Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 548 – 563
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doi:10.3906/mat-1605-8

Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications

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Received: 02.05.2016 • **Accepted/Published Online:** 23.06.2016 • **Final Version:** 22.05.2017

Abstract: Let a, b be two commutative generalized Drazin invertible elements in a Banach algebra; the expressions for the generalized Drazin inverse of the product ab and the sum $a+b$ were studied in some current literature on this subject. In this paper, we generalize these results under the weaker conditions $a^2b = aba$ and $b^2a = bab$. As an application of our results, we obtain some new representations for the generalized Drazin inverse of a block matrix with the generalized Schur complement being generalized Drazin invertible in a Banach algebra, extending some recent works.

Key words: Generalized Drazin inverse, Banach algebra, additive result, block matrix

1. Introduction

The generalized Drazin inverse in a Banach algebra was introduced in [10]. The expressions for the generalized Drazin inverse of the product and the sum were studied by many authors. For instance, in [10], for two commutative generalized Drazin invertible elements a, b in a Banach algebra, Koliha gave the expression of $(ab)^d$. Meanwhile, the representation of $(a+b)^d$ was obtained under the conditions $ab = ba = 0$ in a Banach algebra. Later, Djordjević and Wei [8] gave the expression of $(a+b)^d$ under the assumption $ab = 0$ in the context of the Banach algebra of all bounded linear operators on an arbitrary complex Banach space. In [1], Castro-González and Koliha obtained a formula for $(a+b)^d$ under the conditions $a^\pi b = b, ab^\pi = a, b^\pi aba^\pi = 0$, which are weaker than $ab = 0$ in Banach algebras. In [6], Deng and Wei derived necessary and sufficient conditions for the existence of $(P+Q)^d$ under the condition $PQ = QP$, where P, Q are bounded linear operators. Moreover, the expression of $(P+Q)^d$ was given. In [3], Cvetković-Ilić et al. extended the result of [6] to Banach algebras. More results on generalized Drazin inverse can be found in [2, 4, 7, 8, 12, 14].

In [13], Liu et al. deduced the explicit expressions for the Drazin inverses of the product ab and the sum $a+b$ under the conditions $a^2b = aba$ and $b^2a = bab$, where a and b are complex matrices. In [18], the corresponding results of [13] were studied for the pseudo Drazin inverse (which is a special case of generalized Drazin inverse [17]) in a Banach algebra. In this paper, we will further consider the results of [13] and [18] for the generalized Drazin inverse, which extend [10, Theorem 5.5] and [3, Theorem 2.1].

Another relevant topic is to establish a representation for the generalized Drazin inverse of a block matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in terms of its blocks under certain conditions. The generalized Schur complement $S = D - CA^dB$

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2010 *AMS Mathematics Subject Classification:* 15A09; 46H05; 47A05.

plays an important role in the representation for M^d. Here we list partially some conditions as follows:

(1) S is invertible, $A^\pi BC = 0$, $CA^\pi B = 0$, and $AA^\pi B = A^\pi BD$ (see [5]);

(2) S is invertible, $BCA^\pi = 0$, $CA^\pi B = 0$, and $CAA^\pi = DCA^\pi$ (see [5]);

(3) S is generalized Drazin invertible, $BCA^\pi = 0$, $DCA^\pi = 0$, $S^\pi CA = 0$, and $ABS^\pi = 0$ (see [16]);

(4) S is generalized Drazin invertible, $A^\pi B = 0$, and $S^\pi CA = 0$ (see [15]).

In this paper, we will extend the above results under weaker conditions as applications of our additive result.

2. Preliminaries

Throughout this paper, $\mathscr{A}$ denotes a complex Banach algebra with unity 1. For $a \in \mathscr{A}$, denote the spectrum and the spectral radius of a by $\sigma(a)$ and $r(a)$, respectively. $\mathscr{A}^{-1}$ and $\mathscr{A}^{qnil}$ stand for the sets of all invertible and quasinilpotent elements ($\sigma(a) = \{0\}$) in $\mathscr{A}$, respectively. The commutant of an element $a \in \mathscr{A}$ is defined by comm(a)=$\{b \in \mathscr{A} : ab = ba\}$. In addition, denote by C_n^k the binomial coefficient $\frac{n!}{k!(n-k)!}$ $(0 \le k \le n)$.

For the readers' convenience, we first recall the definitions of some generalized inverses. The generalized Drazin inverse [10] of $a \in \mathscr{A}$ (or Koliha–Drazin inverse of a) is the element $x \in \mathscr{A}$ that satisfies

$$xax = x, \quad ax = xa \quad \text{and} \quad a - a^2x \in \mathscr{A}^{qnil}.$$

Such x, if it exists, is unique and will be denoted by a^d. It is well known that $a \in \mathscr{A}$ has a generalized Drazin inverse if and only if 0 is not an accumulation point of $\sigma(a)$. Let $\mathscr{A}^d$ denote the set of all generalized Drazin invertible elements in $\mathscr{A}$. If $a \in \mathscr{A}^d$, the spectral idempotent a^π of a corresponding to the set $\{0\}$ is given by $a^\pi = 1 - aa^d$. In this case, the resolvent $R(\lambda, a) = (\lambda 1 - a)^{-1}$ has a Laurent series

$$R(\lambda, a) = \sum_{n=1}^{\infty} \lambda^{-n} a^{n-1} a^\pi - \sum_{n=0}^{\infty} \lambda^n (a^d)^{n+1},$$

on some punctured disc $\{\lambda : 0 < |\lambda| < r\}, r > 0$ (see [10, Theorem 5.1]).

The group inverse of $a \in \mathscr{A}$ is the element $x \in \mathscr{A}$ that satisfies

$$axa = a, \quad xax = x \quad \text{and} \quad ax = xa.$$

If the group inverse of a exists, it is unique and denoted by $a^\#$.

Let $p \in \mathscr{A}$ be an idempotent ($p^2 = p$). Then we can represent element $a \in \mathscr{A}$ as

$$a = \begin{bmatrix} a_1 & a_3 \\ a_4 & a_2 \end{bmatrix}_p,$$

where $a_1 = pap$, $a_2 = (1-p)a(1-p)$, $a_3 = pa(1-p)$, and $a_4 = (1-p)ap$.

It is well known that if $a \in \mathscr{A}^d$, then we have the following matrix representations:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathscr{A}p)^{-1}$, and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$.

Now we present two useful lemmas, which play an important role in the sequel.

