Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2017) 41: 756 – 774
© TÜBİTAK
doi:10.3906/mat-1512-62

Arithmetic properties of ℓ-regular overpartition pairs

Megadahalli Siddanaika Mahadeva NAIKA*, **Chandrappa SHIVASHANKAR**
Department of Mathematics, Bangalore University, Central College Campus, Bangalore, Karnataka, India

Received: 14.12.2015 • **Accepted/Published Online:** 26.07.2016 • **Final Version:** 22.05.2017

Abstract: In this paper, we investigate the arithmetic properties of ℓ-regular overpartition pairs. Let $\overline{B}_\ell(n)$ denote the number of ℓ-regular overpartition pairs of n. We will prove the number of Ramanujan-like congruences and infinite families of congruences modulo 3, 8, 16, 36, 48, 96 for $\overline{B}_3(n)$ and modulo 3, 16, 64, 96 for $\overline{B}_4(n)$. For example, we find that for all nonnegative integers α and n, $\overline{B}_3(3^\alpha(3n+2)) \equiv 0 \pmod 3$, $\overline{B}_3(3^\alpha(6n+4)) \equiv 0 \pmod 3$, and $\overline{B}_4(8n+7) \equiv 0 \pmod{64}$.

Key words: Congruences, theta function, overpartition pair, regular partition

1. Introduction

An overpartition of a nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n, and where the first occurrence of a number may be overlined. For example, eight overpartitions of the integer 3 are 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$, and $\overline{1}+1+1$. Due to Corteel and Lovejoy [9], the generating function of overpartition is:

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + \cdots . \tag{1.1}$$

where $\overline{p}(n)$ denotes the number of overpartitions of n. Many mathematicians extensively studied the arithmetic properties of $\overline{p}(n)$. For more information, see [12, 13, 17, 18, 23, 31].

An overpartition pair of a positive integer n was defined by Lovejoy [21] as a pair of overpartitions (μ, λ) where the sum of all the parts is n. Lovejoy denoted the number of overpartition pairs of a positive integer n by $\overline{pp}(n)$; for convenience, define $\overline{pp}(0) = 1$. The generating function for overpartition pairs is

$$\sum_{n=0}^{\infty} \overline{pp}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2} = 1 + 4q + 12q^2 + 32q^3 + 76q^4 + 168q^5 + 352q^6 + \cdots . \tag{1.2}$$

Bringmann and Lovejoy [6] defined a rank for overpartition pairs to study the congruence properties of $\overline{pp}(n)$ and they found many Ramanujan-type congruences for $\overline{pp}(n)$. For example, $\overline{pp}(3n+2) \equiv 0 \pmod 3$. For more details on arithmetic properties of overpartition pairs one can see [8, 15, 19].

*Correspondence: msmnaika@rediffmail.com
2010 *AMS Mathematics Subject Classification:* 11P83, 05A15, 05A17.

For a positive integer $\ell > 1$, a partition is called ℓ-regular if none of the parts are divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function for $b_\ell(n)$ satisfies

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}, \tag{1.3}$$

where

$$f_k := (q^k;q^k)_\infty \quad \text{and} \quad (a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1.$$

The regular partitions have been studied by a number of mathematicians; see [10, 14, 16, 24, 27, 30].

Recently, Lovejoy [22] studied the function $\overline{A}_\ell(n)$, which counts the number of overpartitions of n with no parts divisible by ℓ, and Andrews [2] defined a new function $\overline{C}_{k,i}(n)$ as the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. The generating function of $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}, \tag{1.4}$$

for $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$. With a note that $\overline{A}_3(n) = \overline{C}_{3,1}(n)$. Andrews proved the following two congruences:

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \tag{1.5}$$

Chen et al. [7] and Ahmed and Baruah [1] proved many infinite families of congruences of $\overline{C}_{k,i}(n)$ for different values of k and i.

Following these works, more recently, Shen [25] proved a number of arithmetic properties of $\overline{A}_\ell(n)$ and gave a combinatorial interpretation for $\overline{A}_3(9n+3)$ and $\overline{A}_3(9n+6)$ being divisible by 3. Shen called $\overline{A}_\ell(n)$ the number of ℓ-regular overpartitions of n. The generating function of $\overline{A}_\ell(n)$ is

$$\sum_{n=0}^{\infty} \overline{A}_\ell(n)q^n = \frac{f_2}{f_1^2}\frac{f_\ell^2}{f_{2\ell}} = \frac{\varphi(-q^\ell)}{\varphi(-q)}. \tag{1.6}$$

In this paper, we shall study the arithmetic properties of ℓ-regular overpartition pairs of n. An ℓ-regular overpartition pair of n is a pair of ℓ-regular overpartitions (μ, λ) where the sum of all the parts is n. We have the generating function

$$\sum_{n=0}^{\infty} \overline{B}_\ell(n)q^n = \frac{f_2^2}{f_1^4}\frac{f_\ell^4}{f_{2\ell}^2} = \frac{\varphi(-q^\ell)^2}{\varphi(-q)^2}, \tag{1.7}$$

where $\overline{B}_\ell(n)$ denotes the number of ℓ-regular overpartitions of n.

We will prove the number of arithmetic properties of the partition function $\overline{B}_\ell(n)$ for $\ell = 3, 4$.

2. Preliminary results

We define Ramanujan's general theta function $f(a,b)$ for $|ab|<1$ as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$$
$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

The important cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{2.3}$$

$$f(-q) := f(-q^2,-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty, \tag{2.4}$$

and

$$\chi(-q) := (q;q^2)_\infty = \frac{f_1}{f_2}. \tag{2.5}$$

The following dissection formula of Hirschhorn and Sellers [13] plays an important role in our work.

