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On certain semigroups of full contraction maps of a finite chain

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Abstract: Let $X_n = \{1, 2, \ldots, n\}$ with its natural order and let $\mathcal{T}_n$ be the full transformation semigroup on X_n. A map $\alpha \in \mathcal{T}_n$ is said to be order-preserving if, for all $x, y \in X_n$, $x \leq y \Rightarrow x\alpha \leq y\alpha$. The map $\alpha \in \mathcal{T}_n$ is said to be a contraction if, for all $x, y \in X_n$, $|x\alpha - y\alpha| \leq |x - y|$. Let $\mathcal{CT}_n$ and $\mathcal{OCT}_n$ denote, respectively, subsemigroups of all contraction maps and all order-preserving contraction maps in $\mathcal{T}_n$. In this paper we present characterisations of Green's relations on $\mathcal{CT}_n$ and starred Green's relations on both $\mathcal{CT}_n$ and $\mathcal{OCT}_n$.

Key words: Full transformation, order-preserving, contraction, Green's relations, starred Green's relations

1. Introduction

The full transformation semigroup on $X_n = \{1, 2, \ldots, n\}$, under its natural order, is denoted by $\mathcal{T}_n$. The importance of the study of $\mathcal{T}_n$, as a naturally occurring semigroup, is justified by its universal property in which every finite semigroup is embeddable in some $\mathcal{T}_n$. This is analogous to Cayley's theorem for symmetric group $\mathcal{S}_n$, of all permutations of X_n, in group theory. Thus, just as the study of alternating and dihedral groups has made a significant contribution to group theory, there is some interest in identifying and studying certain special subsemigroups of $\mathcal{T}_n$. The subsemigroups $\mathcal{O}_n = \{\alpha \in \mathcal{T}_n : x \leq y \Rightarrow x\alpha \leq y\alpha, \text{ for all } x, y \in X_n\}$, of order-preserving elements and $S_n^- = \{\alpha \in \mathcal{T}_n : x\alpha \leq x, \text{for all } x \in X_n\}$, of order-decreasing elements of $\mathcal{T}_n$ have been studied. In [14], Howie showed that every element of $\mathcal{O}_n$ is expressible as a product of idempotents and also obtained formulae for the number of elements and the number of idempotents in $\mathcal{O}_n$. Umar in [22] showed that every element of S_n^- is expressible as a product of idempotents. The rank and idempotent rank of $\mathcal{O}_n$ were computed by Gomes and Howie [12] to be n and $2(n-1)$, respectively. Maximal subsemigroups, maximal idempotent-generated/regular subsemigroups, and locally maximal idempotent-generated subsemigroups of $\mathcal{O}_n$ were described and classified in [24–26]. The results of [26] were simplified in [28]. Maximal regular subsemibands of the two-sided ideals of $\mathcal{O}_n$ were completely described by Zhao [27]. In [8], a description of the endomorphisms of $\mathcal{O}_n$ was presented. Other algebraic properties in the semigroup $\mathcal{O}_n$ and some of its notable subsemigroups and oversemigroups may be found in [3–7,9].

On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ *if and only if* a, b are related by the Green's relation $\mathcal{L}$ in some over semigroup of S. The relation $\mathcal{R}^*$ is defined dually. These relations have played a fundamental role in the study of many important classes of semigroups; see for example the work by Fountain [10, 11]. Moreover, many papers have appeared describing the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ in certain

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subsemigroups of $\mathcal{T}_n$ preserving order and an equivalence relation. Araujo and Konieczny [2] characterised $\mathcal{L}^*$ and $\mathcal{R}^*$ in the subsemigroup of $\mathcal{T}_n$, consisting of all transformations preserving an equivalence relation and a cross-section of the relation. Pei and Zhou [18] characterised $\mathcal{L}^*$ and $\mathcal{R}^*$ in the subsemigroup of $\mathcal{T}_n$, consisting of all transformations preserving an equivalence relation. Similar characterisations of $\mathcal{L}^*$ and $\mathcal{R}^*$ were presented in [16–21]. In this current article we consider an algebra study for the so-called subsemigroups of contraction mappings of $\mathcal{T}_n$. In particular, we present characterisations of both Green's and starred Green's relations for these semigroups.