Lemma 2.1 [1, Theorem 2.3] *Let $p^2 = p$, $x, y \in \mathscr{A}$ and let x and y have the representations*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p}. \tag{1}$$

(i) *If $a \in (p\mathscr{A}p)^d$ and $b \in ((1-p)\mathscr{A}(1-p))^d$, then $x, y \in \mathscr{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & u \\ 0 & b^d \end{bmatrix}_p, \quad y^d = \begin{bmatrix} b^d & 0 \\ u & a^d \end{bmatrix}_{1-p}, \tag{2}$$

where

$$u = \sum_{n=0}^{\infty}(a^d)^{n+2}cb^n b^\pi + \sum_{n=0}^{\infty} a^\pi a^n c(b^d)^{n+2} - a^d cb^d. \tag{3}$$

(ii) *If $x \in \mathscr{A}^d$ [resp. $y \in \mathscr{A}^d$] and $a \in (p\mathscr{A}p)^d$, then $b \in ((1-p)\mathscr{A}(1-p))^d$, and x^d [resp. y^d] is given by (2) and (3).*

Lemma 2.2 [10, Theorem 5.5] *Let $a, b \in \mathscr{A}^d$ be such that ab=ba. Then $ab \in \mathscr{A}^d$ and $(ab)^d = a^d b^d$.*

Next, the commuting property for the generalized Drazin inverse is investigated in a Banach algebra.

Theorem 2.3 *Let $a, b \in \mathscr{A}^d$ and $c \in \mathscr{A}$. If $ca = bc$, then $ca^d = b^d c$.*

Proof Suppose that $a, b \in \mathscr{A}^d$ and $ca = bc$, for any $n \in \mathbb{N}$, we have the following equations:

$$\begin{aligned} bb^d c - bb^d caa^d &= bb^d c(1 - aa^d) = (bb^d)^n c(1 - aa^d) \\ &= (b^d)^n (b^n c)(1 - aa^d) = (b^d)^n (ca^n)(1 - aa^d), \end{aligned}$$

which imply

$$\|bb^d c - bb^d caa^d\|^{\frac{1}{n}} = \|(b^d)^n ca^n (1 - aa^d)\|^{\frac{1}{n}} \le \|b^d\| \|c\|^{\frac{1}{n}} \|a^n(1 - aa^d)\|^{\frac{1}{n}} \xrightarrow{n \to \infty} 0.$$

Thus, $bb^d c = bb^d caa^d$, i.e. $b^d c = b^d caa^d$.

On the other hand, we have that

$$\begin{aligned} ca^d a - b^d caa^d a &= ca^d a - b^d bca^d a = (1 - bb^d)ca^d a = (1 - bb^d)c(a^d a)^n \\ &= (1 - bb^d)(ca^n)(a^d)^n = (1 - bb^d)(b^n c)(a^d)^n. \end{aligned}$$

Then we obtain

$$\|caa^d - b^d caa^d a\|^{\frac{1}{n}} = \|(1 - bb^d)b^n c(a^d)^n\|^{\frac{1}{n}} \le \|b^n(1 - bb^d)\|^{\frac{1}{n}} \|c\|^{\frac{1}{n}} \|a^d\| \xrightarrow{n \to \infty} 0.$$

Thus, $caa^d = b^d caa^d a$, i.e. $ca^d = b^d caa^d$. Therefore, we deduce that $ca^d = b^d c$. □

Corollary 2.4 [10, Theorem 4.4] *Let $a \in \mathscr{A}^d$ and $c \in \mathscr{A}$. If $ca = ac$, then $ca^d = a^d c$.*

The following lemmas will also be useful.

Lemma 2.5 *Let $a, b \in \mathscr{A}^d$ be such that $a^2b = aba$ and $b^2a = bab$. Then*

(i) $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a) \cap comm(a^d)$.

(ii) $\{ba, b^da, ba^d, b^da^d\} \subseteq comm(b) \cap comm(b^d)$.

Proof (i) By Corollary 2.4, it suffices to prove $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a)$.

Since $a^2b = aba$, then $(a^db)a = (a^d)^2aba = (a^d)^2a^2b = a(a^db)$.

Note that $bab^d = b^dba$, and we get $a(ab^d) = a^2b(b^d)^2 = aba(b^d)^2 = a(b^d)^2ba = (ab^d)a$, which implies $a(a^db^d) = (a^d)^2a(ab^d) = (a^d)^2(ab^d)a = (a^db^d)a$.

(ii) It is analogous to the proof of (i). □

Remark 2.6 *In Lemma 2.5, the conditions $a^2b = aba$ and $b^2a = bab$ are weaker than $ab = ba$. Indeed, it is clear that $ab = ba$ can imply $a^2b = aba$ and $b^2a = bab$. However, in general, the converse is false. The following example can illustrate this fact.*

Example 2.7 *Let $\mathscr{A}$ be the Banach algebra of all complex 3×3 matrices, and take*

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Clearly, $a^2b = aba$ and $b^2a = bab$. However, $ab \neq ba$.

Remark 2.8 *We have seen that if $a \in \mathscr{A}^d$, $b \in \mathscr{A}$, and $ab = ba$, then $a^db = ba^d$. However, under the conditions of Lemma 2.5, $a^db = ba^d$ may not be true, which can also be illustrated by the previous Example 2.7. Note that $a^3 = a$ and $b^3 = b$; then $a^d = a$ and $b^d = b$. However, $a^db \neq ba^d$.*

The next result was proved for complex matrices (see [13, Lemma 2.3]). Indeed, it is true in a Banach algebra.

Lemma 2.9 *Let $a, b \in \mathscr{A}$ be such that $a^2b = aba$ and $b^2a = bab$. Then*

$$(a+b)^n = \sum_{i=0}^{n-1} C_{n-1}^i (a^{n-i}b^i + b^{n-i}a^i), \quad \text{where } n \in \mathbb{N}.$$

Next, we establish two crucial auxiliary results.

Lemma 2.10 *Let $a, b \in \mathscr{A}$ be such that $a^2b = aba$ and $b^2a = bab$. Then*

(i) $r(a+b) \leqslant r(a) + r(b)$.

