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f-Biminimal immersions

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Abstract: In the present paper, we define f-biminimal immersions. We consider f-biminimal curves in a Riemannian manifold and f-biminimal submanifolds of codimension 1 in a Riemannian manifold, and we give examples of f-biminimal surfaces. Finally, we consider f-biminimal Legendre curves in Sasakian space forms and give an example.

Key words: f-Biminimal immersion, f-biminimal curve, f-biminimal surface, Legendre curve

1. Introduction and preliminaries

Let (M, g) and (N, h) be two Riemannian manifolds. A map $\varphi : (M, g) \to (N, h)$ is called a *harmonic map* if it is a critical point of the *energy functional*

$$E(\varphi) = \frac{1}{2} \int_{\Omega} \|d\varphi\|^2 \, d\nu_g,$$

where Ω is a compact domain of M. The Euler–Lagrange equation gives the harmonic map equation

$$\tau(\varphi) = tr(\nabla d\varphi) = 0,$$

where $\tau(\varphi) = tr(\nabla d\varphi)$ is called the *tension field* of the map φ [6]. The map φ is said to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2} \int_{\Omega} \|\tau(\varphi)\|^2 \, d\nu_g,$$

where Ω is a compact domain of M [10]. In [10], Jiang obtained the Euler–Lagrange equation of $E_2(\varphi)$. This gives us the biharmonic map equation

$$\tau_2(\varphi) = tr(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.1}$$

which is the *bitension field* of φ, and R^N is the curvature tensor of N, defined by

$$R^N(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z.$$

An f-*harmonic map* with a positive function $f : M \overset{C^\infty}{\to} \mathbb{R}$ is a critical point of f-energy

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$$E_f(\varphi) = \frac{1}{2}\int_\Omega f \left\| d\varphi \right\|^2 d\nu_g,$$

where Ω is a compact domain of M. Using the Euler–Lagrange equation for the f-harmonic map, in [5] and [16] the f-harmonic map equation is obtained by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(gradf) = 0, \tag{1.2}$$

where $\tau_f(\varphi)$ is called the f-*tension field* of the map φ. The map φ is said to be f-*biharmonic* [13] if it is a critical point of the f-bienergy functional

$$E_{2,f}(\varphi) = \frac{1}{2}\int_\Omega f \left\| \tau(\varphi) \right\|^2 d\nu_g,$$

where Ω is a compact domain of M. The Euler–Lagrange equation for the f-biharmonic map is given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla^{\varphi}_{gradf}\tau(\varphi) = 0, \tag{1.3}$$

where $\tau_{2,f}(\varphi)$ is the f-*bitension field* of the map φ [13]. If f is a constant, an f-biharmonic map turns into a biharmonic map.

In [12], Loubeau and Montaldo defined and considered biminimal immersions. They studied biminimal curves in a Riemannian manifold, curves in a space form, and isometric immersions of codimension 1 in a Riemannian manifold.

An immersion φ is called *biminimal* [12] if it is a critical point of the bienergy functional $E_2(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi) \tag{1.4}$$

for any smooth variation of the map $\varphi_t :]-\epsilon, +\epsilon[$, $\varphi_0 = \varphi$, such that $V = \frac{d\varphi_t}{dt}\mid_{t=0} = 0$ is normal to $\varphi(M)$. The Euler–Lagrange equation for a λ-biminimal immersion is

$$[\tau_{2,\lambda}(\varphi)]^\perp = [\tau_2(\varphi)]^\perp - \lambda[\tau(\varphi)]^\perp = 0 \tag{1.5}$$

for some value of $\lambda \in \mathbb{R}$, where $[\cdot]^\perp$ denotes the normal component of $[\cdot]$. An immersion is called *free biminimal* if it is biminimal for $\lambda = 0$ [12].

In [12], Loubeau and Montaldo studied biminimal immersions. In [9], Inoguchi and Lee completely classified biminimal curves in 2-dimensional space forms. In [8], Inoguchi studied biminimal curves and surfaces in contact 3-manifolds. In [13], Lu defined f-biharmonic maps between Riemannian manifolds. In [15], Ou considered f-biharmonic maps and f-biharmonic submanifolds. In [7], Güvenç and the second author studied f-biharmonic Legendre curves in Sasakian space forms. Motivated by the studies [12] and [13], in this paper, we define f-biminimal immersions. We consider f-biminimal curves in a Riemannian manifold. We also consider f-biminimal submanifolds of codimension 1 in a Riemannian manifold and give some examples of f-biminimal surfaces. Furthermore, we give an example for an f-biminimal Legendre curve in a Sasakian space form.

