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Solvability of boundary value problems for coupled impulsive differential equations with one-dimensional p-Laplacians

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Abstract: This paper is concerned with a boundary value problem of impulsive differential systems on the whole line with one-dimensional p-Laplacians. By constructing a weighted Banach space and defining a nonlinear operator, together with Schauder's fixed point theorem, sufficient conditions to guarantee the existence of at least one solution are established (Theorems 3.1–3.3). Two examples are given to illustrate the main results.

Key words: Impulsive differential system on whole line, boundary value problem, increasing odd homeomorphisms, sub-Carathéodory function, discrete Carathéodory function, fixed point theorem

1. Introduction

Boundary-value problems for linear second order ordinary differential equations were initiated by Il'in and Moiseev [17] and studied by many authors; see the textbooks [1, 16], the papers [9, 26, 27], and the references therein.

In recent years, many authors have studied the existence of positive radial solutions for elliptic systems in annular/exterior domains, which is equivalent to that of positive solutions for the corresponding systems of ordinary differential equations (see [12–15, 19, 20] and the references therein). The usual method used is the fixed point theorems of cone expansion/compression type, the upper and lower solutions method, and the fixed point index theory in cones.

In [10, 23], the following system and its special case were discussed:

$$\begin{aligned} &[\phi_p(u'(t))]' + \lambda h_1(t) f(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ &[\phi_p(v'(t))]' + \mu h_2(t) g(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ &u(0) = a \geq 0, v(0) = b \geq 0, u(1) = v(1) = 0, \end{aligned} \tag{1.1}$$

where $\phi_p(x) = |x|^{p-2}x, p > 1$, λ, μ are nonnegative real parameters, $h_i \in C((0,1),(0,\infty)), i = 1,2$, $f, g \in C([0,\infty) \times [0,\infty), [0,\infty))$, h_i may be singular at $t = 0$ and $f(0,0) = g(0,0) = 0$, and $f(u,v) > 0, g(u,v) > 0$ for all $(u,v) > (0,0)$. The existence, nonexistence, and multiplicity of positive solutions for (1.1) were obtained by using the upper and lower solution method and the fixed point index theorem.

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Studies on boundary value problems (BVPs) for second order nonlinear ordinary differential equations on a whole line were addressed in [2–8, 21, 24, 25]. Prototypes of equations of such problems are as follows:

$$(a(t)\phi_p(x'(t)))' + f(t,x(t)) = 0,\ t \in \mathbb{R},$$

$$(a(t)\phi_p(x'(t)))' + f(t,x(t),x'(t)) = 0,\ t \in \mathbb{R},$$

$$(a(t,x(t),x'(t))\phi_p(x'(t)))' + f(t,x(t),x'(t)) = 0,\ t \in \mathbb{R}.$$

The theory of impulsive differential equations describes processes that experience a sudden change in their state at certain moments. Processes with such a character arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. For an introduction to the basic theory of impulsive differential equations, we refer the reader to [18].

In [22], the following boundary value problem for a second order singular differential system on the whole line with impulse effects was studied:

$$\begin{aligned}
&[\rho(t)\Phi_p(x'(t))]' + f(t,x(t),y(t)) = 0, \quad a.e.\ t \in \mathbb{R},\\
&[\varrho(t)\Phi_q(y'(t))]' + g(t,x(t),y(t)) = 0, \quad a.e.\ t \in \mathbb{R}\\
&\lim_{t\to+\infty} x(s) = 0,\ \lim_{t\to-\infty} x(s) = 0,\ \lim_{t\to+\infty} y(s) = 0,\ \lim_{t\to-\infty} y(s) = 0,\\
&\Delta x(t_k) = I_k(t_k,x(t_k),y(t_k)),\ \Delta y(t_k) = J_k(t_k,x(t_k),y(t_k)), k \in \mathbb{Z},
\end{aligned} \tag{1.2}$$

where $\rho,\varrho \in C^0(\mathbb{R},[0,\infty))$, $\rho(t),\varrho(t) > 0$ for all $t \in \mathbb{R}$ with

$$\textstyle\int_{-\infty}^{+\infty} \frac{ds}{\rho(s)} < +\infty,\ \int_{-\infty}^{+\infty} \frac{ds}{\varrho(s)} < +\infty, \tag{1.3}$$

$\Phi_p(x) = |x|^{p-2}x$ and $\Phi_q(x) = |x|^{q-2}x$ are one-dimensional p-Laplacian, f,g defined on $\mathbb{R}^3$ are Carathéodory functions, $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$ with

$$\lim_{k\to-\infty} t_k = -\infty,\ \lim_{k\to+\infty} t_k = +\infty,$$

$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)(k \in \mathbb{Z})$, $\mathbb{Z}$ is the set of all integers, $\{I_k\},\{J_k\}$ with $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}(k \in \mathbb{Z})$ are Carathéodory sequences.