Lemma 2.1 *We have*

$$\frac{1}{\varphi(-q)} = \frac{\varphi(-q^9)}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2\right) \tag{2.6}$$
$$= \frac{1}{\varphi(-q^4)^4}(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3), \tag{2.7}$$

where

$$A(q) = \frac{f_1 f_6^2}{f_2 f_3}.$$

Lemma 2.2 *We have*

$$\psi(q) = f(q^3,q^6) + q\psi(q^9), \tag{2.8}$$
$$\varphi(q) = \varphi(q^9) + 2qf(q^3,q^{15}), \tag{2.9}$$
$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2. \tag{2.10}$$

Proof For the proofs of (2.8) and (2.9), see [5, p. 49, Corollary]. Adding identities (v) and (vi) of [5, p. 40, Entry 25.], we can easily get (2.10). □

Lemma 2.3 *[3] For $1 \leq k \leq 7$, we have*

$$r_k(8n+k) = 2^{k-1}\left(2+\binom{k}{4}\right)t_k(n), \tag{2.11}$$

where $r_k(n)$ denotes the number of representations of n as sum of k squares, and $t_k(n)$ denotes the number of representations of n as the sum of k triangular numbers for nonnegative integers n and k.

Lemma 2.4 *The following 2-dissection holds:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.12}$$

Xia and Yao [29] proved (2.12) by employing an addition formula for theta functions.

Lemma 2.5 *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.13}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{2.14}$$

Proof Hirschhorn et al. [11] proved equation (2.13). In the same paper the authors also proved

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{2.15}$$

Replacing q by $-q$ in (2.15) and using $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$, we obtain (2.14). □

3. Congruence results for $\overline{B}_3(n)$

In this section, we establish several congruences for $\overline{B}_3(n)$.

Lemma 3.1 *We have*

$$\sum_{n=0}^{\infty} \overline{B}_3(3n)q^n = \frac{\varphi(-q^3)^6}{\varphi(-q)^6} + 16q\frac{\varphi(-q^3)^3}{\varphi(-q)^6}A(q)^3, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+1)q^n = 4\frac{\varphi(-q^3)^5}{\varphi(-q)^6}A(q) + 16q\frac{\varphi(-q^3)^2}{\varphi(-q)^6}A(q)^4, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+2)q^n = 12\frac{\varphi(-q^3)^4}{\varphi(-q)^6}A(q)^2. \tag{3.3}$$

Proof Setting $\ell = 3$ in (1.7),

$$\sum_{n=0}^{\infty} \overline{B}_3(n)q^n = \frac{f_2^2}{f_1^4}\frac{f_3^4}{f_6^2} = \frac{\varphi(-q^3)^2}{\varphi(-q)^2}. \tag{3.4}$$

Substituting (2.6) into (3.4), we see that

$$\sum_{n=0}^{\infty}\overline{B}_3(n)q^n = \frac{\varphi(-q^9)^2}{\varphi(-q^3)^6}\left(\varphi(-q^9)^2+2q\varphi(-q^9)A(q^3)+4q^2A(q^3)^2\right)^2$$
$$= \frac{\varphi(-q^9)^2}{\varphi(-q^3)^6}\left(\varphi(-q^9)^4+4q^2\varphi(-q^9)^2A(q^3)^2+16q^4A(q^3)^4\right.$$
$$\left.+4q\varphi(-q^9)^3A(q^3)+8q^2\varphi(-q^9)^2A(q^3)^2+16q^3\varphi(-q^9)A(q^3)^3\right).$$

The lemma follows by extracting the terms involving q^{3n+j} for $j=0,1,2$. □

Theorem 3.1 *For all integers $n\geq 0$ and $\alpha\geq 0$,*

$$\overline{B}_3(3^{\alpha+1}n)\equiv\overline{B}_3(n)\pmod 3, \tag{3.5}$$

$$\overline{B}_3(3^{\alpha}(3n+2))\equiv 0\pmod 3, \tag{3.6}$$

$$\overline{B}_3(3^{\alpha}(6n+4))\equiv 0\pmod 3, \tag{3.7}$$

$$\overline{B}_3(3^{\alpha}(8n+4))\equiv 0\pmod 3, \tag{3.8}$$

$$\sum_{n=0}^{\infty}\overline{B}_3(6n+1)q^n\equiv f_1f_3\pmod 3. \tag{3.9}$$

Proof For any prime p, we have

$$(q;q)^p\equiv(q^p;q^p)\pmod p. \tag{3.10}$$

It is easy to see that

$$\varphi(-q)^3\equiv\varphi(-q^3)\pmod 3. \tag{3.11}$$

In view of (3.4) and (3.11), we see that

$$\sum_{n=0}^{\infty}\overline{B}_3(n)q^n\equiv\frac{\varphi(-q)^6}{\varphi(-q)^2}\equiv\varphi(-q)\varphi(-q^3)\pmod 3. \tag{3.12}$$

Employing (2.9) in (3.12), we have

$$\sum_{n=0}^{\infty}\overline{B}_3(n)q^n\equiv\varphi(-q^3)\varphi(-q^9)-2q\varphi(-q^3)f(-q^3,-q^{15})\pmod 3, \tag{3.13}$$

which implies that

$$\overline{B}_3(3n+2)\equiv 0\pmod 3, \tag{3.14}$$

$$\sum_{n=0}^{\infty}\overline{B}_3(3n)q^n\equiv\varphi(-q)\varphi(-q^3)\pmod 3, \tag{3.15}$$

and

$$\sum_{n=0}^{\infty}\overline{B}_3(3n+1)q^n\equiv\varphi(-q)f(-q,-q^5)\pmod 3. \tag{3.16}$$

Congruences (3.5) and (3.6) follow from (3.12), (3.14), and (3.15) and by induction on α.