Now we give the following definition:

Definition 1.1 *An immersion φ is called f-biminimal if it is a critical point of the f-bienergy functional $E_{2,f}(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-f-bienergy*

$$E_{2,\lambda,f}(\varphi) = E_{2,f}(\varphi) + \lambda E_f(\varphi)$$

for any smooth variation of the map φ_t defined above. Using the Euler–Lagrange equations for f-harmonic and f-biharmonic maps, an immersion is f-biminimal if

$$[\tau_{2,\lambda,f}(\varphi)]^{\perp} = [\tau_{2,f}(\varphi)]^{\perp} - \lambda[\tau_f(\varphi)]^{\perp} = 0 \tag{1.6}$$

for some value of $\lambda \in \mathbb{R}$. We call an immersion free f-biminimal if it is f-biminimal for $\lambda = 0$. If f is a constant, then the immersion is biminimal.

Remark 1.1 *The notions of f-biharmonic submanifolds, biminimal submanifolds, and f-biminimal submanifolds are distinct. We will see details in the examples given in Section 4 and Section 5.*

2. f-Biminimal curves

Let $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ be a curve parametrized by arc length in a Riemannian manifold (M^m, g). We recall the definition of Frenet frames:

Definition 2.1 *[11] The Frenet frame $\{E_i\}_{i=1,2,\dots m}$ associated with a curve $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ is the orthonormalization of the $(m+1)-$tuple*

$$\left\{ \nabla^{(k)}_{\frac{\partial}{\partial t}} d\gamma(\frac{\partial}{\partial t}) \right\}_{k=0,1,\dots,m}$$

described by

$$E_1 = d\gamma(\frac{\partial}{\partial t}),$$

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} E_1 = k_1 E_2,$$

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} E_i = -k_{i-1}E_{i-1} + k_i E_{i+1}, \quad 2 \leq i \leq m-1,$$

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} E_m = -k_{m-1}E_{m-1},$$

where the functions $\{k_1 = k, k_2 = \tau, k_3, ..., k_{m-1}\}$ are called the curvatures of γ. In addition $E_1 = T = \gamma'$ is the unit tangent vector field to the curve.

First, we have the following proposition for an f-biminimal curve in a Riemannian manifold:

Proposition 2.1 *Let M^m be a Riemannian manifold and $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ be an isometric curve. Then γ is f-biminimal if and only if there exists a real number λ such that*

$$f\left\{\left(k_1'' - k_1^3 - k_1 k_2^2\right) - k_1 g(R(E_1, E_2)E_1, E_2)\right\} + \left(f'' - \lambda f\right) k_1 + 2f'k' = 0, \tag{2.1}$$

$$f\left\{(k_1'k_2+(k_1k_2)')-k_1g(R(E_1,E_2)E_1,E_3)\right\}+2f'k_1k_2=0, \tag{2.2}$$

$$f\left\{k_1k_2k_3-k_1g(R(E_1,E_2)E_1,E_4)\right\}=0, \tag{2.3}$$

$$fk_1g(R(E_1,E_2)E_1,E_j)=0,\quad 5\leq j\leq m, \tag{2.4}$$

where R is the curvature tensor of (M^m,g) and $\{E_i\}_{i=1,2,\ldots m}$ is the Frenet frame of γ.