A map α in $\mathcal{T}_n$ is said to be a *contraction* if $|x\alpha - y\alpha| \leq |x-y|$, for all $x, y \in X_n$. The sets of all contraction maps and of all order-preserving contraction maps in $\mathcal{T}_n$ are, respectively, denoted by $\mathcal{CT}_n$ and $\mathcal{OCT}_n$, which are subsemigroups of $\mathcal{T}_n$. The term contraction map first appeared in [13] but algebraic and combinatorial studies of the semigroups $\mathcal{CT}_n$ and $\mathcal{OCT}_n$ were initiated by Dauda [1]. Orders and regularity for both $\mathcal{CT}_n$ and $\mathcal{OCT}_n$ were investigated in [1]. He also characterises Green's relations on $\mathcal{OCT}_n$. Here we investigate Green's relations on $\mathcal{CT}_n$ and starred Green's relations on both $\mathcal{CT}_n$ and $\mathcal{OCT}_n$.

2. Preliminaries

Let $\mathcal{O}_n = \{\alpha \in \mathcal{T}_n \setminus \mathcal{S}_n : (\forall x, y \in X_n)\ x \leq y \Rightarrow x\alpha \leq y\alpha\}$, $\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n \setminus \mathcal{S}_n : (\forall x, y \in X_n)\ |x\alpha - y\alpha| \leq |x-y|\}$, and $\mathcal{OCT}_n = \mathcal{CT}_n \cap \mathcal{O}_n$ be the subsemigroups of $\mathcal{T}_n \setminus \mathcal{S}_n$ consisting of all order-preserving maps, all contraction maps, and all order-preserving contraction maps, respectively.

Definition 2.1 *Let A be a subset of X_n and let $\{A_1, A_2, \ldots, A_r\}$ be a partition of X_n. Then A is called* convex *if, for all $x, y \in X_n$, $(x, y \in A$ and $x \leq z \leq y) \Rightarrow z \in A$. A is called a* transversal *of $\{A_1, A_2, \ldots, A_r\}$ if $|A| = r$ and each A_i $(1 \leq i \leq r)$ contains exactly one point of A. The partition $\{A_1, A_2, \ldots, A_r\}$ is called a* convex partition *if it possesses a convex transversal.*

From the definition of contraction maps, it is easy to notice (which is also noticed in [1, Lemma 3.1.2]) that if $\alpha \in \mathcal{T}_n$ is a contraction, then there exists $s \in X_n$ such that

$$\mathrm{im}(\alpha) = \{s, s+1, \ldots, t-1, t\},$$

in other words, $\mathrm{im}(\alpha)$ is convex.

Each map $\alpha \in \mathcal{O}_n$ can be written as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}, \tag{1}$$

where $\mathrm{im}(\alpha) = \{a_1 < a_2 < \ldots < a_r\}$ and $A_1, A_2, \ldots, A_r$ are equivalence classes under the equivalence $\ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}$. Thus, $x\alpha = a_i$ for all $x \in A_i$ $(1 \leq i \leq r)$. It is then easy to see, from the order-preserving property, that the $\ker(\alpha)$-classes A_i $(1 \leq i \leq r)$ are convex subsets of X_n. We start by characterising contraction maps in $\mathcal{O}_n$.

Lemma 2.1 *$\alpha \in \mathcal{O}_n$ is a contraction if and only if $\mathrm{im}(\alpha)$ is convex.*

Proof Since $\mathcal{O}_n$ is a subsemigroup of $\mathcal{T}_n$ it is clear, from our observation just after Definition 2.1, that $\mathrm{im}(\alpha)$ is convex whenever $\alpha \in \mathcal{O}_n$ is a contraction.

Conversely, suppose that $\text{im}(\alpha) = \{a_1 < a_2 < \ldots < a_r\}$ is convex. Then $a_{i+1} = a_i + 1$ $(1 \leq i \leq r-1)$. Let $x, y \in X_n$ and suppose (without loss of generality) that $x < y$. Then either $x, y \in a_i\alpha^{-1}$ (for some i) or $x \in a_i\alpha^{-1}$ and $y \in a_j\alpha^{-1}$ (for some $i < j$). In the former, we have $|x\alpha - y\alpha| = |a_i - a_i| = 0 < |x - y|$. In the latter, assume that $j = i + k$, where k is any positive integer, so that $|x\alpha - y\alpha| = |a_{i+k} - a_i| = |a_i + k - a_i| = k \leq |x - y|$ since $\ker(\alpha)$-classes $a_i\alpha^{-1}$ $(1 \leq i \leq r)$ are convex. Thus, $|x\alpha - y\alpha| \leq |x - y|$ for all $j \geq i$ and so α is a contraction. $\square$

Next we characterise contraction maps in $\mathcal{T}_n$.

