Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 888 – 895
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doi:10.3906/mat-1605-117

Coefficient bounds for a new subclass of analytic bi-close-to-convex functions by making use of Faber polynomial expansion

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Received: 26.05.2016 • **Accepted/Published Online:** 27.09.2016 • **Final Version:** 25.07.2017

Abstract: Recently, in the literature, we can see quite a few papers about general coefficient bounds for subclasses of bi-univalent functions. However, we can find just a few papers about general coefficient estimates for subclasses of bi-close-to-convex functions. In the present study, we give and look into a new subclass of analytic and bi-close-to-convex functions in the open unit disk. Making use of the Faber series, we have an upper bound for the general coefficient of functions in this class. We also demonstrate the invisible behavior of the beginning coefficients of a special subclass of bi-close-to-convex functions.

Key words: Analytic functions, bi-close-to convex functions, Faber polynomials, bi-univalent functions, coefficient estimates

1. Introduction

We know that a function is *univalent* if it never takes the same value twice. We also know that a function is *bi-univalent* if both it and its inverse are univalent.

Let $\mathcal{A}$ denote the class of functions f that are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For α; $0 \leq \alpha < 1$, we let $S^*(\alpha)$ denote the class of function $g \in S$ that are starlike of order α in $\mathbb{U}$, namely, $Re\left\{\frac{zg'(z)}{g(z)}\right\} > \alpha$ in $\mathbb{U}$ and $C(\alpha)$ indicate the class of functions $f \in S$ that are close-to-convex of order α in $\mathbb{U}$, namely, if a function g is in $S^*(0) = S^*$ so that $Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha$ in $\mathbb{U}$ [6, 10]. We note that $S^*(\alpha) \subset C(\alpha) \subset S$ and that $|a_n| \leq n$ for $f \in S$ by the Bieberbach conjecture [3, 6].

The Koebe 1/4 theorem [6] asserts that the image of $\mathbb{U}$ under each univalent function $f \in \mathcal{A}$ contains the disk of radius $1/4$. According to this, if $F = f^{-1}$ is the inverse of a function $f \in S$, then F has a Maclaurin series expansion in some disk about the origin. Thus every function $f \in S$ has an inverse f^{-1} that satisfies $f^{-1}(f(z)) = z$ for $z \in \mathbb{U}$ and $f(f^{-1}(w)) = w$ for $|w| < 1/4$.

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A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and $F = f^{-1}$ are univalent in $\mathbb{U}$. Similarly, a function $f \in \mathcal{A}$ is said to be bi-close-to-convex of order α if both f and $F = f^{-1}$ are bi-close-to-convex of order α in $\mathbb{U}$. Let Σ define the class of all bi-univalent functions in $\mathbb{U}$ represented by the Taylor–Maclaurin series expansion (1.1). For a short history and examples of functions in the class Σ, see [13] (see also [5, 11, 12, 14]).

For some $(0 \leq \beta < 1,\ 0 \leq \lambda \leq 1)$, we let $\mathcal{T}_\Sigma(\lambda, \beta)$ define the subclass of close-to-convex functions f given by (1.1) if a function $g(z)$ is in S^* such that

$$\Re\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)g(z) + \lambda z g'(z)}\right\} > \beta, \quad (z \in \mathbb{U}). \tag{1.2}$$

In particular, for $\lambda = 0$, we have $\mathcal{T}_\Sigma(0, \beta)$, which was introduced by Hamidi and Jahangiri [8] and they said that the bi-close-to-convex functions considered in their paper are the largest subclass of bi-univalent functions.

Faber polynomials, which are used by us in this paper, play a considerable act in geometric function theory, which was introduced by Faber [7].

Firstly, Lewin [11] considered the class of bi-univalent functions, obtaining the estimate $|a_2| \leq 1.51$. Subsequently, Brannan and Clunie [4] developed Lewin's result to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Accordingly, Netanyahu [12] showed that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [5] defined certain subclasses of bi-univalent function class Σ similar to the usual subclasses. In fact, the aforementioned work by Srivastava et al. [13] essentially revived the investigation of various subclasses of bi-univalent function class Σ in recent years. Lately, many mathematicians found bounds for several subclasses of bi-univalent functions (see [9, 13, 16]). Only a few papers determine general coefficient bounds $|a_n|$ for the analytic bi-close-to-convex functions in the associated documents. In particular, in [8] Hamidi and Jahangiri introduced the class of bi-close-to-convex functions and determined estimates for the general coefficient $|a_n|$ of bi-close-to-convex function under certain gap series condition by using Faber polynomials.