Proof Using equation (1.2), Definition 2.1, and $\tau(\gamma)=k_1E_2$ (see [12]), the f-tension field of γ is

$$\tau_f(\gamma)=fk_1E_2+f'E_1. \tag{2.5}$$

From Definition 2.1, we have

$$\nabla_T\nabla_T T=-k_1^2E_1+k_1'E_2+k_1k_2E_3, \tag{2.6}$$

$$\nabla_T\nabla_T\nabla_T T=-3k_1k_1^{'}E_1+\left(k_1''-k_1^3-k_1k_2^2\right)E_2$$

$$+\left(k_1'k_2+(k_1k_2)'\right)E_3+\left(k_1k_2k_3\right)E_4 \tag{2.7}$$

and

$$\nabla_{grad f}\tau(\gamma)=f^{'}\left\{-k_1^2E_1+k_1'E_2+k_1k_2E_3\right\}. \tag{2.8}$$

Using equations (2.6), (2.7), and (2.8) in equation (1.3), its f-bitension field is

$$\tau_{2,f}(\gamma)=f\left\{\left(-3k_1k_1'\right)E_1+\left(k_1''-k_1^3-k_1k_2^2\right)E_2+\left(k_1'k_2+(k_1k_2)'\right)E_3\right.$$

$$\left.+\left(k_1k_2k_3\right)E_4-k_1R(E_1,E_2)E_1\right\}$$

$$+f''k_1E_2+2f'\left\{-k_1^2E_1+k_1'E_2+k_1k_2E_3\right\}. \tag{2.9}$$

By the use of equations (2.5) and (2.9) in equation (1.6), we find

$$f\left\{\left(k_1''-k_1^3-k_1k_2^2\right)E_2+\left(k_1'k_2+(k_1k_2)'\right)E_3\right.$$

$$\left.+\left(k_1k_2k_3\right)E_4-k_1\left[R(E_1,E_2)E_1\right]^{\perp}\right\}$$

$$+f''k_1E_2+2f'\left\{k_1'E_2+k_1k_2E_3\right\}-\lambda\left\{fk_1E_2\right\}=0. \tag{2.10}$$

Then taking the scalar product of equation (2.10) with E_2,E_3,E_4, and E_j, $5\leq j\leq m$, respectively, we obtain the desired results. □

Now we investigate f-biminimality conditions for a surface or a three-dimensional Riemannian manifold with a constant sectional curvature. We have the following corollary:

Corollary 2.1 1) *A curve γ on a surface of Gaussian curvature G is f-biminimal if and only if its signed curvature k satisfies the equation*

$$f\left(k'' - k^3 + kG\right) + \left(f'' - \lambda f\right)k + 2f'k' = 0 \tag{2.11}$$

for some $\lambda \in \mathbb{R}$.

2) *A curve γ on Riemannian 3-manifold M of constant sectional curvature c is f-biminimal if and only if its curvature k and torsion τ satisfy the system*

$$f\left(k'' - k^3 - k\tau^2 + kc\right) + \left(f'' - \lambda f\right)k + 2f'k' = 0$$

$$f\left(k'\tau + (k\tau)'\right) + 2f'k\tau = 0 \tag{2.12}$$

for some $\lambda \in \mathbb{R}$.

Proof 1) Since γ is a curve on a surface, if γ is f-biminimal then by the use of equation (2.1), we obtain

$$f\left\{k'' - k^3 - kg(R(T,N)T,N)\right\} + \left(f'' - \lambda f\right)k + 2f'k' = 0. \tag{2.13}$$

Then we have

$$g(R(T,N)T,N) = -G. \tag{2.14}$$

Finally, substituting equation (2.14) into equation (2.13), we obtain

$$f\left\{k'' - k^3 + kG\right\} + \left(f'' - \lambda f\right)k + 2f'k' = 0.$$

2) Since γ is a curve on a Riemannian 3-manifold, the Frenet frame of γ is $\{T,\ N = B_2,\ B = B_3\}$, and then equations (2.1) and (2.2) turn into

$$f\left\{k'' - k^3 - k\tau^2 - kg(R(T,N)T,N)\right\} + \left(f'' - \lambda f\right)k + 2f'k' = 0 \tag{2.15}$$

and

$$f\left\{k'\tau + (k\tau)' - kg(R(T,N)T,B)\right\} + 2f'k\tau = 0. \tag{2.16}$$

Since M has constant sectional curvature we have

$$g(R(T,N)T,N) = -c \tag{2.17}$$

and

$$g(R(T,N)T,B) = 0. \tag{2.18}$$

Finally, substituting equations (2.17) and (2.18) into equations (2.15) and (2.16), respectively, we get

$$f\left\{k'' - k^3 - k\tau^2 + kc\right\} + \left(f'' - \lambda f\right)k + 2f'k' = 0$$

and

$$f\left\{k'\tau + (k\tau)'\right\} + 2f'k\tau = 0.$$

This completes the proof. □

Remark 2.1 *In Proposition 2.1 and Corollary 2.1, if we take f as a constant, we obtain Proposition 2.2 and Corollary 2.4 in [12].*