(ii) *If both a and b are quasinilpotent, then $a+b$ is quasinilpotent.*

Proof (i) Take any $\alpha > r(a)$ and $\beta > r(b)$. Let $a_1 = \frac{1}{\alpha}a$ and $b_1 = \frac{1}{\beta}b$. Then $r(a_1) < 1$ and $r(b_1) < 1$. From

Lemma 2.9, we have that

$$\begin{aligned}
\|(a+b)^{n+1}\| &= \|\sum_{i=0}^{n} C_n^i (a^{n+1-i}b^i + b^{n+1-i}a^i)\| \\
&= \|a\sum_{i=0}^{n} C_n^i a^{n-i}b^i + b\sum_{i=0}^{n} C_n^i b^{n-i}a^i\| \\
&\leq \|a\|\sum_{i=0}^{n} C_n^i \|a^{n-i}\|\|b^i\| + \|b\|\sum_{i=0}^{n} C_n^i \|b^{n-i}\|\|a^i\| \\
&= (\|a\|+\|b\|)\sum_{i=0}^{n} C_n^i \|a^i\|\|b^{n-i}\| \\
&= (\|a\|+\|b\|)\sum_{i=0}^{n} C_n^i \alpha^i\beta^{n-i}\|a_1^i\|\|b_1^{n-i}\|.
\end{aligned}$$

For each n, choose $n', n'' \in \mathbb{N}$ such that $n'+n''=n$ and $\|a_1^{n'}\|\|b_1^{n''}\| = \max_{0\leq i\leq n}\|a_1^i\|\|b_1^{n-i}\|$, then we have

$$\|(a+b)^{n+1}\| \leq (\|a\|+\|b\|)(\alpha+\beta)^n\|a_1^{n'}\|\|b_1^{n''}\|,$$

which implies

$$\begin{aligned}
r(a+b) &= \lim_{n\to\infty}(\|(a+b)^{n+1}\|^{\frac{1}{n+1}})^{\frac{n+1}{n}} = \lim_{n\to\infty}\|(a+b)^{n+1}\|^{\frac{1}{n}} \\
&\leq (\alpha+\beta)\lim_{n\to\infty}(\|a\|+\|b\|)^{\frac{1}{n}}\liminf_{n\to\infty}\|a_1^{n'}\|^{\frac{1}{n}}\|b_1^{n''}\|^{\frac{1}{n}} \\
&= (\alpha+\beta)\liminf_{n\to\infty}\|a_1^{n'}\|^{\frac{1}{n}}\|b_1^{n''}\|^{\frac{1}{n}}.
\end{aligned}$$

According to the proof of [9, Lemma 1.2.13], we obtain $r(a+b)\leq\alpha+\beta$, which yields $r(a+b)\leq r(a)+r(b)$.

(ii) This can be obtained by (i). □

Lemma 2.11 *Let $a,b\in\mathscr{A}$ be such that $a^2b=aba$ or $b^2a=bab$. Then*

(i) $r(ab)\leqslant r(a)r(b)$.

(ii) *If either a or b is quasinilpotent, then ab is quasinilpotent.*

Proof (i) Note the symmetry of $a^2b=aba$ and $b^2a=bab$, it suffices to prove the case $a^2b=aba$.

Assume $a^2b=aba$; then $(ab)^n=a^nb^n$ for $n\in\mathbb{N}$ by induction. Therefore,

$$\|(ab)^n\|^{\frac{1}{n}} = \|a^nb^n\|^{\frac{1}{n}} \leq \|a^n\|^{\frac{1}{n}}\|b^n\|^{\frac{1}{n}}.$$

Let $n\to\infty$; then we obtain that $r(ab)\leqslant r(a)r(b)$.

(ii) This follows from (i) directly. □

3. Main results

In this section, for $a,b\in\mathscr{A}^d$, we will investigate the representations of $(ab)^d$ and $(a+b)^d$ under the new conditions $a^2b=aba$ and $b^2a=bab$.

We start with a theorem that is an extension of [10, Theorem 5.5].

Theorem 3.1 *Let $a,b\in\mathscr{A}^d$ be such that $a^2b=aba$ and $b^2a=bab$. Then $ab\in\mathscr{A}^d$ and $(ab)^d=a^db^d$.*

Proof We consider the matrix representations of a and b relative to the idempotent $p = aa^d$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p,$$

where $a_1 \in (p\mathscr{A}p)^{-1}$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$.

The condition $a^2b = aba$ expressed in matrix form yields

$$\begin{bmatrix} a_1^2b_1 & a_1^2b_3 \\ a_2^2b_4 & a_2^2b_2 \end{bmatrix}_p = a^2b = aba = \begin{bmatrix} a_1b_1a_1 & a_1b_3a_2 \\ a_2b_4a_1 & a_2b_2a_2 \end{bmatrix}_p.$$

Thus, we have $a_1^2b_3 = a_1b_3a_2$, i.e. $b_3 = a_1^{-1}b_3a_2$, which implies $b_3 = a_1^{-n}b_3a_2^n$ for any $n \in \mathbb{N}$. Since $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$, then

$$\|b_3\|^{\frac{1}{n}} = \|a_1^{-n}b_3a_2^n\|^{\frac{1}{n}} \leqslant \|a_1^{-1}\|\|b_3\|^{\frac{1}{n}}\|a_2^n\|^{\frac{1}{n}} \xrightarrow{n\to\infty} 0.$$

Hence, $b_3 = 0$. Similarly, from $a_2b_4 = a_2^2b_4a_1^{-1}$, it follows that $a_2b_4 = 0$. In addition, we can get $a_1b_1 = b_1a_1$ and $a_2^2b_2 = a_2b_2a_2$. Then we have

$$b = \begin{bmatrix} b_1 & 0 \\ b_4 & b_2 \end{bmatrix}_p \text{ and } ab = \begin{bmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{bmatrix}_p.$$

Next, we prove that $b_1 \in (p\mathscr{A}p)^d$ and $b_1^d = aa^db^daa^d$ by the definition of generalized Drazin inverse. Note that $b_1 = aa^dbaa^d = aa^db$ and $aa^db^daa^d = aa^db^d$ by Lemma 2.5(i). Therefore, we need to prove $b_1^d = aa^db^d$.

Let $v = aa^db^d$. Then we have

(1) $b_1v = aa^dbaa^db^d = aa^dbb^d = aa^db^daa^db = vb_1$.

(2) $vb_1v = aa^db^daa^dbaa^db^d = aa^db^dbaa^db^d = aa^dbab^da^db^d = aa^dba^dab^db^d = aa^db^d = v$.