Theorem 2.2 *Let α be an element of $\mathcal{T}_n$ of height r, where $r \leq n$. Then α is contraction if and only if*

(i) $\text{im}(\alpha)$ is a convex subset of X_n, and

(ii) for each $i \in \text{im}(\alpha)$ and each $x \in i\alpha^{-1}$, if $x - 1 \in k\alpha^{-1}$ and $x + 1 \in t\alpha^{-1}$, then $k, t \in \Phi_i$, where

$$\Phi_i = \begin{cases} \{i, i+1\} & \text{if } i = 1 \\ \{i-1, i, i+1\} & \text{if } 1 < i < r \\ \{i-1, i\} & \text{if } i = r. \end{cases}$$

Proof Suppose that α in $\mathcal{T}_n$ is a contraction. Then, by [1, Lemma 3.1.2], part (i) holds, that is, $\text{im}(\alpha)$ is convex. Now suppose that, for each $i \in \text{im}(\alpha)$ and each $x \in i\alpha^{-1}$, $x - 1 \in s\alpha^{-1}$ and $x + 1 \in t\alpha^{-1}$. We need to show that $s, t \in \Phi_i$. Suppose that either $s \notin \Phi_i$ or $t \notin \Phi_i$. Then

$$|x\alpha - (x-1)\alpha| = |i - s| > 1 = |x - (x-1)|$$

or

$$|(x+1)\alpha - x\alpha| = |t - i| > 1|(x+1) - x|,$$

so that, in both cases, α cannot be a contraction. This is a contradiction to the choice of α. Thus both s and t must be in Φ_i.

Conversely, suppose that $\alpha \in \mathcal{T}_n$ satisfies the two conditions of the theorem and let $x, y \in X_n$. If both x and y belong to the same block of α, then

$$|x\alpha - y\alpha| = 0 \leq |x - y|.$$

On the other hand, if x and y belong to different blocks of α, say $x \in s\alpha^{-1}$ and $y \in t\alpha^{-1}$, where $s, t \in \text{im}(\alpha)$ and $s \neq t$, it is then not so hard to see that the two conditions of the theorem ensure that

$$|x\alpha - y\alpha| = |s - t| \leq |x - y|.$$

Thus, α is a contraction. $\square$

3. Green's relations

For the definition of Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$, and $\mathcal{J}$ on a semigroup see [15]. As in [23], we shall throughout this and the next sections write $\mathcal{K}(S)$ to emphasise that $\mathcal{K}$ is a relation on a semigroup S. In this section we characterise the relations $\mathcal{L}, \mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ on $\mathcal{CT}_n$.

Let $\text{Ker}(\alpha)$ be the set of all the equivalence classes of the equivalence relation $\ker(\alpha)$ on X_n, that is $\text{Ker}(\alpha) = X_n/\ker(\alpha)$.

Theorem 3.1 *Let $\alpha, \beta \in \mathcal{CT}_n$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}(\mathcal{CT}_n)$ *if and only if* $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$*, and both* $\mathrm{Ker}(\alpha)$ *and* $\mathrm{Ker}(\beta)$ *are convex partitions of* X_n*;*

(ii) $(\alpha, \beta) \in \mathcal{R}(\mathcal{CT}_n)$ *if and only if* $\ker(\alpha) = \ker(\beta)$*;*

(iii) $(\alpha, \beta) \in \mathcal{D}(\mathcal{CT}_n)$ *if and only if* $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$*, and both* $\mathrm{Ker}(\alpha)$ *and* $\mathrm{Ker}(\beta)$ *are convex partitions of* X_n*.*

Proof (i) Suppose that $(\alpha, \beta) \in \mathcal{L}(\mathcal{CT}_n)$, then

$$\delta\beta = \alpha \quad \text{and} \quad \gamma\alpha = \beta \quad \text{for some} \quad \delta, \gamma \in \mathcal{CT}_n^1.$$

This clearly implies that $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$. Therefore, $\mathrm{im}(\gamma)$ and $\mathrm{im}(\delta)$ must be transversal of $\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta)$, respectively. However, since $\delta, \gamma \in \mathcal{CT}_n^1$ it follows, by Theorem 2.2(i), that $\mathrm{im}(\delta)$ and $\mathrm{im}(\gamma)$ are convex subsets of X_n. Thus, $\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta)$ are convex partitions of X_n.