Now assume that $M^2 \subset \mathbb{R}^3$ is a surface of revolution obtained by rotating the arc length parametrized curve $\alpha(u) = (h(u), 0, g(u))$ in the xz-plane around the z-axis. Then it can be easily seen that the Gaussian curvature G of the surface of revolution is

$$G = -\frac{h''(u)}{h(u)}. \tag{2.19}$$

The Gaussian curvature G depends only on u; that is, G is constant along any parallel. This implies that if the Gaussian curvature is constant along a curve, then either the curve is a parallel or the curve lies in a part of the surface with constant Gaussian curvature [4]. From equation (2.19) and equation (2.11), it is easy to see that if a parallel of M is f-biminimal then f is a constant, which means that the parallel is biminimal. Biminimal curves in a surface of revolution was studied by Aykut in [1]. Hence, we can state the following result:

Proposition 2.2 *An f-biminimal parallel in a surface of revolution is biminimal.*

3. Codimension-1 f-biminimal submanifolds

Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension 1. We shall denote by B, η, A, Δ, and $H_1 = H\eta$ the second fundamental form, the unit normal vector field, the shape operator, the Laplacian, and the mean curvature vector field of φ (H the mean curvature function), respectively.

Then we have the following proposition:

Proposition 3.1 *Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension 1 and $H_1 = H\eta$ its mean curvature vector. Then φ is f-biminimal if and only if*

$$\Delta H - H\left\|B\right\|^2 + HRicci(\eta,\eta) + \left(\frac{\Delta f}{f} - \lambda\right)H + 2grad\ln f\,(H) = 0 \tag{3.1}$$

for some value of λ in $\mathbb{R}$.

Proof Assume that φ is f-biminimal. Let $\{e_i\}$, $1 \leq i \leq m$ be a local geodesic orthonormal frame at $p \in M$. Then using equation (1.2), the f-tension field of φ is

$$\tau_f(\varphi) = fmH\eta + d\varphi(gradf) \tag{3.2}$$

and using equation (1.3) and the definitions of $\tau(\varphi)$ and $\tau_2(\varphi)$ in [12], its f-bitension field is

$$\tau_{2,f}(\varphi) = f\left\{m(\Delta H)\eta + 2m\sum_{i=1}^{m} e_i(H)\nabla^{\varphi}_{e_i}\eta - mH\Delta^{\varphi}\eta\right.$$

$$\left.-mH\sum_{i=1}^{m} R^N(d\varphi(e_i),\eta)d\varphi(e_i)\right\} + \Delta f(mH\eta) + 2m\nabla^{\varphi}_{gradf}H\eta. \tag{3.3}$$

Then taking the scalar product of equations (3.2) and (3.3) with η, respectively, we find

$$g(\tau_f(\varphi),\eta) = fmH \tag{3.4}$$

and

$$g(\tau_{2,f}(\varphi),\eta) = f\left\{m(\Delta H) + 2m\sum_{i=1}^{m} e_i(H)g(\nabla^{\varphi}_{e_i}\eta,\eta) - mHg(\Delta^{\varphi}\eta,\eta)\right.$$

$$\left. -mHg(\sum_{i=1}^{m} R^N(d\varphi(e_i),\eta)d\varphi(e_i),\eta)\right\} + \Delta f(mH) + 2mg(\nabla^{\varphi}_{gradf}H\eta,\eta). \tag{3.5}$$