This paper is a continuation of [22]. We consider the following boundary value problem for a second order singular differential system on the whole line with impulse effects:

$$\begin{aligned}
&[\Phi(\rho(t)x'(t))]' + f(t,x(t),y(t)) = 0, \quad a.e.\ t \in \mathbb{R},\\
&[\Psi(\varrho(t)y'(t))]' + g(t,x(t),y(t)) = 0, \quad a.e.\ t \in \mathbb{R}\\
&\lim_{t\to-\infty} x(s) = 0,\ \lim_{t\to+\infty} \rho(t)x'(t) = 0,\\
&\lim_{t\to-\infty} y(s) = 0,\ \lim_{t\to+\infty} \varrho(t)y'(t) = 0,\\
&\Delta x(t_k) = I_0(t_k,x(t_k),y(t_k)),\ \Delta\Phi(\rho(t_k)x'(t_k)) = I_1(t_k,x(t_k),y(t_k)), k \in \mathbb{Z},\\
&\Delta y(t_k) = J_0(t_k,x(t_k),y(t_k)),\ \Delta\Psi(\varrho(t_k)y'(t_k)) = J_1(t_k,x(t_k),y(t_k)), k \in \mathbb{Z},
\end{aligned} \tag{1.4}$$

where

(i) $\rho, \varrho \in C^0(\mathbb{R}, (0,\infty))$ with

$$\int_{-\infty}^{0} \frac{ds}{\rho(s)} < +\infty,\ \int_{0}^{+\infty} \frac{ds}{\rho(s)} = +\infty,\ \int_{-\infty}^{0} \frac{ds}{\varrho(s)} < +\infty,\ \int_{0}^{+\infty} \frac{ds}{\varrho(s)} = +\infty, \tag{1.5}$$

(ii) $\Phi(x) = |x|^{p_1}x$ and $\Psi(x) = |x|^{p_2}x$ are one-dimensional p-Laplacians,

(iii) $\mathbb{R}$ is the set of all real numbers, $\mathbf{Z}$ the set of all integers, $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$ with

$$\lim_{k\to-\infty} t_k = -\infty,\ \lim_{k\to+\infty} t_k = +\infty,$$

(iv) the operator Δ is defined by

$$\Delta x(t_k) = x(t_k^+) - x(t_k),\ \ \Delta y(t_k) = y(t_k^+) - y(t_k),$$

$$\Delta\Phi(\rho(t_k)x'(t_k)) = \Phi(\rho(t_k^+)x'(t_k^+)) - \Phi(\rho(t_k)x'(t_k)),$$

$$\Delta\Psi(\varrho(t_k)y'(t_k)) = \Psi(\varrho(t_k^+)y'(t_k^+)) - \Psi(\varrho(t_k)y'(t_k))(k \in \mathbf{Z}),$$

(v) f, g defined on $\mathbb{R}^3$ sub-Carathéodory functions (see Definition 2.1),

(vi) $I_1, J_1 : \{t_s : s \in \mathbf{Z}\} \times \mathbb{R}^2 \mapsto \mathbb{R}$ are discrete Carathéodory functions (see Definition 2.2), $I_0 : \{t_s : s \in \mathbf{Z}\} \times \mathbb{R}^2 \mapsto \mathbb{R}$ is a discrete $\rho-$Carathéodory function (see Definition 2.3), $J_0 : \{t_s : s \in \mathbf{Z}\} \times \mathbb{R}^2 \mapsto \mathbb{R}$ is a discrete $\varrho-$Carathéodory function (see Definition 2.4).