From [4, Eq. (3.17)], we have

$$\varphi(-q)f(-q,-q^5) = \psi(q^2)f(q^2,q^4) - 3q\frac{f_{12}^3}{f_4}. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+1)q^n \equiv \psi(q^2)f(q^2,q^4) \pmod 3,$$

which yields

$$\overline{B}_3(6n+4) \equiv 0 \pmod 3$$

and

$$\sum_{n=0}^{\infty} \overline{B}_3(6n+1)q^n \equiv \psi(q)f(q,q^2) \pmod 3. \tag{3.18}$$

However,

$$f(q,q^2) = \frac{f_2 f_3^2}{f_1 f_6} \tag{3.19}$$

and for any integer $m \geq 1$, we have

$$f_{3m} \equiv f_m^3 \pmod 3. \tag{3.20}$$

Combining (2.3), (3.18), and (3.19), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(6n+1)q^n \equiv \frac{f_2^3 f_3^2}{f_1^2 f_6} \pmod 3. \tag{3.21}$$

Congruence (3.9) follows from (3.20) and (3.21).

Again, from equation (3.12), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(n)q^n \equiv \varphi(-q)^4 \equiv \sum_{n=0}^{\infty}(-1)^n r_4(n)q^n \pmod 3, \tag{3.22}$$

which implies that

$$\overline{B}_3(n) \equiv (-1)^n r_4(n) \pmod 3.$$

Utilizing (2.11) in the above equation with $k=4$, we obtain

$$\overline{B}_3(8n+4) \equiv 0 \pmod 3. \tag{3.23}$$

Congruence (3.8) follows from (3.23) and (3.5). □

Theorem 3.2 *For any prime* $p \equiv 5 \pmod 6$, $\alpha \geq 1$, *and* $n \geq 0$, *we have*

$$\overline{B}_3(6p^{2\alpha}n + (6i+p)p^{2\alpha-1}) \equiv 0 \pmod 3, \tag{3.24}$$

where $i = 1, 2, ..., p-1$.

Proof Utilizing (2.4), we can rewrite (3.9) as

$$\sum_{n=0}^{\infty} \overline{B}_3(6n+1)q^n \equiv \sum_{k,m=-\infty}^{\infty} (-1)^{k+m} q^{k(3k+1)/2+3m(3m+1)/2} \pmod 3. \tag{3.25}$$

Consider the congruence equation

$$\frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} \equiv \frac{4p^2-4}{24} \pmod p, \tag{3.26}$$

which is equivalent to

$$(6k+1)^2 + 3 \cdot (6m+1)^2 \equiv 0 \pmod p.$$

Since $(\frac{-3}{p}) = -1$ for $p \equiv 5 \pmod 6$, we deduce that

$$6k+1 \equiv 6m+1 \equiv 0 \pmod p.$$

If $p \equiv 1 \pmod 6$, then $k \equiv m \equiv \dfrac{p-1}{6} \pmod p$. Letting

$$k = rp + \tfrac{p-1}{6} \quad \text{and} \quad m = sp + \tfrac{p-1}{6},$$

we have

$$\frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} = \frac{p^2-1}{6} + p^2\left(\frac{3r^2+r}{2}\right) + 3p^2\left(\frac{3s^2+s}{2}\right).$$

If $p \equiv -1 \pmod 6$, then $k \equiv m \equiv \dfrac{-p-1}{6} \pmod p$. Letting

$$k = -rp - \tfrac{p+1}{6} \quad \text{and} \quad m = -sp - \tfrac{p+1}{6},$$

we have

$$\frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} = \frac{p^2-1}{6} + p^2\left(\frac{3r^2+r}{2}\right) + 3p^2\left(\frac{3s^2+s}{2}\right).$$

By the above analysis, extracting the terms in which the powers of q are $pn + \frac{p^2-1}{6}$ from (3.25) and dividing the resulting equation by $q^{\frac{p^2-1}{6}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(6pn+p^2)q^{pn} \equiv \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{p^2(3r^2+r)/2+3p^2(3s^2+s)/2} \pmod 3,$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(6p^2n+p^2)q^n \equiv f_1 f_3 \pmod 3, \tag{3.27}$$

and for $n \geq 0$,

$$\overline{B}_3(6p^2n + 6pi + p^2) \equiv 0 \pmod{3}, \tag{3.28}$$

where i is an integer and $1 \leq i \leq p-1$. Combining (3.27) and (3.9), we see that for $n \geq 0$,

$$\overline{B}_3(6p^2n + p^2) \equiv \overline{B}_3(6n+1) \pmod{3}. \tag{3.29}$$

By (3.29) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$\overline{B}_3(6p^{2\alpha}n + p^{2\alpha}) \equiv \overline{B}_3(6n+1) \pmod{3}. \tag{3.30}$$

Replacing n by $p^2n + pi + \frac{p^2-1}{6}$ in (3.30) and using (3.28), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$\overline{B}_3(6p^{2\alpha+2}n + 6p^{2\alpha+1}i + p^{2\alpha+2}) \equiv 0 \pmod{3}. \tag{3.31}$$

Congruence (3.24) follows from (3.31). □

Theorem 3.3 *For all nonnegative integers α and n,*

$$\overline{B}_3(6 \cdot 4^{\alpha+1}n + 2 \cdot 4^{\alpha+1}) \equiv \overline{B}_3(6n+2) \pmod{36}, \tag{3.32}$$

$$\overline{B}_3(6 \cdot 4^{\alpha+1}n + 5 \cdot 4^{\alpha+1}) \equiv 0 \pmod{36}. \tag{3.33}$$