By use of the Weingarten formula, we have

$$\nabla^{\varphi}_{gradf}H\eta = (gradf(H))\eta + H\nabla^{\varphi}_{gradf}\eta$$

$$= (gradf(H))\eta + H(-A_\eta gradf + \nabla^{\perp}_{gradf}\eta)$$

$$= (gradf(H))\eta - HA_\eta gradf.$$

Hence, taking the scalar product of the above equation with η, we obtain

$$g(\nabla^{\varphi}_{gradf}H\eta,\eta) = gradf(H). \tag{3.6}$$

Moreover, we have

$$g(\nabla^{\varphi}_{e_i}\eta,\eta) = \frac{1}{2}e_i g(\eta,\eta) = 0 \tag{3.7}$$

and

$$g(\sum_{i=1}^{m} R^N(d\varphi(e_i),\eta)d\varphi(e_i),\eta) = -Ricci(\eta,\eta). \tag{3.8}$$

Using the definition of the Laplacian, we get

$$g(\Delta^{\varphi}\eta,\eta) = \sum_{i=1}^{m} g(-\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\eta + \nabla^{\varphi}_{\nabla_{e_i}e_i}\eta,\eta)$$

$$= \sum_{i=1}^{m} g(\nabla^{\varphi}_{e_i}\eta,\nabla^{\varphi}_{e_i}\eta) = \|B\|^2. \tag{3.9}$$

By use of equations (3.6), (3.7), (3.8), and (3.9) in equation (3.5), we have

$$g(\tau_{2,f}(\varphi),\eta) = f\left\{m(\Delta H) - mH\|B\|^2 + mRicci(\eta,\eta)\right\}$$

$$+\Delta f(mH) + 2mgradf(H). \tag{3.10}$$

Finally, substituting equations (3.4) and (3.10) in equation (1.6), we obtain (3.1).

Conversely, assume that (3.1) holds on M^m. If we take the product of equation (3.1) with mf we have

$$mf\Delta H - mfH\|B\|^2 + mfHRicci(\eta,\eta)$$

$$+(m\Delta f - mf\lambda)H + 2mgradf(H) = 0. \tag{3.11}$$

It is easy to see that

$$(\tau_{2,f}(\varphi))^{\perp} = f\left\{m(\Delta H) - mH\|B\|^2 - mHRicci(\eta,\eta)\right\}$$
$$+\Delta f(mH) + 2mgradf\,(H) \tag{3.12}$$

and

$$(\tau_f(\varphi))^{\perp} = fmH. \tag{3.13}$$

In view of equations (3.12) and (3.13), equation (3.11) turns into

$$(\tau_{2,f}(\varphi))^{\perp} - \lambda\,(\tau_f(\varphi))^{\perp} = 0,$$

which means that M^m is f-biminimal. This proves the proposition. □

Corollary 3.1 *Let $\varphi : M^m \longrightarrow N^{m+1}(c)$ be an isometric immersion of a Riemannian manifold $N^{m+1}(c)$ of constant curvature c. Then φ is f-biminimal if and only if there exists a real number λ such that*

$$\Delta H - \left(m^2H^2 - s + m(m-2)c - \frac{\Delta f}{f} + \lambda\right)H - 2grad\ln f\,(H) = 0, \tag{3.14}$$

where H is the mean curvature function and s the scalar curvature of M^m. In addition, let $\varphi : M^2 \longrightarrow N^3(c)$ be an isometric immersion from a surface to a three-dimensional space form. Then φ is f-biminimal if and only if

$$\Delta H - 2\left(2H^2 - G - \frac{1}{2}\frac{\Delta f}{f} + \frac{1}{2}\lambda\right)H - grad\ln f\,(H) = 0 \tag{3.15}$$

for some $\lambda \in \mathbb{R}$.

Proof Let $\{e_i\}$, $1 \leq i \leq m$ be a local geodesic orthonormal frame of M^m, $\{k_1, k_2, ..., k_m\}$ its principal curvatures, and B its second fundamental form. Then using the proof of Corollary 3.2. in [12], we have

$$\|B\|^2 = m^2H^2 - s + m(m-1)c$$

and

$$Ricci(\eta,\eta) = mc.$$

By use of Proposition 3.1, we obtain

$$\Delta H - \left(m^2H^2 - s + m(m-2)c - \frac{\Delta f}{f} + \lambda\right)H - 2grad\ln f\,(H) = 0. \tag{3.16}$$

For $\varphi : M^2 \longrightarrow N^3(c)$, substituting $m = 2$ into equation (3.16), we get the result. □