In this study, we let $f \in \mathcal{T}_\Sigma(\lambda, \beta)$ and $F = f^{-1} \in \mathcal{T}_\Sigma(\lambda, \beta)$. We make use of the Faber series to determine the general Taylor–Maclaurin coefficients $|a_n|$ of functions in a certain subclass of bi-close-to-convex functions under some condition. We demonstrate the unpredictability of the beginning coefficients behavior of bi-starlike functions. For some special cases, the coefficient estimates for the functions in this class are the same as the coefficient estimates of the bi-close-to-convex functions considered in [8], which are the largest subclass of bi-univalent functions.

We need the following theorem from Airault and Bouali [1] for proving our main results.

Theorem 1.1 Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. The inverse function of f, $f^{-1}(f(z)) = z$ is given in terms of the Faber polynomials of f with

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^n = w + \sum_{n=2}^{\infty} A_n w^n, \tag{1.3}$$

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3$$

$$+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4$$

$$+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5+(-n+2)a_3^2]$$

$$+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]+\sum_{j\geq 7}a_2^{n-j}V_j,$$

where V_j is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [2]. The first few terms of K_{n-1}^{-n} are

$$K_1^{-2} = -2a_2$$

$$K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, an expansion of $K_{n-1}^{p}(a_2, a_3, ..., a_n)$ is given by

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + \cdots + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1}, \quad (p \in \mathbb{Z}),$$

where $\mathbb{Z} = \{0, \mp 1, \mp 2, \cdots\}$ and $D_{n-1}^p = D_{n-1}^p(a_2, a_3, \ldots)$ and by [15],

$$D_{n-1}^m = D_{n-1}^m(a_2, a_3, \ldots a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_1!...\mu_{n-1}!}, \quad for, \; m \leq n.$$

and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_{n-1}$ satisfying the following conditions:

$$\begin{cases} \mu_1 + \mu_2 + ... + \mu_{n-1} = m \\ \mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n-1. \end{cases}$$

(see, for details, [1, 2]). It is clear that

$$D_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}.$$

2. Main results

Firstly, we state the general coefficient estimate of functions in $\mathcal{T}_{\Sigma}(\lambda, \beta)$ as follows.

Theorem 2.1 For $0 \leq \beta < 1$ and $0 \leq \lambda \leq 1$, let $f \in \mathcal{T}_{\Sigma}(\lambda, \beta)$ and $F = f^{-1} \in \mathcal{T}_{\Sigma}(\lambda, \beta)$. If $a_k = 0$ for $2 \leq k \leq n-1$, then

$$|a_n| \leq \frac{2(1-\beta)}{n\left[1+(n-1)\lambda\right]} + 1. \tag{2.1}$$

Proof Let $f \in \mathcal{T}_{\Sigma}(\lambda, \beta)$ given by (1.1). Therefore, there is a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$ so that

$$\Re\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)g(z) + \lambda z g'(z)}\right\} > \beta, \quad z \in \mathbb{U}.$$

By using Faber polynomial expansion, we obtain

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)g(z)+\lambda zg'(z)} = 1+\sum_{n=2}^{\infty}[(1+(n-1)\lambda)(na_n-b_n)$$

$$\left. +\sum_{s=1}^{n-2}(1+(n-s-1)\lambda)K_s^{-1}[(1+\lambda)b_2,(1+2\lambda)b_3,...,(1+s\lambda)b_{s+1}]((n-s)a_{n-s}-b_{n-s})\right] z^{n-1}. \tag{2.2}$$

Now for the inverse map $F=f^{-1}$ there exists a function $G(w)=w+\sum_{n=2}^{\infty}B_n w^n \in S^*$ so that

$$\Re\left\{\frac{wF'(w)+\lambda w^2F''(w)}{(1-\lambda)G(w)+\lambda wG'(w)}\right\}>\beta, \quad z\in\mathbb{U}.$$