Conversely, suppose that $\mathrm{im}(\alpha) = \mathrm{im}(\beta) = \{c_1, c_2, \ldots, c_r\}$ and $\mathrm{Ker}(\alpha)$, $\mathrm{Ker}(\beta)$ are convex partitions of X_n. Let $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ be convex transversal of $\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta)$, respectively, arranged in a way that $a_i \in c_i\alpha^{-1}$ and $b_i \in c_i\beta^{-1}$ for each $1 \leq i \leq r$. Define maps δ and γ by $\ker(\delta) = \ker(\alpha)$, $\ker(\gamma) = \ker(\beta)$, $(c_i\alpha^{-1})\delta = b_i$, and $(c_i\beta^{-1})\gamma = a_i$, for each $1 \leq i \leq r$. Then $\delta, \gamma \in \mathcal{CT}_n$ and $\delta\beta = \alpha$, $\gamma\alpha = \beta$ so that $(\alpha, \beta) \in \mathcal{L}(\mathcal{CT}_n)$.

(ii) Suppose that $(\alpha, \beta) \in \mathcal{R}(\mathcal{CT}_n)$; then

$$\beta\delta = \alpha \quad \text{and} \quad \alpha\gamma = \beta \quad \text{for some} \quad \delta, \gamma \in \mathcal{CT}_n^1.$$

From this it follows that $\ker(\alpha) = \ker(\beta)$.

Conversely, suppose that $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta) = \{C_1, C_2, \ldots, C_r\}$. Then, since $\alpha, \beta \in \mathcal{CT}_n$, we may (without loss of generality) write

$$\alpha = \begin{pmatrix} C_1 & C_2 & \cdots & C_r \\ i & i+1 & \cdots & i+r-1 \end{pmatrix} \quad \text{and} \qquad \beta = \begin{pmatrix} C_1 & C_2 & \cdots & C_r \\ j & j+1 & \cdots & j+r-1 \end{pmatrix}$$

for some $i, j \in X_n$. Then the maps

$$\delta = \begin{pmatrix} \{1, 2, \ldots, j\} & j+1 & \cdots & j+r-2 & \{j+r-1, j+r, \ldots, n\} \\ i & i+1 & \cdots & i+r-2 & i+r-1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} \{1, 2, \ldots, i\} & i+1 & \cdots & i+r-2 & \{i+r-1, i+r, \ldots, n\} \\ j & j+1 & \cdots & j+r-2 & j+r-1 \end{pmatrix}$$

are in $\mathcal{CT}_n^1$ and satisfy $\beta\delta = \alpha$, $\alpha\gamma = \beta$ so that $(\alpha, \beta) \in \mathcal{R}(\mathcal{CT}_n)$.

(iii) Suppose that $(\alpha, \beta) \in \mathcal{D}(\mathcal{CT}_n)$; then $(\alpha, \gamma) \in \mathcal{L}(\mathcal{CT}_n)$ and $(\gamma, \beta) \in \mathcal{R}(\mathcal{CT}_n)$, for some $\gamma \in \mathcal{CT}_n$. Using Theorem 3.1, we have that $\mathrm{im}(\alpha) = \mathrm{im}(\gamma)$, $\ker(\gamma) = \ker(\beta)$, and $\mathrm{Ker}(\alpha)$, $\mathrm{Ker}(\gamma)$ are convex partitions of X_n. This implies that $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$ and $\mathrm{Ker}(\alpha)$, $\mathrm{Ker}(\beta)$ are convex partitions of X_n.

