally compact, it is C-normal, see [2]. Now a subspace C of $\Psi(\mathcal{A})$ is countably compact if and only if $C \cap \mathcal{A}$ is finite and the set $(\omega \cap C) \setminus (\bigcup_{A \in (C \cap \mathcal{A})} A)$ is finite. This means that any countably compact subspace is compact. Thus, by Corollary 2.3(b), $\Psi(\mathcal{A})$ is CC-normal.

3. CC-normality and other properties

Let us recall some definitions.

Definition 3.1 *A subset A of a space X is called a closed domain* of X [4] (also called *regularly closed, κ-closed*) if $A = \overline{\text{int}A}$. A space X is called *mildly normal* [14] (also called *κ-normal* [12]) if for any two disjoint closed domains A and B of X there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [6,7]. A space X is called *almost normal* [13] if for any two disjoint closed subsets A and B of X, one of which is a closed domain, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [8]. A subset A of a space X is called *π-closed* [7] if A is a finite intersection of closed domains of X. A space X is called *π-normal* [7] if for any two disjoint closed subsets A and B of X, one of which is π-closed, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A space X is called *quasinormal* [17] if for any two disjoint π-closed subsets A and B of X, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [7].

It is clear from the definitions that

normal $\Longrightarrow$ π-normal $\Longrightarrow$ almost normal $\Longrightarrow$ mildly normal.

normal $\Longrightarrow$ π-normal $\Longrightarrow$ quasinormal $\Longrightarrow$ mildly normal.

Example 3.2 *Consider $(\mathbb{R}, \mathcal{CF})$, where $\mathcal{CF}$ is the finite complement topology [15]. Since the only closed domains of $(\mathbb{R}, \mathcal{CF})$ are $\emptyset$ and $\mathbb{R}$, $(\mathbb{R}, \mathcal{CF})$ is π-normal and hence quasinormal, almost normal, and mildly normal, but $(\mathbb{R}, \mathcal{CF})$ is not CC-normal because it is countably compact, being a compact, nonnormal space.*

Here is an example of a CC-normal space that is not π-normal.

Example 3.3 *The modified Dieudonné plank X of Example 1.4 is CC-normal but neither quasinormal nor π-normal.*

Proof X is not normal because A and B are closed disjoint subsets that cannot be separated by two disjoint open sets. Note that $\text{int}(A) = \emptyset = \text{int}(B)$. Thus, A and B are not closed domains. We will show that A and B are π-closed sets. Let $E = \{n < \omega_0 : n \text{ is even}\}$ and $O = \{n < \omega_0 : n \text{ is odd}\}$. Let $C = \{\langle \alpha, n\rangle : \alpha < \omega_2, n \in E\} = \omega_2 \times E$ and $D = \{\langle \alpha, n\rangle : \alpha < \omega_2, n \in O\} = \omega_2 \times O$. Then C and D are both open in X, being subsets of N. Thus, $\overline{C}$ and $\overline{D}$ are both closed domains in X, being closures of open sets. Now $\overline{C} = C \cup B \cup \{\langle \omega_2, n\rangle \in A : n \in E\}$ and $\overline{D} = D \cup B \cup \{\langle \omega_2, n\rangle \in A : n \in O\}$, and hence $\overline{C} \cap \overline{D} = B$. Thus, B is π-closed. Now let K and L be subsets of ω_2 such that $K \cap L = \emptyset$, $K \cup L = \omega_2$, and the cofinality of K and L is ω_2; for instance, let K be the set of limit ordinals in ω_2 and L be the set of successor ordinals in ω_2. Let $G = \{\langle \alpha, n\rangle : \alpha \in K, n < \omega_0\} = K \times \omega_0$ and $H = \{\langle \alpha, n\rangle : \alpha \in L, n < \omega_0\} = L \times \omega_0$. Then G and H are both open in X being subsets of N. Thus $\overline{G}$ and $\overline{H}$ are both closed domains in X, being closures of open sets. Now $\overline{G} = G \cup A \cup \{\langle \alpha, \omega_0\rangle \in B : \alpha \in K\}$ and $\overline{H} = H \cup A \cup \{\langle \alpha, \omega_0\rangle \in B : \alpha \in L\}$, and hence $\overline{G} \cap \overline{H} = A$. Thus A is π-closed. Therefore, the modified Dieudonné plank X is CC-normal but neither quasinormal nor π-normal. □