According to (2.2), the Faber polynomial series of the inverse map $F=f^{-1}$ is $F(w)=w+\sum_{n=2}^{\infty}A_n w^n$. Thus we have

$$\frac{wF'(w)+\lambda w^2F''(w)}{(1-\lambda)G(w)+\lambda wG'(w)} = 1+\sum_{n=2}^{\infty}[(1+(n-1)\lambda)(nA_n-B_n)$$

$$\left. +\sum_{s=1}^{n-2}(1+(n-s-1)\lambda)K_s^{-1}[(1+\lambda)B_2,(1+2\lambda)B_3,...,(1+s\lambda)B_{s+1}]((n-s)A_{n-s}-B_{n-s})\right] w^{n-1}. \tag{2.3}$$

Nevertheless, since $f\in\mathcal{T}_\Sigma(\lambda,\beta)$ and $F=f^{-1}\in\mathcal{T}_\Sigma(\lambda,\beta)$, we know that there are two positive real part functions:

$$\mathtt{p}(z)=1+\sum_{n=1}^{\infty}c_n z^n$$

and

$$\mathtt{q}(w)=1+\sum_{n=1}^{\infty}d_n w^n,$$

where $Re\,(\mathtt{p}(z))>0$ and $Re\,(\mathtt{q}(w))>0$ in $\mathbb{U}$ so that

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)g(z)+\lambda zg'(z)} = \beta+(1-\beta)\mathtt{p}(z)$$

$$=1+(1-\beta)\sum_{n=1}^{\infty}c_n z^n \tag{2.4}$$

and

$$\frac{wF'(w)+\lambda w^2F''(w)}{(1-\lambda)G(w)+\lambda wG'(w)}=\beta+(1-\beta)\mathtt{q}(w)$$

$$=1+(1-\beta)\sum_{n=1}^{\infty}d_nw^n. \tag{2.5}$$

Comparing the coefficients of (2.2) with (2.4) and considering the Caratheodory Lemma, $|c_n|\leq 2$ and $|d_n|\leq 2$ yield

$$\left[(1+(n-1)\lambda)(na_n-b_n)+\sum_{s=1}^{n-2}(1+(n-s-1)\lambda)K_s^{-1}[(1+\lambda)b_2,(1+2\lambda)b_3,...,(1+s\lambda)b_{s+1}]((n-s)a_{n-s}-b_{n-s})\right]$$

$$=(1-\beta)c_{n-1}. \tag{2.6}$$

In a similar way, comparing the coefficients of (2.3) with (2.5), we obtain

$$\left[(1+(n-1)\lambda)(nA_n-B_n)+\sum_{s=1}^{n-2}(1+(n-s-1)\lambda)K_s^{-1}[(1+\lambda)B_2,(1+2\lambda)B_3,...,(1+s\lambda)B_{s+1}]((n-s)A_{n-s}-B_{n-s})\right]$$

$$=(1-\beta)d_{n-1}. \tag{2.7}$$

Note that for the special case $n=2$, (2.6) and (2.7) respectively, yield

$$(1+\lambda)(2a_2-b_2)=(1-\beta)c_1$$

and

$$-(1+\lambda)(2a_2+B_2)=(1-\beta)d_1.$$

Now, solving for a_2 and taking modulus, we have

$$|a_2|\leq\frac{(2+\lambda-\beta)}{(1+\lambda)}.$$

For $a_k=0,\ 2\leq k\leq n-1$, we note that (2.6) and (2.7) respectively, yield

$$(1+(n-1)\lambda)(na_n-b_n)=(1-\beta)c_{n-1} \tag{2.8}$$

$$-(1+(n-1)\lambda)(na_n+B_n)=(1-\beta)d_{n-1}. \tag{2.9}$$

It is known that, with respect to the Caratheodory Lemma [6], $|c_n|\leq 2$ and $|d_n|\leq 2$ for $n\in\mathbb{N}$. Just taking the absolute values of (2.8) and (2.9) for $|b_n|\leq n$ and $|B_n|$, we get

$$|a_n|\leq\frac{2(1-\beta)}{n\left[1+(n-1)\lambda\right]}+1$$

which evidently finishes the proof of Theorem 2.1.

If we take $\lambda = 0$ in Theorem 2.1, we obtain Corollary 2.2, which was proved by Hamidi and Jahangiri [8].