Proof Using definitions of $\varphi(-q)$ and $A(q)$, we can rewrite (3.3) as

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+2)q^n = 12\frac{f_2^4 f_3^6}{f_1^{10}}. \tag{3.34}$$

Applying (3.20) in (3.34), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+2)q^n \equiv 12\frac{f_2^4 f_3^3}{f_1} \pmod{36}.$$

Substituting (2.13) into the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+2)q^n \equiv 12\frac{f_2^2 f_4^3 f_6^2}{f_{12}} + 12q\frac{f_2^4 f_{12}^3}{f_4} \pmod{36},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(6n+2)q^n \equiv 12\frac{f_1^2 f_2^3 f_3^2}{f_6} \equiv 12\frac{f_3^3}{f_1} \pmod{36}. \tag{3.35}$$

Substituting (2.13) into (3.35), we find that

$$\sum_{n=0}^{\infty} \overline{B}_3(6n+2)q^n \equiv 12\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 12q\frac{f_{12}^3}{f_4} \pmod{36}, \tag{3.36}$$

which yields

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+8)q^n \equiv 12\frac{f_6^3}{f_2} \pmod{36},$$

which implies that

$$\overline{B}_3(24n+8) \equiv \overline{B}_3(6n+2) \pmod{36} \tag{3.37}$$

and

$$\overline{B}_3(24n+20) \equiv 0 \pmod{36}. \tag{3.38}$$

Congruences (3.32) and (3.33) follow from (3.37), (3.38), and induction on α. □

Theorem 3.4 *For any prime $p \equiv 5 \pmod 6$, $\alpha \geq 1$, and $n \geq 0$, we have*

$$\overline{B}_3(12p^{2\alpha}n + (12i+2p)p^{2\alpha-1}) \equiv 0 \pmod{36}, \tag{3.39}$$

where $i = 1, 2, ..., p-1$.

Proof Utilizing (3.20) in (3.36), we find that

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+2)q^n \equiv 12 f_1 f_3 \pmod{36}. \tag{3.40}$$

We omit the remaining part of the proof because it is exactly similar to the proof of (3.24). □

Theorem 3.5 *For all integers $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{B}_3(3 \cdot 4^{\alpha+1}n + 11 \cdot 4^{\alpha}) \equiv 0 \pmod{48}, \tag{3.41}$$

$$\overline{B}_3(24n+14) \equiv 0 \pmod{48}, \tag{3.42}$$

$$\overline{B}_3(24n+23) \equiv 0 \pmod{96}, \tag{3.43}$$

$$\overline{B}_3(48n+14) \equiv 0 \pmod{96}. \tag{3.44}$$

Proof By mathematical induction it is easy to see that

$$f_{2m} \equiv f_m^2 \pmod{2}, \tag{3.45}$$

$$f_{2m}^2 \equiv f_m^4 \pmod{2^2}, \tag{3.46}$$

$$f_{2m}^4 \equiv f_m^8 \pmod{2^3}. \tag{3.47}$$

Employing (3.47) in (3.34), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(3n+2)q^n \equiv 12\frac{f_3^6}{f_1^2} \equiv 12\sum_{n=0}^{\infty} A_3(n)q^n \pmod{96}, \tag{3.48}$$

which implies that

$$\overline{B}_3(3n+2) \equiv 12 A_3(n) \pmod{96}, \tag{3.49}$$

where $A_3(n)$ denotes the number of bipartitions with 3-cores of n. Congruence (3.41) follows from Theorem 2.3 of [20]. Congruence (3.43) follows from Theorem 2.6 (Eq. 2.16) of [20]. Finally, congruences (3.42) and (3.44) follow from Theorem 1.5 of [28]. □

Theorem 3.6 *For all nonnegative integers α and n,*

$$\overline{B}_3(4\cdot 3^{2\alpha+2}n+3^{2\alpha+2})\equiv \overline{B}_3(4n+1)\pmod{2^4}, \tag{3.50}$$

$$\overline{B}_3(4\cdot 3^{2\alpha+2}n+7\cdot 3^{2\alpha+1})\equiv 0\pmod{2^4}, \tag{3.51}$$

$$\overline{B}_3(4\cdot 3^{2\alpha+2}n+11\cdot 3^{2\alpha+1})\equiv 0\pmod{2^4}. \tag{3.52}$$

Proof Setting $\ell=3$ in (1.7), we have

$$\sum_{n=0}^{\infty}\overline{B}_3(n)q^n=\frac{f_2^2}{f_1^4}\frac{f_3^4}{f_6^2}. \tag{3.53}$$

Substituting (2.12) into (3.53), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{B}_3(n)q^n&=\frac{f_2^2}{f_6^2}\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right)^2\\&=\frac{f_4^8f_{12}^4}{f_2^8f_8^2f_{24}^2}+4q^2\frac{f_4^2f_6^2f_8^2f_{24}^2}{f_2^6f_{12}^2}+4q\frac{f_4^5f_6f_{12}}{f_2^7}.\end{aligned} \tag{3.54}$$

Extracting the terms involving q^{2n+1} from both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_3(2n+1)q^n=4\frac{f_2^5f_3f_6}{f_1^7}, \tag{3.55}$$

but

$$\frac{f_2^5f_3f_6}{f_1^7}\equiv\frac{f_2^3f_3f_6}{f_1^3}\pmod{2^2}. \tag{3.56}$$

Combining (3.55) and (3.56), we have

$$\sum_{n=0}^{\infty}\overline{B}_3(2n+1)q^n\equiv 4\frac{f_2^3f_3f_6}{f_1^3}\pmod{2^4}. \tag{3.57}$$