Example 3.4 *$\mathbb{R}$ with the particular point topology $\mathcal{T}_p$, see [15], where the particular point is $p \in \mathbb{R}$, is not CC-normal. Recall that $\mathcal{T}_p = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : p \in U\}$. It is well known that $(\mathbb{R}, \mathcal{T}_p)$ is neither T_1 nor normal and if $A \subseteq \mathbb{R}$, then $\{\{x, p\} : x \in A\}$ is an open cover for A; thus, a subset A of $\mathbb{R}$ is countably compact if and only if it is finite. To see that $(\mathbb{R}, \mathcal{T}_p)$ is not CC-normal, suppose that $(\mathbb{R}, \mathcal{T}_p)$ is CC-normal. Let Y be a normal space and $f : \mathbb{R} \longrightarrow Y$ be a bijection such that the restriction $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each countably compact subspace C of $(\mathbb{R}, \mathcal{T}_p)$. For the space Y, we have only two cases:*

Case 1: Y is T_1. Take $C = \{x, p\}$, where $x \neq p$. Then C is a countably compact subspace of $(\mathbb{R}, \mathcal{T}_p)$. By assumption $f_{|_C} : C \longrightarrow f(C) = \{f(x), f(p)\}$ is a homeomorphism. Since $f(C)$ is a finite subspace of Y and Y is T_1, $f(C)$ is a discrete subspace of Y. Thus, we obtain that $f_{|_C}$ is not continuous, which is a contradiction, as $f_{|_C}$ is a homeomorphism.

Case 2: Y is not T_1. We claim that the topology on Y is coarser than the particular point topology on Y with $f(p)$ as its particular point. To prove this claim, we suppose not. Then there exists a nonempty open set $U \subset Y$ such that $f(p) \notin U$. Pick $y \in U$ and let $x \in \mathbb{R}$ be the unique real number such that $f(x) = y$. Consider $\{x, p\}$. Note that $x \neq p$ because $f(x) = y \in U$, $f(p) \notin U$, and f is one-to-one. Consider $f_{|_{\{x,p\}}} : \{x, p\} \longrightarrow \{y, f(p)\}$. Now $\{y\}$ is open in the subspace $\{y, f(p)\}$ of Y because $\{y\} = U \cap \{y, f(p)\}$, *but* $f^{-1}(\{y\}) = \{x\}$ *and* $\{x\}$ *is not open in the subspace* $\{x, p\}$ *of* $(\mathbb{R}, \mathcal{T}_p)$*, which means* $f_{|_{\{x,p\}}}$ *is not continuous. This is a contradiction, and our claim is proved. However, any topology coarser than the particular point topology has no disjoint nonempty open sets and therefore cannot be normal, so we get a contradiction as* Y *is assumed to be normal. Therefore,* $(\mathbb{R}, \mathcal{T}_p)$ *is not* CC*-normal.*

Since the only closed domains in $(\mathbb{R}, \mathcal{T}_p)$ *are* $\emptyset$ *and* $\mathbb{R}$*,* $(\mathbb{R}, \mathcal{T}_p)$ *is almost normal.*

Recall that a topological space $(X, \mathcal{T})$ is called *submetrizable* if there exists a metric d on X such that the topology $\mathcal{T}_d$ on X generated by d is coarser than $\mathcal{T}$, i.e. $\mathcal{T}_d \subseteq \mathcal{T}$, see [5]. A topological space $(X, \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4 [2].

The space $\omega_1 + 1$ is CC-normal being T_2-compact but it is not submetrizable. Indiscrete spaces with more than one element and $(\mathbb{R}, \mathcal{CC})$ are examples of CC-normal spaces that are not epinormal because they are not Hausdorff. It was proved in [2] that any submetrizable space is C-normal and any epinormal space is C-normal. We still do not know if submetrizability implies CC-normality or if epinormality implies CC-normality or not.

We discovered that the Alexandroff duplicate space of a CC-normal space is CC-normal. Recall that the Alexandroff duplicate space $A(X)$ of a space X is defined as follows: let X be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in X' by x' and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Then $\mathcal{B} = \{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ will generate a unique topology on $A(X)$ such that $\mathcal{B}$ is its neighborhood system. $A(X)$ with this topology is called the *Alexandroff duplicate of* X [1,4].