(3) Note that $b_1 - b_1^2v = aa^db(1-bb^d)$. By induction and Lemma 2.5, we have that $(aa^db(1-bb^d))^n = aa^db^n(1-bb^d)$ for any $n \in \mathbb{N}$. Since $b(1-bb^d) \in \mathscr{A}^{qnil}$, then

$$\|(b_1 - b_1^2v)^n\|^{\frac{1}{n}} = \|aa^db^n(1-bb^d)\|^{\frac{1}{n}} \leq \|aa^d\|^{\frac{1}{n}}\|b^n(1-bb^d)\|^{\frac{1}{n}} \xrightarrow{n\to\infty} 0.$$

Thus $b_1 - b_1^2v \in (p\mathscr{A}p)^{qnil}$. Hence, $b_1^d = v$. Similarly, we have that $b_2^d = b^d(1-aa^d)$.

According to the equation $a_1b_1 = b_1a_1$ and Lemma 2.2, we have that $a_1b_1 \in (p\mathscr{A}p)^d$ and $(a_1b_1)^d = a_1^{-1}b_1^d$. Observe that $a_2^2b_2 = a_2b_2a_2$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$; applying Lemma 2.11(ii) to the elements a_2, b_2, we get $a_2b_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$, i.e. $(a_2b_2)^d = 0$.

Finally, applying Lemma 2.1(i), we have $ab \in \mathscr{A}^d$ and

$$(ab)^d = \begin{bmatrix} (a_1b_1)^d & 0 \\ 0 & (a_2b_2)^d \end{bmatrix}_p = \begin{bmatrix} a_1^{-1}b_1^d & 0 \\ 0 & 0 \end{bmatrix}_p = a^db^d.$$

□

Remark 3.2 *(1) From Lemma 2.2 and Corollary 2.4, we can see that* $(ab)^d = a^d b^d = b^d a^d$ *for commutative generalized Drazin invertible elements* $a, b \in \mathscr{A}$*. However, in general,* $(ab)^d \neq b^d a^d$ *under the conditions of Theorem 3.1. For example, let* a, b *be the same as the elements in Example 2.7. Clearly,*

$$ab = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (ab)^d.$$

However, $(ab)^d \neq b^d a^d$*.*

(2) In Theorem 3.1, if we replace $b^2 a = bab$ *with* $ba^2 = aba$*, then we can conclude that* $(ab)^d = a^d b^d = b^d a^d$*. The proof of the previous result is similar to the proof of Theorem 3.1 and so we omit the proof. The following example shows that the conditions* $a^2 b = aba$ *and* $ba^2 = aba$ *are weaker than* $ab = ba$*. Let* $\mathscr{A} = M_2(\mathbb{C})$ *and take*

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then we can get that $a^2 b = aba$ *and* $ba^2 = aba$*. However,* $ab \neq ba$*.*

Next, we present our main result, which recovers [3, Theorem 2.1].

Theorem 3.3 *Let* $a, b \in \mathscr{A}^d$ *be such that* $a^2 b = aba$ *and* $b^2 a = bab$*. Then the following conditions are equivalent:*

(i) $a + b \in \mathscr{A}^d$.

(ii) $1 + a^d b \in \mathscr{A}^d$.

(iii) $c = aa^d(a+b)bb^d \in \mathscr{A}^d$.

In this case,

$$\begin{aligned}(a+b)^d &= a^d(1+a^d b)^d + a^\pi b (a^d)^2((1+a^d b)^d)^2 + \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi \\ &\quad + \sum_{n=0}^{\infty}(n+1)b^\pi a (b^d)^{n+2}(-a)^n a^\pi,\end{aligned} \tag{4}$$

$$\begin{aligned}(a+b)^d &= c^d + \sum_{n=0}^{\infty}(a^d)^{n+1}(-b)^n b^\pi + a^\pi b (c^d)^2 + \sum_{n=0}^{\infty} a^\pi b c^d (a^d)^{n+1}(-b)^n b^\pi \\ &\quad + \sum_{n=0}^{\infty} a^\pi b (a^d)^{n+1}(-b)^n b^\pi c^d + \sum_{n=0}^{\infty}(n+1)a^\pi b (a^d)^{n+2}(-b)^n b^\pi \\ &\quad + \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi + \sum_{n=0}^{\infty}(n+1)b^\pi a (b^d)^{n+2}(-a)^n a^\pi,\end{aligned} \tag{5}$$

$$(1+a^d b)^d = a^\pi + a^2 a^d (a+b)^d \text{ and } (aa^d(a+b)bb^d)^d = aa^d(a+b)^d bb^d. \tag{6}$$

Proof As in the proof of Theorem 3.1, we have that

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } b = \begin{bmatrix} b_1 & 0 \\ b_4 & b_2 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathscr{A}p)^{-1}$, and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$. Moreover, we have $a_1b_1 = b_1a_1$, $a_2b_4 = 0$, $a_2^2b_2 = a_2b_2a_2$, $b_1^d = aa^db^d$, and $b_2^d = b^d(1-aa^d)$. From the condition $b^2a = bab$, it follows that $b_2^2a_2 = b_2a_2b_2$ and $b_2b_4 = 0$.

Let $p_1 = b_1b_1^d$ and $p_2 = b_2b_2^d$. Then $p_1p = pp_1 = p_1$ and $p_2(1-p) = (1-p)p_2 = p_2$ by Lemma 2.5. We now consider the matrix representations of b_1 and b_2 relative to idempotents p_1 and p_2, respectively. We have that

$$b_1 = \begin{bmatrix} b_1' & 0 \\ 0 & b_2' \end{bmatrix}_{p_1} \text{ and } b_2 = \begin{bmatrix} b_1'' & 0 \\ 0 & b_2'' \end{bmatrix}_{p_2},$$

where $b_1' \in (p_1\mathscr{A}p_1)^{-1}$, $b_1'' \in (p_2\mathscr{A}p_2)^{-1}$, $b_2' \in ((p-p_1)\mathscr{A}(p-p_1))^{qnil}$, and $b_2'' \in ((1-p-p_2)\mathscr{A}(1-p-p_2))^{qnil}$.