Remark 3.1 *In Proposition 3.1 and Corollary 3.1, if we take f as a constant, we obtain Proposition 3.1 and Corollary 3.2 in [12].*

4. Examples of f-biminimal surfaces

In the present section, we give some examples of f-biminimal surfaces. To obtain examples of free f-biminimal surfaces, similar to Theorem 2.3 in [15], we state the following theorem:

Theorem 4.1 *$\varphi : (M^2, g) \longrightarrow (N^n, h)$ is a free f-biminimal map if and only if $\varphi : (M^2, f^{-1}g) \longrightarrow (N^n, h)$ is a free biminimal map.*

Proof Using equation (1.6), $\varphi : (M^2, g) \longrightarrow (N^n, h)$ is a free f-biminimal map if and only if

$$[\tau_{2,f}(\varphi, g)]^{\perp} = f\,[\tau_2(\varphi, g)]^{\perp} + \Delta f\,[\tau(\varphi, g)]^{\perp} + 2\left[\nabla^{\varphi}_{gradf}\tau(\varphi, g)\right]^{\perp} = 0,$$

which is equivalent to

$$[\tau_2(\varphi, g)]^{\perp} + \left(\Delta \ln f + \parallel grad \ln f \parallel^2\right)[\tau(\varphi)]^{\perp} + 2\left[\nabla^{\varphi}_{grad\ln f}\tau(\varphi)\right]^{\perp} = 0.$$

Furthermore, by Corollary 1 in [14], the relationship between the bitension field $[\tau_2(\varphi, g)]^{\perp}$ and that of map $\varphi : (M^2, \bar{g} = F^{-2}g) \longrightarrow (N^n, h)$ is given by

$$[\tau_2(\varphi, \bar{g})]^{\perp} = F^4\,[\tau_2(\varphi, g)]^{\perp} + \left(\Delta \ln F^2 + \parallel grad \ln F^2 \parallel^2\right)[\tau(\varphi)]^{\perp} + 2\left[\nabla^{\varphi}_{grad\ln F^2}\tau(\varphi)\right]^{\perp} = 0.$$

Then map $\varphi : (M^2, \bar{g} = F^{-2}g) \longrightarrow (N^n, h)$ is free biminimal if and only if

$$[\tau_2(\varphi, g)]^{\perp} + \left(\Delta \ln F^2 + \parallel grad \ln F^2 \parallel^2\right)[\tau(\varphi)]^{\perp} + 2\left[\nabla^{\varphi}_{grad\ln F^2}\tau(\varphi)\right]^{\perp} = 0. \tag{4.1}$$

Substituting $F^2 = f$ into equation (4.1), we obtain the result. □

Examples

1. Let us consider the cone on a free biminimal curve on $\mathbb{S}^2$ with

$$\varphi : \left(\mathbb{S}^2, d\theta^2\right) \longrightarrow \left(\mathbb{R}^3 \setminus \{0\} = \mathbb{R}^+ \times_{t^2} \mathbb{S}^2, dt^2 + t^2 d\theta^2\right).$$

Then it is a free biminimal surface [12], where $\times_{t^2}$ denotes the warped product. Hence, from Theorem 4.1, $\varphi : \left(\mathbb{S}^2, f d\theta^2\right) \longrightarrow \left(\mathbb{R}^3 \setminus \{0\} = \mathbb{R}^+ \times_{t^2} \mathbb{S}^2, dt^2 + t^2 d\theta^2\right)$ is a free f-biminimal surface.

2. Let $\beta : I \longrightarrow \mathbb{R}^2$ be the logarithmic spiral whose curvature $k = \frac{1}{\sqrt{2s}}$ and $\alpha : I \longrightarrow \mathbb{R}^3$ be a helix of the cylinder on the plane curve β with its Frenet frame $\{T, N, B\}$. Then the envelope S of α parametrized by $X : \left(\mathbb{R}^2, g\right) \longrightarrow \left(\mathbb{R}^3, \widetilde{g}\right)$, $X(u, s) = \alpha(s) + u(B + T)$ is a free biminimal surface [12]. Hence, from Theorem 4.1, $X : \left(\mathbb{R}^2, fg\right) \longrightarrow \left(\mathbb{R}^3, \widetilde{g}\right)$ is a free f-biminimal surface.