Substituting (2.14) into (3.57), we obtain

$$\sum_{n=0}^{\infty}\overline{B}_3(2n+1)q^n\equiv 4\frac{f_4^6f_6^4}{f_2^6f_{12}^2}+12q\frac{f_4^2f_6^2f_{12}^2}{f_2^4}\pmod{2^4}, \tag{3.58}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{B}_3(4n+1)q^n\equiv 4\frac{f_2^6f_3^4}{f_1^6f_6^2}\pmod{2^4}, \tag{3.59}$$

but

$$\frac{f_2^6 f_3^4}{f_1^6 f_6^2} \equiv \frac{f_2^4}{f_1^2} \equiv \psi(q)^2 \pmod{2^2}. \tag{3.60}$$

In view of (3.59) and (3.60), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(4n+1)q^n \equiv 4\psi(q)^2 \pmod{2^4}. \tag{3.61}$$

Substituting (2.8) into (3.61), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(4n+1)q^n \equiv 4f(q^3,q^6)^2 + 4q^2\psi(q^9)^2 + 8qf(q^3,q^6)\psi(q^9) \pmod{2^4}, \tag{3.62}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+9)q^n \equiv 4\psi(q^3)^2 \pmod{2^4}, \tag{3.63}$$

which yields

$$\overline{B}_3(36n+9) \equiv \overline{B}_3(4n+1) \pmod{2^4}, \tag{3.64}$$

$$\overline{B}_3(36n+21) \equiv 0 \pmod{2^4}, \tag{3.65}$$

and

$$\overline{B}_3(36n+33) \equiv 0 \pmod{2^4}. \tag{3.66}$$

Congruences (3.50), (3.51), and (3.52) follow from (3.64), (3.65), (3.66), and induction on α. □

Theorem 3.7 *For any prime $p \equiv 3 \pmod 4$, $\alpha \geq 1$, and $n \geq 0$, we have*

$$\overline{B}_3(12p^{2\alpha}n + (12i+p)p^{2\alpha-1}) \equiv 0 \pmod{2^4}, \tag{3.67}$$

where $i = 1, 2, ..., p-1$.

Proof Extracting the terms involving q^{3n} from both sides of (3.62), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+1)q^n \equiv 4f(q,q^2)^2 \pmod{2^4}, \tag{3.68}$$

but

$$f(q,q^2)^2 = \frac{f_2^2 f_3^4}{f_1^2 f_6^2} \equiv f_1^2 \pmod 4. \tag{3.69}$$

Combining (3.68) and (3.69), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+1)q^n \equiv 4f_1^2 \pmod{2^4}. \tag{3.70}$$

Utilizing (2.4), we can rewrite (3.70) as

$$\sum_{n=0}^{\infty}\overline{B}_3(12n+1)q^n \equiv 4\sum_{k,m=-\infty}^{\infty}(-1)^{k+m}q^{k(3k+1)/2+m(3m+1)/2} \pmod{2^4}. \tag{3.71}$$

Consider the congruence equation

$$\frac{3k^2+k}{2}+\frac{3m^2+m}{2}\equiv\frac{p^2-1}{12} \pmod{p}, \tag{3.72}$$

which is equivalent to

$$(6k+1)^2+(6m+1)^2\equiv 0 \pmod{p}.$$

Since $(\frac{-1}{p})=-1$ for $p\equiv 3 \pmod 4$, we deduce that

$$6k+1\equiv 6m+1\equiv 0 \pmod{p}.$$

If $p\equiv 1 \pmod 6$, then $k\equiv m\equiv \dfrac{p-1}{6} \pmod p$. Letting

$$k=rp+\tfrac{p-1}{6} \quad\text{and}\quad m=sp+\tfrac{p-1}{6},$$

we have

$$\frac{3k^2+k}{2}+\frac{3m^2+m}{2}=\frac{p^2-1}{12}+p^2\left(\frac{3r^2+r}{2}\right)+p^2\left(\frac{3s^2+s}{2}\right).$$

If $p\equiv -1 \pmod 6$, then $k\equiv m\equiv \dfrac{-p-1}{6} \pmod p$. Letting

$$k=-rp-\tfrac{p+1}{6} \quad\text{and}\quad m=-sp-\tfrac{p+1}{6},$$

we have

$$\frac{3k^2+k}{2}+\frac{3m^2+m}{2}=\frac{p^2-1}{12}+p^2\left(\frac{3r^2+r}{2}\right)+p^2\left(\frac{3s^2+s}{2}\right).$$

By the above analysis, extracting the terms in which the powers of q are $pn+\frac{p^2-1}{12}$ from (3.71) and dividing the resulting equation by $q^{\frac{p^2-1}{12}}$, we obtain

$$\sum_{n=0}^{\infty}\overline{B}_3(12pn+p^2)q^{pn}\equiv 4\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}q^{p^2(3r^2+r)/2+p^2(3s^2+s)/2} \pmod{2^4},$$

which implies that

$$\sum_{n=0}^{\infty}\overline{B}_3(12p^2n+p^2)q^n\equiv 4f_1^2 \pmod{2^4}, \tag{3.73}$$

and for $n\geq 0$,

$$\overline{B}_3(12p^2n+12pi+p^2)\equiv 0 \pmod{2^4}, \tag{3.74}$$

where i is an integer and $1 \leq i \leq p-1$. Combining (3.73) and (3.70), we see that for $n \geq 0$,

$$\overline{B}_3(12p^2n + p^2) \equiv \overline{B}_3(12n+1) \pmod{2^4}. \tag{3.75}$$