Note that $p_1a_1(p-p_1) = b_1b_1^da_1(p-b_1b_1^d) = b_1a_1b_1^d(p-b_1b_1^d) = b_1a_1(b_1^d - b_1^db_1b_1^d) = 0$. Similarly, $(p-p_1)a_1p_1 = 0$ and $p_2a_2(1-p-p_2) = 0$. Thus, we get the following matrix representations:

$$a_1 = \begin{bmatrix} a_1' & 0 \\ 0 & a_2' \end{bmatrix}_{p_1} \text{ and } a_2 = \begin{bmatrix} a_1'' & 0 \\ a_4'' & a_2'' \end{bmatrix}_{p_2}.$$

Note that $a_2^2b_2 = a_2b_2a_2$ and $b_2^2a_2 = b_2a_2b_2$; as in the proof of Theorem 3.1, we have that $b_1''a_1'' = a_1''b_1''$, $(b_2'')^2a_2'' = b_2''a_2''b_2''$ and $(a_2'')^2b_2'' = a_2''b_2''a_2''$. Moreover, $(a_1'')^d = p_2a_2^d = 0$ and $(a_2'')^d = a_2^d(1-p-p_2) = 0$, which imply a_1'' and a_2'' are quasinilpotent. Besides these, $b_2''a_4'' = a_2''a_4'' = 0$.

Next, we will prove that $a_2 + b_2 \in ((1-p)\mathscr{A}(1-p))^d$. Observe that

$$a_2 + b_2 = \begin{bmatrix} a_1'' + b_1'' & 0 \\ a_4'' & a_2'' + b_2'' \end{bmatrix}_{p_2}.$$

Since $a_1'' + b_1'' = b_1''(p_2 + (b_1'')^{-1}a_1'')$ and a_1'' is quasinilpotent, we have that $a_1'' + b_1''$ is invertible in subalgebra $p_2\mathscr{A}p_2$ and

$$(a_1'' + b_1'')^{-1} = (b_1'')^{-1}(p_2 + (b_1'')^{-1}a_1'')^{-1} = (b_1'')^{-1}(p_2 + \sum_{n=1}^{\infty}(b_1'')^{-n}(-a_1'')^n).$$

Note that $(b_1'')^{-1} = b_2^d = b^d(1-aa^d)$. By induction, we can obtain that $(b_1'')^{-n} = (b^d)^n(1-aa^d)$ for any $n \in \mathbb{N}$. In addition, we verify that

$$a_1'' = p_2a_2p_2 = b_2b_2^da_2b_2b_2^d = b_2b_2^da_2 = (ba^\pi)(b^da^\pi)(a^\pi a) = bb^da^\pi a,$$

which implies $(-a_1'')^n = bb^d(-a)^na^\pi$ for any $n \in \mathbb{N}$ by induction. Note that $a^\pi bb^da^\pi = bb^da^\pi$ and $p_2 = b_2b_2^d =$

$ba^\pi b^d a^\pi = bb^d a^\pi$; then we get

$$\begin{aligned}(a_1'' + b_1'')^{-1} &= b^d a^\pi (bb^d a^\pi + \sum_{n=1}^{\infty} (b^d)^n a^\pi (bb^d (-a)^n a^\pi) \\ &= b^d a^\pi (bb^d a^\pi + \sum_{n=1}^{\infty} (b^d)^n bb^d (-a)^n a^\pi) \\ &= b^d a^\pi bb^d a^\pi + b^d a^\pi \sum_{n=1}^{\infty} (b^d)^n (-a)^n a^\pi \\ &= b^d a^\pi + \sum_{n=1}^{\infty} (b^d)^{n+1} (-a)^n a^\pi \\ &= \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi.\end{aligned}$$

Applying Lemma 2.10(ii) to the element a_2'', b_2'', we have that $a_2'' + b_2''$ is quasinilpotent, i.e. $(a_2'' + b_2'')^d = 0$. Lemma 2.1(i) ensures that $a_2 + b_2 \in ((1-p)\mathscr{A}(1-p))^d$ and

$$(a_2 + b_2)^d = \begin{bmatrix} (a_1'' + b_1'')^{-1} & 0 \\ x & 0 \end{bmatrix}_{p_2},$$

where $x = a_4''(a_1'' + b_1'')^{-2}$. Note that

$$a_4'' = (1 - p - p_2) a_2 p_2 = (b^\pi a^\pi)(a^\pi a)(bb^d a^\pi) = b^\pi a a^\pi bb^d a^\pi = b^\pi a bb^d a^\pi.$$

Because $a^\pi (b^d)^n a^\pi = (b^d)^n a^\pi$ for any $n \in \mathbb{N}$, then

$$\begin{aligned}x &= b^\pi abb^d a^\pi (\sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi)^2 \\ &= b^\pi abb^d a^\pi (\sum_{n=0}^{\infty} (n+1)(b^d)^{n+2} (-a)^n a^\pi) \\ &= b^\pi a \sum_{n=0}^{\infty} (n+1)(b^d)^{n+2} (-a)^n a^\pi.\end{aligned}$$

Therefore, we can obtain

$$(a_2 + b_2)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi + b^\pi a \sum_{n=0}^{\infty} (n+1)(b^d)^{n+2} (-a)^n a^\pi.$$

Since

$$a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_4 & a_2 + b_2 \end{bmatrix}_p,$$

by Lemma 2.1, we have that $a + b \in \mathscr{A}^d$ if and only if $a_1 + b_1 \in (p\mathscr{A}p)^d$. In this case, we have

$$(a + b)^d = \begin{bmatrix} (a_1 + b_1)^d & 0 \\ y & (a_2 + b_2)^d \end{bmatrix}_p,$$

where $y = b_4((a_1 + b_1)^d)^2$.

(i) $\Leftrightarrow$ (ii) From

$$1 + a^d b = \begin{bmatrix} p + a_1^{-1} b_1 & 0 \\ 0 & 1 - p \end{bmatrix}_p,$$

it follows that $1+a^d b \in \mathscr{A}^d$ if and only if $p+a_1^{-1}b_1 \in (p\mathscr{A}p)^d$. By Lemma 2.2, we have that $a_1+b_1 = a_1(p+a_1^{-1}b_1) \in (p\mathscr{A}p)^d$ if and only if $p+a_1^{-1}b_1 \in (p\mathscr{A}p)^d$. Hence, $a+b \in \mathscr{A}^d$ if and only if $1+a^d b \in \mathscr{A}^d$. In this case, we have

$$(1+a^d b)^d = \begin{bmatrix} (p+a_1^{-1}b_1)^d & 0 \\ 0 & 1-p \end{bmatrix}_p .$$

Moreover, we deduce that

$$(a_1+b_1)^d = a_1^{-1}(p+a_1^{-1}b_1)^d = a^d((1+a^d b)^d-(1-p)) = a^d(1+a^d b)^d.$$

By a straightforward computation, we obtain that the equation (4) holds.