Theorem 3.5 *If* X *is* CC*-normal, then its Alexandroff duplicate* $A(X)$ *is also* CC*-normal.*

Proof Let X be any CC-normal space. Pick a normal space Y and a bijective function $f : X \longrightarrow Y$ such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each countably compact subspace $C \subseteq X$. Consider the Alexandroff duplicate spaces $A(X)$ and $A(Y)$ of X and Y respectively. It is well known that the Alexandroff duplicate of a normal space is normal [1], and hence $A(Y)$ is also normal. Define $g : A(X) \longrightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$. If $a \in X'$, let b be the unique element in X such that $b' = a$, and then define $g(a) = (f(b))'$. Then g is a bijective function. Now a subspace $C \subseteq A(X)$ is countably compact if and only if $C \cap X$ is countably compact in X and for each open set U in X with $C \cap X \subseteq U$, we have that $(C \cap X') \setminus U'$ is finite. Let $C \subseteq A(X)$ be any countably compact subspace. We show that $g_{|_C} : C \longrightarrow g(C)$ is a homeomorphism. Let $a \in C$ be arbitrary. If $a \in C \cap X'$, let $b \in X$ be the unique element such that $b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$ we have that $\{a\}$ is open in C and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in C \cap X$, let W be any open set in Y such that $g(a) = f(a) \in W$. Consider $H = (W \cup (W' \setminus \{(f(a))'\})) \cap g(C)$, which is a basic open neighborhood of $f(a)$ in $g(C)$. Since $f_{|_{C\cap X}} : C \cap X \longrightarrow f(C \cap X)$ is a homeomorphism, there exists an open set U in X with $a \in U$ and $f_{|_{C\cap X}}(U \cap C) \subseteq W \cap f(C \cap X)$. Now $(U \cup (U' \setminus \{a'\})) \cap C = G$ is open in C such that $a \in G$ and $g_{|_C}(G) \subseteq H$. Therefore, $g_{|_C}$ is continuous. Now we show that $g_{|_C}$ is open. Let $K \cup (K' \setminus \{k'\})$, where $k \in K$ and K is open in X, be any basic open set in $A(X)$; then $(K \cap C) \cup ((K' \cap C) \setminus \{k'\})$ is a basic open set in C. Since $X \cap C$ is countably compact in X, $g_{|_C}(K \cap (X \cap C)) = f_{|_{X\cap C}}(K \cap (X \cap C))$ is open in $Y \cap f(C \cap X)$ as $f_{|_{X\cap C}}$ is a homeomorphism. Thus $g_{|_C}(K \cap C)$ is open in $Y \cap f(X \cap C)$. Also, $g((K' \cap C) \setminus \{k'\})$ is open in $Y' \cap g(C)$ being a set of isolated points. Thus, $g_{|_C}$ is an open function. Therefore, $g_{|_C}$ is a homeomorphism. □

The following problems are still open:

1. Is CC-normality hereditary with respect to closed subspaces?

2. If X is a Dowker space, is $X \times I$ then CC-normal?

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References

[1] Alexandroff PS, Urysohn PS. Mémoire sur les espaces topologiques compacts. Amsterdam, the Netherlands: Koninklijke Akademie van Wetenschappen, 1929 (in French).

[2] AlZahrani S, Kalantan L. C-normal topological property. Filomat 2017; 31: 407-411.

[3] Engelking R. On the double circumference of Alexandroff. Bull Acad Pol Sci Ser Astron Math Phys 1968; 16; 8: 629-634.

[4] Engelking R. General Topology. Warsaw, Poland: PWN, 1977.

[5] Gruenhage G. Generalized metric spaces. In: Kunen K, editor. Handbook of Set-Theoretic Topology. Amsterdam, the Netherlands: North-Holland, 1984, pp. 428-434.

[6] Kalantan L. Results about κ-normality. Topol Appl 2002; 125: 47-62.

[7] Kalantan L. π-normal topological spaces. Filomat 2008; 22: 173-181.

[8] Kalantan L, Allahabi F. On almost normality. Demonstratio Mathematica 2008; 41: 961-968.

[9] Kalantan L, Saeed M. L-normality. Topol P (in press).

[10] Kalantan L, Szeptycki P. κ-normality and products of ordinals. Topol Appl 2002; 123: 537-545.

[11] Mrówka S. On completely regular spaces. Fund Math 1954; 41: 105-106.

[12] Shchepin EV. Real valued functions and spaces close to normal. Sib J Math 1972; 13: 1182-1196.

[13] Singal M, Arya S. Almost normal and almost completely regular spaces. Kyungpook Math J 1970; 25: 141-152.

[14] Singal M, Singal AR. Mildly normal spaces. Kyungpook Math J 1973; 13: 29-31.

[15] Steen L, Seebach JA. Counterexamples in Topology. Mineola, NY, USA: USA; Dover Publications, 1995.

[16] van Douwen EK. The integers and topology. In: Kunen K, editor. Handbook of Set-Theoretic Topology. Amsterdam, the Netherlands: North-Holland, 1984, pp. 111-167.

[17] Zaitsev V. On certain classes of topological spaces and their bicompactifications. Dokl Akad Naur SSSR 1968; 178: 778-779.