Conversely, suppose that $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$, and both $\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta)$ are convex partitions of X_n. Then we can choose $\gamma \in \mathcal{CT}_n$ such that $\ker(\gamma) = \ker(\beta)$ and $\mathrm{im}(\gamma) = \mathrm{im}(\alpha)$. It is then clear that $(\alpha, \gamma) \in \mathcal{L}(\mathcal{CT}_n)$ and $(\gamma, \beta) \in \mathcal{R}(\mathcal{CT}_n)$, so that $(\alpha, \beta) \in \mathcal{D}(\mathcal{CT}_n)$. $\square$

4. Starred Green's relations

Recall that on a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $(a,b) \in \mathcal{L}^*$ *if and only if* a,b are related by the Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. These relations also have the following characterisations (see [10])

$$\mathcal{L}^*(S) = \{(a,b) : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\} \tag{2}$$

and

$$\mathcal{R}^*(S) = \{(a,b) : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb\}. \tag{3}$$

The join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$.

Theorem 4.1 *Let* $S \in \{\mathcal{CT}_n, \mathcal{OCT}_n\}$ *and let* $\alpha, \beta \in S$. *Then*

(i) $(\alpha,\beta) \in \mathcal{L}^*(S)$ *if and only if* $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$,

(ii) $(\alpha,\beta) \in \mathcal{R}^*(S)$ *if and only if* $\ker(\alpha) = \ker(\beta)$,

(iii) $(\alpha,\beta) \in \mathcal{H}^*(S)$ *if and only if* $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$ *and* $\ker(\alpha) = \ker(\beta)$,

(iv) $(\alpha,\beta) \in \mathcal{D}^*(S)$ *if and only if* $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$.

Proof (i) Suppose that $(\alpha,\beta) \in \mathcal{L}^*(S)$. Let $\mathrm{im}(\alpha) = \{a_1, \ldots, a_r\}$, where (by [1, Lemma 3.1.2], or Lemma 2.1) $a_{i+1} = a_i + 1$ for each $i = 1, \ldots, n-1$. Then

$$\alpha \cdot \begin{pmatrix} \{1,\ldots,a_1\} & a_2 & \cdots & a_{r-1} & \{a_r,\ldots,n\} \\ a_1 & a_2 & \cdots & a_{r-1} & a_r \end{pmatrix} = \alpha \cdot 1_{X_n}$$

and, by Equation (2), if and only if

$$\beta \cdot \begin{pmatrix} \{1,\ldots,a_1\} & a_2 & \cdots & a_{r-1} & \{a_r,\ldots,n\} \\ a_1 & a_2 & \cdots & a_{r-1} & a_r \end{pmatrix} = \beta \cdot 1_{X_n}$$

which implies that $\mathrm{im}(\beta) \subseteq \{a_1, \ldots, a_r\} = \mathrm{im}(\alpha)$. Similarly, we can show that $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\beta)$, and so $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$.

Conversely, suppose that $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$. Then $(\alpha,\beta) \in \mathcal{L}(\mathcal{T}_n)$ and, since $\mathcal{T}_n$ is an oversemigroup of S, it follows from definition that $(\alpha,\beta) \in \mathcal{L}^*(S)$.

(ii) Suppose that $(\alpha,\beta) \in \mathcal{R}^*(S)$. Then

$$\begin{aligned} (x,y) \in \ker(\alpha) &\iff x\alpha = y\alpha \\ &\iff \begin{pmatrix} X_n \\ x \end{pmatrix} \cdot \alpha = \begin{pmatrix} X_n \\ y \end{pmatrix} \cdot \alpha \\ &\iff \begin{pmatrix} X_n \\ x \end{pmatrix} \cdot \beta = \begin{pmatrix} X_n \\ y \end{pmatrix} \cdot \beta \quad \text{(by Equation(3))} \\ &\iff x\beta = y\beta \\ &\iff (x,y) \in \ker(\beta). \end{aligned}$$

Hence $\ker(\alpha) = \ker(\beta)$.

Similarly, the converse part is clear.

(iii) This follows from parts (i) and (ii).

(iv) Suppose $(\alpha,\beta)\in\mathcal{D}^*(S)$. Then, by [15, Proposition 1.5.11], for some $n\in\mathbb{N}$, there exist elements $\delta_1,\delta_2,\ldots,\delta_{2n-1}\in S$ such that

$$(\alpha,\delta_1)\in\mathcal{L}^*(S),(\delta_1,\delta_2)\in\mathcal{R}^*(S),(\delta_2,\delta_3)\in\mathcal{L}^*(S),\ldots,(\delta_{2n-1},\beta)\in\mathcal{R}^*(S).$$

Now, by parts (i) and (ii) of the theorem, we have $|\mathrm{im}(\alpha)|=|\mathrm{im}(\delta_1)|=|X_n/\ker(\delta_1)|=|X_n/\ker(\delta_2)|=|\mathrm{im}(\delta_2)|=|\mathrm{im}(\delta_3)|=\cdots=|X_n/\ker(\delta_{2n-1})|=|X_n/\ker(\beta)|=|\mathrm{im}(\beta)|$.