3. The circular cylinder $\varphi : D = \{(u, v) \in (0, 2\pi) \times \mathbb{R}\} \longrightarrow \mathbb{R}^3$ with $\varphi(u, v) = (r\cos u, r\sin u, v)$ is an f-biminimal surface for $f(u) = C_1 e^{\sqrt{-1-\lambda r^2}u} + C_2 e^{-\sqrt{-1-\lambda r^2}u}$, where C_1 and C_2 are real constants. It is easy to see that this surface with $f(u) = C_1 e^{\sqrt{-1-\lambda r^2}u} + C_2 e^{-\sqrt{-1-\lambda r^2}u}$ is not an f-biharmonic surface because if φ is f-biharmonic, then using Theorem 3.2 of [15] we get $\lambda = 0$. Then the function f is indefinite, so this surface can not be f-biharmonic and free f-biminimal. Moreover, using Proposition 3.1 of [12], we obtain that φ cannot be biminimal unless $\lambda = -\frac{1}{r^2}$. This shows that the f-biharmonicity, biminimality, and f-biminimality of φ are different.

5. f-Biminimal Legendre curves in Sasakian space forms

Let $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ be a contact metric manifold. If the Nijenhuis tensor of φ equals $-2d\eta\otimes\xi$, then $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ is called a *Sasakian manifold* [2]. If a Sasakian manifold has constant φ-sectional curvature c, then it is called a *Sasakian space form*. The curvature tensor of a Sasakian space form is given by

$$R(X,Y)Z=\frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\}+\frac{c-1}{4}\{g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X$$
$$+2g(X,\varphi Y)\varphi Z+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X$$
$$+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\} \tag{5.1}$$

for all $X,Y,Z\in TM$ [3].

A submanifold of a Sasakian manifold is called an *integral submanifold* if $\eta(X)=0$ for every tangent vector X. A 1-dimensional integral submanifold of a Sasakian manifold is called a *Legendre curve* of M. Hence, a curve $\gamma:I\longrightarrow M=\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ is called a Legendre curve if $\eta(T)=0$, where T is the tangent vector field of γ [3].

We can state the following theorem:

Theorem 5.1 *Let $\gamma:(a,b)\longrightarrow M$ be a nongeodesic Legendre Frenet curve of osculating order r in a Sasakian space form $M=\left(M^{2m+1},\varphi,\xi,\eta,g\right)$. Then γ is f-biminimal if and only if the following three equations hold:*

$$k_1''-k_1^3-k_1k_2^2+\frac{(c+3)}{4}k_1+2k_1'\frac{f'}{f}+k_1\frac{f''}{f}-\lambda k_1+\frac{3(c-1)}{4}\left[k_1g(\varphi T,E_2)^2\right]^{\perp}=0,$$

$$k_1'k_2+(k_1k_2)'+2k_1k_2\frac{f'}{f}+\frac{3(c-1)}{4}\left[k_1g(\varphi T,E_2)g(\varphi T,E_3)\right]^{\perp}=0,$$

and

$$k_1k_2k_3+\frac{3(c-1)}{4}\left[k_1g(\varphi T,E_2)g(\varphi T,E_4)\right]^{\perp}=0.$$

Proof Let $M=\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ be a Sasakian space form and $\gamma:(a,b)\longrightarrow M$ a Legendre Frenet curve of osculating order r. Differentiating

$$\eta(T)=0$$

and using Definition 2.1, we obtain

$$\eta(E_2)=0. \tag{5.2}$$

Then using equations (5.1) and (5.2), we have

$$R(T,\nabla_TT)T=-k_1\frac{(c+3)}{4}E_2-3k_1\frac{(c-1)}{4}g(\varphi T,E_2)\varphi T. \tag{5.3}$$

By use of equations (2.5), (2.9), and (5.3) in equation (1.6), we find

$$\left(k_1''-k_1^3-k_1k_2^2+\frac{(c+3)}{4}k_1+2k_1'\frac{f'}{f}+k_1\frac{f''}{f}-\lambda k_1\right)E_2+\left(k_1'k_2+(k_1k_2)'+2k_1k_2\frac{f'}{f}\right)E_3$$

$$+ (k_1 k_2 k_3) E_4 + \frac{3(c-1)}{4} [k_1 g(\varphi T, E_2)\varphi T]^{\perp} = 0. \tag{5.4}$$