By (3.75) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$\overline{B}_3(12p^{2\alpha}n + p^{2\alpha}) \equiv \overline{B}_3(12n+1) \pmod{2^4}. \tag{3.76}$$

Replacing n by $p^2n + pi + \frac{p^2-1}{6}$ in (3.76) and using (3.74), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$\overline{B}_3(12p^{2\alpha+2}n + 12p^{2\alpha+1}i + p^{2\alpha+2}) \equiv 0 \pmod{2^4}. \tag{3.77}$$

Congruence (3.67) follows from (3.77). □

Theorem 3.8 *For all nonnegative integers α and n:*

$$\overline{B}_3(4\cdot 3^{\alpha+1}n + 2\cdot 3^{\alpha+1}) \equiv \overline{B}_3(4n+2) \pmod{2^3}, \tag{3.78}$$

$$\overline{B}_3(4\cdot 3^{\alpha+1}n + 10\cdot 3^{\alpha}) \equiv 0 \pmod{2^3}, \tag{3.79}$$

$$\overline{B}_3(48n+2) \equiv \begin{cases} 4 \pmod{2^3}, & \text{if } n = P_k, \\ 0 \pmod{2^3}, & \text{otherwise.} \end{cases} \tag{3.80}$$

$$\overline{B}_3(12(4n+i)+2) \equiv 0 \pmod{2^3}, \tag{3.81}$$

where $i = 1, 2, 3$ and P_k is either of the generalized pentagonal numbers $k(3k \pm 1)/2$.

Proof From equation (3.54), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(2n)q^n = \frac{f_2^8 f_6^4}{f_1^8 f_4^2 f_{12}^2} + 4q\frac{f_2^2 f_3^2 f_4^2 f_{12}^2}{f_1^6 f_6^2}.$$

Applying (3.45) and (3.47) in the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(2n)q^n \equiv \frac{f_2^4 f_6^4}{f_4^2 f_{12}^2} + 4q\frac{f_4^2 f_{12}^2}{f_2 f_6} \pmod{2^3},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(4n+2)q^n \equiv 4\frac{f_2^2 f_6^2}{f_1 f_3} \equiv 4\psi(q)\psi(q^3) \pmod{2^3}. \tag{3.82}$$

By substituting (2.8) into (3.82), we find that

$$\sum_{n=0}^{\infty} \overline{B}_3(4n+2)q^n \equiv 4(f(q^3,q^6)\psi(q^3) + q\psi(q^3)\psi(q^9)) \pmod{2^3}, \tag{3.83}$$

which yields

$$\overline{B}_3(12n+6) \equiv \overline{B}_3(4n+2) \pmod{2^3}, \tag{3.84}$$

$$\overline{B}_3(12n+10) \equiv 0 \pmod{2^3}. \tag{3.85}$$

Congruences (3.78) and (3.79) follow from (3.84), (3.85), and induction on α.

Again, from (3.83), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+2)q^n \equiv 4\psi(q)f(q,q^2) \pmod{2^3}. \tag{3.86}$$

Using the definitions of $\psi(q)$ and $f(q,q^2)$, we see that

$$\psi(q)f(q,q^2) = \frac{f_2^3 f_3^2}{f_1^2 f_6} \equiv f_4 \pmod{2}. \tag{3.87}$$

By combining (3.86) and (3.87), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(12n+2)q^n \equiv 4f_4 \pmod{2^3}. \tag{3.88}$$

Congruences (3.80) and (3.81) follow from (3.88). □

4. Congruence results for $\overline{B}_4(n)$

In this section, we prove several congruences for $\overline{B}_4(n)$.

Theorem 4.1 *For any nonnegative integer n:*

$$\overline{B}_4(12n+2) \equiv 0 \pmod{3}, \tag{4.1}$$

$$\overline{B}_4(12n+4) \equiv 0 \pmod{3}, \tag{4.2}$$

$$\overline{B}_4(12n+8) \equiv 0 \pmod{3}, \tag{4.3}$$

$$\overline{B}_4(12n+10) \equiv 0 \pmod{3}, \tag{4.4}$$

$$\overline{B}_4(12n+7) \equiv 0 \pmod{96}, \tag{4.5}$$

$$\overline{B}_4(12n+11) \equiv 0 \pmod{96}, \tag{4.6}$$

$$\overline{B}_4(36n+15) \equiv 0 \pmod{96}, \tag{4.7}$$

$$\overline{B}_4(36n+27) \equiv 0 \pmod{96}. \tag{4.8}$$

Proof Setting $\ell = 4$ in (1.7), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(n)q^n = \frac{f_2^2}{f_1^4}\frac{f_4^4}{f_8^2} = \frac{\varphi(-q^4)^2}{\varphi(-q)^2}. \tag{4.9}$$

Substituting (2.7) into (4.9), we see that

$$\sum_{n=0}^{\infty} \overline{B}_4(n)q^n = \frac{1}{\varphi(-q^4)^6}(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3)^2.$$

After simplification, choosing the terms for which the powers of q are the form q^{4n+i} for $i = 0, 1, 2, 3$, we find that

$$\sum_{n=0}^{\infty} \overline{B}_4(4n)q^n = \frac{\varphi(q)^6}{\varphi(-q)^6} + 48q\frac{\varphi(q)^2\psi(q^2)^4}{\varphi(-q)^6}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+1)q^n = 4\frac{\varphi(q)^5\psi(q^2)}{\varphi(-q)^6} + 64q\frac{\varphi(q)\psi(q^2)^5}{\varphi(-q)^6}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+2)q^n = 12\frac{\varphi(q)^4\psi(q^2)^2}{\varphi(-q)^6} + 64q\frac{\psi(q^2)^6}{\varphi(-q)^6}, \tag{4.12}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+3)q^n = 32\frac{\varphi(q)^3\psi(q^2)^3}{\varphi(-q)^6}. \tag{4.13}$$