Corollary 2.2 For $0 \leq \beta < 1$ let the function $f \in S$ be bi-close-to-convex of order β in $\mathbb{U}$. If $a_k = 0$ for $2 \leq k \leq n-1$, then

$$|a_n| \leq 1 + \frac{2(1-\alpha)}{n}.$$

In particular, we let $f(z) = g(z)$ in (1.2). Thus, for the bi-starlike case, we have the class $\mathcal{S}^*_\Sigma(\lambda, \beta)$, which is a subclass of $\mathcal{T}_\Sigma(\lambda, \beta)$. Then we give Theorem 3, which proves the unpredictability of the coefficient behaviour of this subclass of bi-univalent functions.

Theorem 2.3 Let $f \in \mathcal{S}^*_\Sigma(\lambda, \beta)$ $(0 \leq \beta < 1,\ 0 \leq \lambda \leq 1)$ be given by (1.1) and $F = f^{-1} \in \mathcal{S}^*_\Sigma(\lambda, \beta)$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{1-\lambda^2+2\lambda}}; & 0 \leq \beta < \frac{1+2\lambda-3\lambda^2}{2(1+2\lambda-\lambda^2)} \\ \frac{2(1-\beta)}{1+\lambda}; & \frac{1+2\lambda-3\lambda^2}{2(1+2\lambda-\lambda^2)} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{2(1-\beta)}{-\lambda^2+2\lambda+1}; & 0 \leq \beta < \frac{1+2\lambda-3\lambda^2}{2(1+2\lambda-\lambda^2)} \\ \frac{(1-\beta)(3-2\beta)}{1+2\lambda}; & \frac{1+2\lambda-3\lambda^2}{2(1+2\lambda-\lambda^2)} \leq \beta < 1. \end{cases}$$

Proof We note that the function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ will be equal to the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the proof of Theorem 2.1 for the bi-starlike case. Namely, $a_n = b_n$ in there.

Replacing n by 2 in (2.6) and (2.7), respectively, we have

$$(1+\lambda)a_2 = (1-\beta)c_1 \tag{2.10}$$

and

$$-(1+\lambda)a_2 = (1-\beta)d_1. \tag{2.11}$$

By taking the absolute values of either (2.10) or (2.11) we obtain

$$|a_2| \leq 2\frac{(1-\beta)}{(1+\lambda)}. \tag{2.12}$$

Replacing n by 3 in (2.6) and (2.7), respectively, we have

$$2(1+2\lambda)a_3 - (1+\lambda)^2 a_2^2 = (1-\beta)c_2 \tag{2.13}$$

and

$$-2(1+2\lambda)a_3 + (3+6\lambda-\lambda^2)a_2^2 = (1-\beta)d_2. \tag{2.14}$$

Adding the above two equations (2.13) and (2.14), taking absolute values and solving for $|a_2|$, gives

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(1+2\lambda-\lambda^2)}}. \tag{2.15}$$

As a result of this, we obtain the bound on $|a_2|$ upon noting that $\sqrt{\frac{2(1-\beta)}{(1+2\lambda-\lambda^2)}} < 2(1-\beta)$, if $\beta < \frac{1+2\lambda-3\lambda^2}{2(1+2\lambda-\lambda^2)}$.

When we solve the equations (2.13) and (2.14) for a_3 and then taking absolute values we obtain

$$|a_3| \leq \frac{2(1-\beta)}{(1+2\lambda-\lambda^2)}. \tag{2.16}$$

Substituting $a_2 = \frac{(1-\beta)}{(1+\lambda)}c_1$ in equation (2.13) we obtain $2(1+2\lambda)a_3 = (1-\beta)\left|c_2 + (1-\beta)c_1^2\right|$, calculating their absolute values, and implementing the Caratheodory Lemma, we have

$$|a_3| = \frac{(1-\beta)\left[c_2 + (1-\beta)c_1^2\right]}{2(1+2\lambda)} \leq \frac{(1-\beta)(3-2\beta)}{1+2\lambda}.$$

Thus the proof of Theorem 2.3 was completed.

If we take $\lambda = 0$ in Theorem 2.1, we have Corollary 2.4, which was proved by Hamidi and Jahangiri [8].

Corollory 2.4 For $0 \leq \beta < 1$ let $f \in S^*(\beta)$ and $F = f^{-1} \in S^*(\beta)$. Then

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)}; & 0 \leq \beta < \frac{1}{2} \\ 2(1-\beta); & \frac{1}{2} \leq \beta < 1. \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\beta); & 0 \leq \beta < \frac{1}{2} \\ (1-\beta)(3-2\beta); & \frac{1}{2} \leq \beta < 1. \end{cases}$$

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