Then taking the scalar product of equation (5.4) with $E_2,\ E_3, and\ E_4$, respectively, we obtain the desired results. □

Let us recall some notions about the Sasakian space form $\mathbb{R}^{2m+1}(-3)$ [3]:

Let us take $M = \mathbb{R}^{2m+1}$ with the standard coordinate functions $(x_1, ..., x_m, y_1, ..., y_m, z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y_i dx_i)$, the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$, and the tensor field φ given by

$$\varphi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

The Riemannian metric is $g = \eta \otimes \eta + \frac{1}{4}\sum\limits_{i=1}^{m}\left((dx_i)^2 + (dy_i)^2\right)$. Then $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is a Sasakian space form with constant φ-sectional curvature $c = -3$ and it is denoted by $\mathbb{R}^{2m+1}(-3)$. The vector fields

$$X_i = 2\frac{\partial}{\partial y_i},\ X_{i+m} = \varphi X_i = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}),\ 1 \le i \le m,\ \xi = 2\frac{\partial}{\partial z}, \tag{5.5}$$

form a g-orthonormal basis and the Levi-Civita connection is calculated as

$$\nabla_{X_i} X_j = \nabla_{X_{i+m}} X_{j+m} = 0,\ \nabla_{X_i} X_{j+m} = \delta_{ij}\xi,\ \nabla_{X_{i+m}} X_j = -\delta_{ij}\xi,$$

$$\nabla_{X_i}\xi = \nabla_{\xi} X_i = -X_{m+i},\ \nabla_{X_{i+m}}\xi = \nabla_{\xi} X_{i+m} = X_i$$

(see [2]).

Now let us produce an example of f-biminimal Legendre curves in $\mathbb{R}^5(-3)$:

Example Let $\gamma = (\gamma_1, ..., \gamma_5)$ be a unit speed Legendre curve in $\mathbb{R}^5(-3)$. The tangent vector field of γ is

$$T = \frac{1}{2}\left\{\gamma_3' X_1 + \gamma_4' X_2 + \gamma_1' X_3 + \gamma_2' X_4 + (\gamma_5' - \gamma_1'\gamma_3 - \gamma_2'\gamma_4)\,\xi\right\}.$$

Using the above equation, since γ is a unit speed Legendre curve, we have $\eta(T) = 0$ and $g(T,T) = 1$; that is,

$$\gamma_5' = \gamma_1'\gamma_3 + \gamma_2'\gamma_4$$

and

$$(\gamma_1')^2 + ... + (\gamma_5')^2 = 4.$$

For a Legendre curve, we can use the Levi-Civita connection and equation (5.5) to write

$$\nabla_T T = \frac{1}{2}\left(\gamma_3'' X_1 + \gamma_4'' X_2 + \gamma_1'' X_3 + \gamma_2'' X_4\right), \tag{5.6}$$

$$\varphi T = \frac{1}{2}\left(-\gamma_1' X_1 - \gamma_2' X_2 + \gamma_3' X_3 + \gamma_4' X_4\right). \tag{5.7}$$

Equations (5.6) and (5.7) and $\varphi T \perp E_2$ hold if and only if

$$\gamma_1'\gamma_3'' + \gamma_2'\gamma_4'' = \gamma_3'\gamma_1'' + \gamma_4'\gamma_2''.$$

Finally, we can give the following explicit example:

Let us take $\gamma(t) = (\sin 2t, -\cos 2t, 0, 0, 1)$ in $\mathbb{R}^5(-3)$. Using the above equations and Theorem 5.1, γ is an f-biminimal Legendre curve with osculating order $r = 2$, $k_1 = 2$, $f = e^t$, $\varphi T \perp E_2$. We can easily check that the conditions of Theorem 5.1 are verified. Using Theorem 3.1 of [7], the curve γ is not f-biharmonic. For $\lambda \neq -4$, it is easy to see that γ is not biminimal. Hence, the biminimality and f-biminimality of γ are different unless $\lambda = -4$.

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