With aid of (3.10), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(4n)q^n \equiv \frac{\varphi(q^3)^2}{\varphi(-q^3)^2} \pmod 3, \tag{4.14}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+2)q^n \equiv q\frac{\psi(q^6)^2}{\varphi(-q^3)^2} \pmod 3, \tag{4.15}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+3)q^n \equiv 32\frac{\varphi(q^3)\psi(q^6)}{\varphi(-q^3)^2} \pmod{96}. \tag{4.16}$$

Equating the coefficients of q^{3n+1} and q^{3n+2} of (4.14), we obtain the congruences (4.2) and (4.3). Extracting the terms containing q^{3n} and q^{3n+2} from (4.15), we arrive at (4.1) and (4.4). Similarly, equating the coefficients of q^{3n+1} and q^{3n+2} of (4.16), we obtain the congruences (4.5) and (4.6).

Again from (4.16), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(12n+3)q^n \equiv 32\frac{\varphi(q)\psi(q^2)}{\varphi(-q)^2} \equiv 32\frac{f_6^2}{f_3^2} \pmod{96}.$$

Congruences (4.7) and (4.8) follow from the above equation. □

Theorem 4.2 *For all integers $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{B}_4(8n+7) \equiv 0 \pmod{64}, \tag{4.17}$$

$$\overline{B}_4(8 \cdot 7^{2\alpha}n + 3 \cdot 7^{2\alpha}) \equiv \overline{B}_4(8n+3) \pmod{64}, \tag{4.18}$$

$$\overline{B}_4(8 \cdot 7^{2\alpha+2}n + j \cdot 7^{2\alpha+1}) \equiv 0 \pmod{64}, \tag{4.19}$$

where $j \in \{5, 13, 29, 37, 45, 53\}$.

Proof By definitions of $\varphi(q)$ and $\psi(q)$, we can rewrite (4.13) as

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+3)q^n = 32\frac{f_2^{18}}{f_1^{18}}. \tag{4.20}$$

From (3.10) with $p = 2$, we have

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(4n+3)}{32} q^n \equiv f_2^9 \pmod 2. \tag{4.21}$$

This yields congruence (4.17) by equating q^{2n+1}. Again, by equating q^{2n} from both sides of (4.21), we see that

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8n+3)}{32} q^n \equiv f_1^9 \equiv \left(\frac{f_2^2}{f_1}\right)^3 \pmod 2. \tag{4.22}$$

Combining (2.3) and (4.22), we have

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8n+3)}{32} q^n \equiv \psi(q)^3 \pmod 2. \tag{4.23}$$

From Entry 17(iv) on page 303 in Berndt's book [5], we have the 7-dissection

$$\psi(q) = f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3 f(q^7, q^{42}) + q^6 \psi(q^{49}). \tag{4.24}$$

Employing (4.24) in (4.23), we obtain

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8n+3)}{32} q^n \equiv (f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3 f(q^7, q^{42}) + q^6 \psi(q^{49}))^3 \pmod 2.$$

After simplification, extracting the terms containing q^{7n+4} from both sides of the resulting equation, we get

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8 \cdot 7n + 8 \cdot 4 + 3)}{32} q^n \equiv q^2 \psi(q^7)^3 \pmod 2,$$

which implies that

$$\overline{B}_4(8 \cdot 7^2 n + 8 \cdot 18 + 3) \equiv \overline{B}_4(8n+3) \pmod{64}, \tag{4.25}$$

and

$$\overline{B}_4(8 \cdot 7(7n+i) + 8 \cdot 4 + 3) \equiv 0 \pmod{64} \tag{4.26}$$

for $i = 0, 1, 3, 4, 5, 6$.

Congruences (4.18) and (4.19) follow from (4.25), (4.26), and induction on α. □

Theorem 4.3 *For all nonnegative integers α and n:*

$$\overline{B}_4(4 \cdot 3^{2\alpha+2} n + 3^{2\alpha+2}) \equiv \overline{B}_4(4n+1) \pmod{2^4}, \tag{4.27}$$

$$\overline{B}_4(4 \cdot 3^{2\alpha+2} n + 7 \cdot 3^{2\alpha+1}) \equiv 0 \pmod{2^4}, \tag{4.28}$$

$$\overline{B}_4(4 \cdot 3^{2\alpha+2} n + 11 \cdot 3^{2\alpha+1}) \equiv 0 \pmod{2^4}. \tag{4.29}$$

Proof Following (4.11), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+1)q^n \equiv 4\frac{f_2^{30}}{f_1^{22}f_4^{8}} \pmod{2^4}, \tag{4.30}$$

but

$$\frac{f_2^{30}}{f_1^{22}f_4^{8}} \equiv \frac{f_2^{4}}{f_1^{2}} \pmod{4}. \tag{4.31}$$

Combining (4.30) and (4.31), we see that

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+1)q^n \equiv 4\psi(q)^2 \pmod{2^4}.$$

Substituting (2.8) in the above equation, simplifying, and extracting the terms of the form q^{3n+2} from both sides of the resulting equation, we get

$$\sum_{n=0}^{\infty} \overline{B}_4(12n+9)q^n \equiv 4\psi(q^3)^2 \pmod{2^4},$$

which implies that

$$\overline{B}_4(36n+9) \equiv 4\overline{B}_4(4n+1) \pmod{2^4}, \tag{4.32}$$

$$\overline{B}_4(36n+21) \equiv 0 \pmod{2^4}, \tag{4.33}$$

and

$$\overline{B}_4(36n+33) \equiv 0 \pmod{2^4}. \tag{4.34}$$

The theorem follows from congruences (4.32), (4.33), and (4.34) and by induction on α. □

5. Congruence result for $\overline{B}_5(n)$

In this section, we establish one congruence relation for $\overline{B}_5(n)$.