(i) $\Leftrightarrow$ (iii) From $a_1 \in (p\mathscr{A}p)^{-1}$, we have $a_1' \in (p_1\mathscr{A}p_1)^{-1}$ and $a_2' \in ((p-p_1)\mathscr{A}(p-p_1))^{-1}$. Note that $a_2'b_2' = b_2'a_2'$ and b_2' is quasinilpotent; then $a_2'+b_2' = a_2'((p-p_1)+(a_2')^{-1}b_2')$ is invertible in subalgebra $(p-p_1)\mathscr{A}(p-p_1)$ and $(a_2'+b_2')^{-1} = \sum\limits_{n=0}^{\infty}(a^d)^{n+1}(-b)^n b^\pi$, which is similar to the proof of the expression for $(a_1''+b_1'')^{-1}$. Since

$$a_1+b_1 = \begin{bmatrix} a_1'+b_1' & 0 \\ 0 & a_2'+b_2' \end{bmatrix}_{p_1},$$

we have $a_1+b_1 \in (p\mathscr{A}p)^d$ if and only if $a_1'+b_1' \in (p_1\mathscr{A}p_1)^d$. In this case,

$$(a_1+b_1)^d = (a_1'+b_1')^d + (a_2'+b_2')^{-1}.$$

The following matrix representations

$$c = aa^d(a+b)bb^d = \begin{bmatrix} (a_1+b_1)b_1b_1^d & 0 \\ 0 & 0 \end{bmatrix}_p \text{ and } (a_1+b_1)b_1b_1^d = \begin{bmatrix} a_1'+b_1' & 0 \\ 0 & 0 \end{bmatrix}_{p_1}$$

yield the equality $c = a_1'+b_1'$. Therefore, we conclude that $a+b \in \mathscr{A}^d$ if and only if $c \in \mathscr{A}^d$. In this case, we have

$$y = a^\pi b((c^d)^2 + \sum_{n=0}^{\infty} c^d(a^d)^{n+1}(-b)^n b^\pi + \sum_{n=0}^{\infty}(a^d)^{n+1}(-b)^n b^\pi c^d + \sum_{n=0}^{\infty}(n+1)(a^d)^{n+2}(-b)^n b^\pi),$$

and the equation (5) holds. Finally, the equation (6) can be obtained by an elemental computation. □

Next, we consider some specializations of our main result.

Corollary 3.4 [3, Theorem 2.1] *Let $a,b \in \mathscr{A}^d$ be such that $ab = ba$ Then $a+b \in \mathscr{A}^d$ if and only if $1+a^d b \in \mathscr{A}^d$. In this case,*

$$(a+b)^d = a^d(1+a^d b)^d bb^d + \sum_{n=0}^{\infty} b^\pi(-b)^n(a^d)^{n+1} + \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi. \quad (7)$$

Proof Only the expression for $(a+b)^d$ needs a proof. It follows directly from (6) that $(aa^d(a+b)bb^d)^d = a^d(1+a^d b)^d bb^d$. Note that $a^\pi a^d = 0$ and $b^\pi b^d = 0$; then the equation (7) holds by (5). □

Corollary 3.5 *Let $a, b \in \mathscr{A}^d$ be such that $a^2b = aba$ and $b^2a = bab$.*

(i) *If $1 \notin \sigma(-a^db)$ (or $\sigma(a^db) = \{0\}$), then $a + b \in \mathscr{A}^d$,*

$$(a+b)^d = a^d(1+a^db)^{-1} + a^\pi b(a^d)^2(1+a^db)^{-2} + \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi + \sum_{n=0}^{\infty}(n+1)b^\pi a(b^d)^{n+2}(-a)^n a^\pi,$$

and

$$(1+a^db)^{-1} = a^\pi + a^2a^d(a+b)^d.$$

(ii) *If $\sigma(b) = \{0\}$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = a^d(1+a^db)^{-1} + a^\pi b(a^d)^2(1+a^db)^{-2} = \sum_{n=0}^{\infty}(a^d)^{n+1}(-b)^n + \sum_{n=0}^{\infty}(n+1)a^\pi b(a^d)^{n+2}(-b)^n.$$

Proof (i) This follows from Theorem 3.3 directly.

(ii) Since $\sigma(b) = \{0\}$, then $b \in \mathscr{A}^{qnil}$, i.e. $b^d = 0$, which implies $aa^d(a+b)bb^d = 0$. Thus, we have that $a + b \in \mathscr{A}^d$ by Theorem 3.3. To show that $1 + a^db \in \mathscr{A}^{-1}$, it suffices to prove that $a^db \in \mathscr{A}^{qnil}$. From Lemma 2.5(i), it follows that $(a^d)^2b = a^dba^d$, which yields $a^db \in \mathscr{A}^{qnil}$ by Lemma 2.11(ii). The expressions of $(a+b)^d$ can be obtained by the equations (4) and (5). □

4. Applications to block matrices

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_p \in \mathscr{A} \tag{8}$$

relative to idempotent $p \in \mathscr{A}$, $a \in (p\mathscr{A}p)^d$, and let $s = d - ca^db \in ((1-p)\mathscr{A}(1-p))^d$ be the generalized Schur complement of a in x.

In this section, we get some representations for the generalized Drazin inverse of a block matrix x with applications of our previous result.

For future reference we state two known results.

Lemma 4.1 [1, Example 4.5] *Let $a, b \in \mathscr{A}^d$. If $ab = 0$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = \sum_{n=0}^{\infty}(b^d)^{n+1}a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

Lemma 4.2 [11, Lemma 2.1] *Let x be defined as in (8). Then the following statements are equivalent:*

(i) *$x \in \mathscr{A}^d$ and $x^d = r$, where*

$$r = \begin{bmatrix} a^d + a^dbs^dca^d & -a^dbs^d \\ -s^dca^d & s^d \end{bmatrix}; \tag{9}$$

(ii) *$a^\pi bs^d = a^dbs^\pi$, $s^\pi ca^d = s^dca^\pi$, and* $y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} \in \mathscr{A}^{qnil}$.

Note that, in Lemma 4.2, if $y=0$, then we can check that $xrx=x$, and so that we have the following corollary.