Conversely, suppose that $|\mathrm{im}(\alpha)|=|\mathrm{im}(\beta)|$ and let

$$\alpha=\begin{pmatrix}A_1 & A_2 & \cdots & A_r\\ a_1 & a_2 & \cdots & a_r\end{pmatrix}\quad\text{and}\quad\beta=\begin{pmatrix}B_1 & B_2 & \cdots & B_r\\ b_1 & b_2 & \cdots & b_r\end{pmatrix}$$

where $a_{i+1}=a_i+1$, $b_{i+1}=b_i+1$ for each $i=1,2,\ldots,r-1$. Then the map

$$\gamma=\begin{pmatrix}B_1 & B_2 & \cdots & B_r\\ a_1 & a_2 & \cdots & a_r\end{pmatrix}$$

is in S and, by parts (i) and (ii), $(\alpha,\gamma)\in\mathcal{L}^*(S)$ and $(\gamma,\beta)\in\mathcal{R}^*(S)$ so that, by [15, Proposition 1.5.11], $(\alpha,\beta)\in\mathcal{D}^*(S)$. □

The $\mathcal{L}^*-class$ containing an element a is denoted by L^*_a and corresponding notations are used for the remaining starred relations. We define a *left(right)* $*-ideal$ of a semigroup S to be a left(right) ideal I of S for which $L^*_a\subseteq I$ ($R^*_a\subseteq I$) for all $a\in I$. A subset I of S is a $*-ideal$ if it is both left and right $*-ideals$ of S. The principal $*-ideal$, $J^*(a)$, generated by $a\in S$ is the intersection of all $*-ideals$ of S to which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that: $a\mathcal{J}^*b$ if and only if $J^*(a)=J^*(b)$.

Now we are going to show that on the semigroup $S\in\{\mathcal{CT}_n,\mathcal{OCT}_n\}$, $\mathcal{D}^*=\mathcal{J}^*$ but first we record the following lemma from [11].

Lemma 4.2 *Let a,b be elements of a semigroup S. Then $b\in J^*(a)$ if and only if there are elements $a_0,a_1,\ldots,a_n\in S$, $x_1,\ldots,x_n,y_1,\ldots,y_n\in S^1$ such that $a=a_0,b=a_n$ and $(a_i,x_ia_{i-1}y_i)\in\mathcal{D}^*(S)$ for $i=1,\ldots,n$.*

Immediately we adopt the method used in [23] to have

Lemma 4.3 *Let $S\in\{\mathcal{CT}_n,\mathcal{OCT}_n\}$. Then for each $\alpha,\beta\in S$, $\alpha\in J^*(\beta)$ implies $|\mathrm{im}(\alpha)|\le|\mathrm{im}(\beta)|$.*

Proof Let $\alpha\in J^*(\beta)$, then by Lemma 4.2, there exist $\beta_0,\ldots,\beta_n\in S$, $\delta_1,\ldots,\delta_n$, $\gamma_1,\ldots,\gamma_n\in S^1$ such that $\beta=\beta_0,\alpha=\beta_n$ and $(\beta_i,\delta_i\beta_{i-1}\gamma_i)\in\mathcal{D}^*(S)$, for $i=1,\ldots,n$. However, by Theorem 4.1(iv), this implies that

$$|\mathrm{im}(\beta_i)|=|\mathrm{im}(\delta_i\beta_{i-1}\gamma_i)|\le|\mathrm{im}(\beta_{i-1})|$$

for all $i=1,\ldots,n$, which implies $|\mathrm{im}(\alpha)|\le|\mathrm{im}(\beta)|$ as required. □

The fact that $\mathcal{D}^*\subseteq\mathcal{J}^*$ together with Lemma 4.3 gives the following result.

Theorem 4.4 *On the semigroup $S\in\{\mathcal{CT}_n,\mathcal{OCT}_n\}$, $\mathcal{D}^*=\mathcal{J}^*$.*

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