Theorem 5.1 *For each nonnegative integer α and n, we have*

$$\overline{B}_5(5^{\alpha}n) \equiv \overline{B}_5(n) \pmod{5}. \tag{5.1}$$

Proof Setting $\ell = 5$ in (1.7), we have

$$\sum_{n=0}^{\infty} \overline{B}_5(n)q^n = \frac{\varphi(-q^5)^2}{\varphi(-q)^2}. \tag{5.2}$$

Following (3.10), it is easy to see that

$$\varphi(-q)^5 \equiv \varphi(-q^5) \pmod{5}. \tag{5.3}$$

From (5.3), (5.2) becomes

$$\sum_{n=0}^{\infty} \overline{B}_5(n)q^n \equiv \varphi(-q)^8 \equiv \sum_{n=0}^{\infty}(-1)^n r_8(n)q^n \pmod 5. \tag{5.4}$$

From [26, Lemma 3], for any $p \geq 3$, we have

$$r_8(pn) \equiv r_8(n) \pmod{p^3}. \tag{5.5}$$

The theorem follows from (5.4) and (5.5) with $p = 5$. □

Acknowledgment

The authors would like to thank the anonymous referees for their helpful suggestions and comments, which improved the original version of the manuscript.

References

[1] Ahmed Z, Baruah ND. New congruences for Andrews' singular overpartitions. Int J Number Theory 2015; 11: 2247-2264.

[2] Andrews GE. Singular overpartitions. Int J Number Theory 2015; 11: 1523-1533.

[3] Barrucand P, Cooper S, Hirschhorn MD. Relations between squares and triangles. Discrete Math 2002; 242: 245-247.

[4] Baruah ND, Nath K. Some results on 3-cores. P Am Math Soc 2014; 142: 441-448.

[5] Berndt BC. Ramanujan's Notebooks. Part III. New York, NY, USA: Springer-Verlag, 1991.

[6] Bringmann K, Lovejoy J. Rank and congruences for overpartition pairs. Int J Number Theory 2008; 4: 303-322.

[7] Chen SC, Hirschhorn MD, Sellers JA. Arithmetic properties of Andrews' singular overpartitions. Int J Number Theory 2015; 11: 1463-1476.

[8] Chen WYC, Lin BLS. Arithmetic properties of overpartition pairs. Acta Arith 2012; 151: 263-277.

[9] Corteel S, Lovejoy J. Overpartitions. T Am Math Soc 2004; 356: 1623-1635.

[10] Cui SP, Gu NSS. Arithmetic properties of ℓ-regular partitions. Adv Appl Math 2013; 51: 507-523.

[11] Hirschhorn MD, Garvan F, Borwein J. Cubic analogs of the Jacobian cubic theta function $\theta(z,q)$. Canad J Math 1993; 45: 673-694.

[12] Hirschhorn MD, Sellers JA. An infinite family of overpartition congruences modulo 12. Integers 2005; 5: A20.

[13] Hirschhorn MD, Sellers JA. Arithmetic relations for overpartitions. J Combin Math Combin Comput 2005; 53: 65-73.

[14] Hirschhorn MD, Sellers JA. Elementary proofs of parity results for 5-regular partitions. Bull Aust Math Soc 2010; 81: 58-63.

[15] Keister D, Sellers JA, Vary R. Some arithmetic properties of overpartition k-tuples. Integers 2009; 9: A17.

[16] Keith WJ. Congruences for 9-regular partitions modulo 3. Ramanujan J 2014; 35: 157-164.

[17] Kim B. The overpartition function modulo 128. Integers 2008; 8: A38.

[18] Kim B. A short note on the overpartition function. Discrete Math 2009; 309: 2528-2532.

[19] Kim B. Overpartition pairs modulo powers of 2. Discrete Math 2011; 311: 835-840.

[20] Lin BLS. Some results on bipartitions with 3-core. J Number Theory 2014; 139: 44-52.

[21] Lovejoy J. Overpartition pairs. Ann Inst Fourier 2006; 56: 781-794.

[22] Lovejoy J. Gordon's theorem for overpartitions. J Comb Theory A 2003; 103: 393-401.

[23] Mahlburg K. The overpartition function modulo small powers of 2. Discrete Math 2004; 286: 263-267.

[24] Penniston D. Arithmetic of ℓ-regular partition functions. Int J Number Theory 2008; 4: 295-302.

[25] Shen EYY. Arithmetic properties of ℓ-regular overpartition. Int J Number Theory 2016; 12: 841-852.

[26] Wang L. A short proof of a conjecture by Hirschhorn and Sellers on overpartitions. arXiv:1407.5430 [math.NT].

[27] Webb JJ. Arithmetic of the 13-regular partition function modulo 3. Ramanujan J 2011; 25: 49-56.

[28] Xia EXW. Arithmetic properties of bipartitions with 3-cores. Ramanujan J 2015; 38: 529-548.

[29] Xia EXW, Yao OXM. Analogues of Ramanujan's partition identities. Ramanujan J 2013; 31: 373-396.

[30] Xia EXW, Yao OXM. Parity results for 9-regular partitions. Ramanujan J 2014; 34: 109-117.

[31] Yao OXM, Xia EXW. New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions. J Number Theory 2013; 133: 1932-1949.