Corollary 4.3 *Let x be defined as in (8). If $a^\pi bs^d = a^d bs^\pi$, $s^\pi ca^d = s^d ca^\pi$, and* $y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} = 0$*, then $x \in \mathscr{A}^\#$ and*

$$x^\# = \begin{bmatrix} a^\# + a^\# bs^\# ca^\# & -a^\# bs^\# \\ -s^\# ca^\# & s^\# \end{bmatrix}.$$

Remark 4.4 *For item (ii) of Lemma 4.2, we can see that $a^\pi bs^d = a^d bs^\pi$ is equivalent to $a^\pi bs^d = a^d bs^\pi = 0$. Moreover, $s^\pi ca^d = s^d ca^\pi$ is equivalent to $s^\pi ca^d = s^d ca^\pi = 0$. Now, we drop any one of the four equations $a^\pi bs^d = 0$, $a^d bs^\pi = 0$, $s^\pi ca^d = 0$, $s^d ca^\pi = 0$ and replace the quasinilpotency by the generalized Drazin invertibility of y. Here, we only give the one of the four cases. Similarly, we can prove the others.*

Theorem 4.5 *Let x be defined as in (8). If $a^\pi bs^d = 0$, $s^\pi ca^d = 0$, $s^d ca^\pi = 0$, and* $y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} \in \mathscr{A}^d$*, then $x \in \mathscr{A}^d$ and*

$$x^d = \begin{bmatrix} a^\pi & -a^d bs^\pi \\ 0 & s^\pi \end{bmatrix} y^d + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} p & a^d bs^\pi \\ 0 & 1-p \end{bmatrix} y^n y^\pi, \tag{10}$$

where r is defined as in (9).

Proof From the condition $s^\pi ca^d = 0$, we have $s^\pi ca^\pi + ss^d c = c$ and $s^\pi s + ss^d d = d$. Then we can write

$$x = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & s^\pi s \end{bmatrix} + \begin{bmatrix} a^2 a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix} := y + z.$$

The equations $a^\pi a^d = 0$ and $a^\pi bs^d = 0$ imply $yz = 0$.

To show that $z \in \mathscr{A}^d$, we consider the following decomposition:

$$z = \begin{bmatrix} 0 & aa^d bs^\pi \\ 0 & ss^d ds^\pi \end{bmatrix} + \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d c & ss^d dss^d \end{bmatrix} := z_1 + z_2.$$

Clearly, $z_1 z_2 = 0$ and $z_1^2 = 0$.

Next, we will prove that $z_2 \in \mathscr{A}^d$. Let $z_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, where $a_2 = a^2 a^d$, $b_2 = aa^d bss^d$, $c_2 = ss^d c$, and $d_2 = ss^d dss^d$. It is clear that a_2 is group invertible, $a_2^\# = a^d$, and $a_2^\pi = a^\pi$. Note that $s_2 := d_2 - c_2 a_2^\# b_2 = ss^d dss^d - ss^d ca^d bss^d = s^2 s^d$, which gives s_2 is group invertible, $s_2^\# = s^d$, and $s_2^\pi = s^\pi$. Furthermore, we can deduce that $a_2^\pi b_2 s_2^\# = 0$, $a_2^\# b_2 s_2^\pi = 0$, $s_2^\pi c_2 a_2^\# = 0$, $s_2^\# c_2 a_2^\pi = s^d ca^\pi = 0$, and $y_2 := \begin{bmatrix} a_2 a_2^\pi & a_2^\pi b_2 \\ s_2^\pi c_2 a_2^\pi & s_2 s_2^\pi \end{bmatrix} = 0$. By Corollary 4.3, we obtain that z_2 is group invertible and $z_2^\# = r$, where r is defined as in (9).

It follows directly from Lemma 4.1 that $z \in \mathscr{A}^d$ and $z^d = r + r^2 z_1$. By a direct computation, we have $z^\pi = \begin{bmatrix} a^\pi & -a^d b s^\pi \\ 0 & s^\pi \end{bmatrix}$ and $zz^\pi = 0$. Thus, z is group invertible.

Finally, we deduce that $x \in \mathscr{A}^d$ by Lemma 4.1 again. In addition, the equation (10) holds. □

In the following result, we give a new representation for the generalized Drazin inverse of block matrix x in (8) in terms of a^d and s^d.

Theorem 4.6 *Let x be defined as in (8). If $aa^\pi bc = 0, ca^\pi bc = 0$, $a^\pi bca^\pi b = 0$, $s^\pi ca = 0$, $a^2 a^\pi b + bca^\pi b = aa^\pi bd$, and $caa^\pi b + dca^\pi b = ca^\pi bd$, then $x \in \mathscr{A}^d$ and*

$$\begin{aligned} x^d =& w + \sum_{n=1}^{\infty} w^{n+1} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1} a^\pi & 0 \end{bmatrix} - 2 \sum_{n=1}^{\infty} w^{n+2} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & ca^{n-1} a^\pi b \end{bmatrix} \\ &+ \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} \left(w^2 + \sum_{n=1}^{\infty} w^{n+2} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1} a^\pi & 0 \end{bmatrix} \right), \end{aligned} \tag{11}$$

where

$$w^k = r^k \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1-p \end{bmatrix} + \sum_{n=1}^{\infty} r^{n+k} \begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix}, \quad k \in \mathbb{N}, \tag{12}$$

and r is defined as in (9).

Proof Since $aa^d b + a^\pi b = b$, then

$$x = \begin{bmatrix} a & aa^d b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} := x_1 + x_2.$$

By a computation, the hypotheses imply $x_1^2 x_2 = x_1 x_2 x_1$ and $x_2^2 x_1 = x_2 x_1 x_2$.

We must show that $x_1 \in \mathscr{A}^d$. Let

$$x_1 = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix} + \begin{bmatrix} a^2 a^d & aa^d b \\ caa^d & d \end{bmatrix} := x_1' + x_1'';$$

then $x_1' x_1'' = 0$.

In order to prove that $x_1'' \in \mathscr{A}^d$, we can write $x_1'' := \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, where $a_1 = a^2 a^d, b_1 = aa^d b, c_1 = caa^d$ and $d_1 = d$. Obviously, a_1 is group invertible, $a_1^\# = a^d$, and $a_1^\pi = a^\pi$. Besides these, we can obtain that $s_1 := d_1 - c_1 a_1^\# b_1 = d - ca^d b = s \in \mathscr{A}^d$. Moreover, we clearly have that $a_1^\pi b_1 s_1^d = a^\pi aa^d bs^d = 0$, $s_1^\pi c_1 a_1^\# = s^\pi ca^d = 0$, and $s_1^d c_1 a_1^\pi = s^d caa^d a^\pi = 0$. Let $y_1 := \begin{bmatrix} a_1 a_1^\pi & a_1^\pi b_1 \\ s_1^\pi c_1 a_1^\pi & s_1 s_1^\pi \end{bmatrix}$, then $y_1 = \begin{bmatrix} 0 & 0 \\ 0 & ss^\pi \end{bmatrix} \in \mathscr{A}^{qnil}$. Therefore, according to Theorem 4.5, we have that $x_1'' \in \mathscr{A}^d$ and $(x_1'')^d = w$, where

$$w = r \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1-p \end{bmatrix} + \sum_{n=1}^{\infty} r^{n+1} \begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix}.$$

Observe that $\sigma(x_1') \subseteq \sigma(aa^\pi)\cup\{0\}$ and $aa^\pi \in \mathscr{A}^{qnil}$; then $x_1' \in \mathscr{A}^{qnil}$, i.e. $(x_1')^d = 0$. Applying Lemma 4.1, we deduce that $x_1 \in \mathscr{A}^d$ and

$$x_1^d = w + \sum_{n=1}^{\infty} w^{n+1} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1}a^\pi & 0 \end{bmatrix}.$$

From the equality $x_2^2 = 0$, it follows that $x_2^d = 0$, which yields $x_1 x_1^d (x_1 + x_2) x_2 x_2^d = 0 \in \mathscr{A}^d$. Applying Theorem 3.3, we obtain that $x \in \mathscr{A}^d$ and

$$x^d = x_1^d - (x_1^d)^2 x_2 + x_1^\pi x_2 (x_1^d)^2 - 2x_1^\pi x_2 (x_1^d)^3 x_2.$$

Note that $x_2(x_1^d)^3 x_2 = x_2^2 (x_1^d)^3 = 0$ by Lemma 2.5. Then

$$x^d = x_1^d - (x_1^d)^2 x_2 + x_1^\pi x_2 (x_1^d)^2 = x_1^d - 2(x_1^d)^2 x_2 + x_2 (x_1^d)^2. \tag{13}$$

Next, we prove the expression of x^d. Note that, for $n \in \mathbb{N}$,

$$\begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix} r = 0 \quad \text{and} \quad \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1-p \end{bmatrix} r = r;$$

then the equation (12) holds. By substituting the expression of x_1^d into the equation (13) and using the following equalities

$$\begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1}a^\pi & 0 \end{bmatrix} r = 0 \quad \text{and} \quad w \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} = 0,$$

we can get the equation (11). □

From Theorem 4.6, we can obtain the following corollary, which recovers [5, Theorem 8] for a 2×2 operator matrix.

Corollary 4.7 *Let x be defined as in (8). If $a^\pi bc = 0$, $ca^\pi b = 0$, $aa^\pi b = a^\pi bd$, and $s = d - ca^d b$ is invertible, then $x \in \mathscr{A}^d$ and*

$$x^d = \left(r - \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r^2 \right) \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^\pi a^n & 0 \end{bmatrix} \right), \tag{14}$$

where r is defined as in (9) with $s^d = s^{-1}$.

Proof As in the proof of Theorem 4.6. Note that $x_1 x_2 = x_2 x_1$; then $x_1^d x_2 = x_2 x_1^d$. Thus $x^d = x_1^d - (x_1^d)^2 x_2$. By a computation, we can get the equation (14). □

Remark 4.8 *Theorem 4.6 generalizes [15, Theorem 2.3], where an expression for x^d is given under the conditions $a^\pi b = 0$ and $s^\pi ca = 0$. Indeed, $a^\pi b = 0$ and $s^\pi ca = 0$ can imply the conditions of Theorem 4.6. However, in general, the converse is false. The following example can illustrate this fact.*

Example 4.9 *Let $\mathscr{A}$ be the Banach algebra of all complex 3×3 matrices, and take*

$$x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } c = d = 0.$$

Obviously,

$$a^d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } a^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can see that the conditions of Theorem 4.6 hold. However, $a^\pi b \neq 0$.

Following the same strategy as in the proof of Theorem 4.6, we derive another formula for x^d. Here we omit the proof.

Theorem 4.10 *Let x be defined as in (8). If $bca^\pi b = 0, dca^\pi b = 0$, $ca^\pi bca^\pi = 0$, $s^\pi ca = 0$, $d^2ca^\pi + cbca^\pi = dcaa^\pi$, and $abca^\pi + bdca^\pi = bcaa^\pi$, then $x \in \mathscr{A}^d$ and*

$$\begin{aligned} x^d =& w + \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & a^{n-1}a^\pi b \\ 0 & 0 \end{bmatrix} w^{n+1} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & ca^{n-1}a^\pi b \end{bmatrix} w^{n+2} \\ & - 2\left(w^2 + \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & a^{n-1}a^\pi b \\ 0 & 0 \end{bmatrix} w^{n+2}\right) \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix}, \end{aligned} \tag{15}$$

where w^k is defined as in (12) for $k \in \mathbb{N}$, and r is defined as in (9).

Now, we state a special case of Theorem 4.10, which also generalizes [5, Theorem 9] for a 2×2 operator matrix.

Corollary 4.11 *Let x be defined as in (8). If $bca^\pi = 0$, $ca^\pi b = 0$, $caa^\pi = dca^\pi$, and $s = d - ca^d b$ is invertible, then $x \in \mathscr{A}^d$ and*

$$x^d = r + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & ca^n a^\pi b \end{bmatrix} r^{n+3},$$

where r is defined as in (9) with $s^d = s^{-1}$.

Remark 4.12 *Theorem 4.10 extends [16, Theorem 3.2], where the generalized Drazin inverse of x is considered in the case that $bca^\pi = 0$, $dca^\pi = 0$, $s^\pi ca = 0$, and $abs^\pi = 0$. In fact, Example 4.9 can also illustrate that the conditions of Theorem 4.10 are weaker than those of [16, Theorem 3.2].*

Acknowledgments

The authors would like to thank the referees and editors for their helpful suggestions for the improvement of this paper. This research was supported by the National Natural Science Foundation of China (No. 11371089), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120092110020), the Natural Science Foundation of Jiangsu Province (No. BK20141327), the Fundamental Research Funds for the Central Universities, and the Foundation of Graduate Innovation Program of Jiangsu Province (No. KYZZ15-0049). The second author is supported by the Ministry of Education and Science, Republic of Serbia, grant no. 174007.

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