The purpose of this paper is to establish sufficient conditions for the existence of at least one solution of BVP (1.4). The technical tool used in this paper is the well-known Schauder fixed point theorem. For applying Schauder fixed point theorem, the most crucial thing to prove is operator compactness. Since the problem is considered on the whole line, we need to prove the operator compactness by showing the equi-continuity of the image of the bounded set on each subinterval (there are infinitely many subintervals), the equi-convergence as $t \to t_i (i = 0, \pm1, \pm2, \dots)$, and the equi-convergence as $t \to -\infty$ and as $t \to +\infty$. One sees that (1.5) (more general) is different from (1.3). Thus this paper is a continuation of [22].

By a solution of BVP (1.4) we mean a couple of functions (x, y) with $x, y \in C^1(t_k, t_{k+1}](k \in \mathbf{Z})$ such that both

$$[\Phi(\rho x')]' : t \mapsto [\Phi(\rho(t)x'(t))]' \text{ and } [\Psi(\varrho y')]' : t \mapsto [\Psi(\varrho(t)y'(t))]'$$

are measurable on $\mathbb{R}$, and the limits

$$\lim_{t\to-\infty} x(t),\ \lim_{t\to-\infty} y(t),\ \lim_{t\to+\infty} \rho(t)x'(t) \text{ and } \lim_{t\to+\infty} \varrho(t)y'(t)$$

exist (finite), and all equations in (1.4) are satisfied.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2 and the main results are presented in Section 3. An example is given in Section 4.

2. Preliminary results

In this section, we present some background definitions in Banach spaces and state an important fixed point theorem. The preliminary results are given as well. Denote

$$\sigma(t) = 1 + \int_{-\infty}^{t} \frac{du}{\rho(u)},\ \tau(t) = 1 + \int_{-\infty}^{t} \frac{du}{\varrho(u)},\ t \in \mathbb{R}.$$

Definition 2.1 $h:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ *is called a sub-Carathédory function if it satisfies*

(i) $t\mapsto h\left(t,\sigma(t)u,\tau(t)v\right)$ *is measurable on* $\mathbb{R}$ *for* $(u,v)\in\mathbb{R}^2$,

(ii) $(u,v)\mapsto h\left(t,\sigma(t)u,\tau(t)v\right)$ *is continuous for a.e.* $t\in\mathbb{R}$,

(iii) *for each* $r>0$, *there exist nonnegative function* $\phi_r\in L^1(\mathbb{R})$ *such that* $|u|,|v|\le r$ *implies*

$$|h\left(t,\sigma(t)u,\tau(t)v\right)|\le\phi_r(t),t\in\mathbb{R}.$$

Definition 2.2 $K:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ *is called a discrete Carathédory function if it satisfies*

(i) $(u,v)\mapsto K\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)$ *is continuous for all* $s\in\mathbb{Z}$,

(ii) *for each* $r>0$, *there exist nonnegative constants* $M_{s,r}\ge 0$ *with* $\sum\limits_{s=-\infty}^{+\infty}M_{s,r}<+\infty$ *such that* $|u|,|v|\le r$ *implies*

$$|K\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)|\le M_{s,r}\text{ for all }s\in\mathbb{Z}.$$

Definition 2.3 $H:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ *is called a discrete* ρ*-Carathédory function if it satisfies*

(i) $(u,v)\mapsto H\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)$ *is continuous for all* $s\in\mathbb{Z}$,

(ii) *for each* $r>0$, *there exist nonnegative constants* $N_{s,r}\ge 0$ *with*

$$\sum_{j=-\infty}^{s}N_{j,r}<+\infty\text{ for all }s\in\mathbb{Z},$$

$$\lim_{s\to+\infty}\frac{N_{s,r}}{\int_{t_s}^{t_{s+1}}\frac{du}{\rho(u)}}=\lim_{s\to+\infty}\frac{N_{s,r}}{\int_{t_{s-1}}^{t_s}\frac{du}{\rho(u)}}=0$$

such that $|u|,|v|\le r$ *implies*

$$|H\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)|\le N_{k,r}\text{ for all }s\in\mathbb{Z}.$$

Definition 2.4 $H:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ *is called a discrete* ϱ*-Carathédory function if it satisfies*

(i) $(u,v)\mapsto H\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)$ *is continuous for all* $s\in\mathbb{Z}$,

(ii) *for each* $r>0$, *there exists nonnegative constants* $N_{s,r}\ge 0$ *with*

$$\sum_{j=-\infty}^{s}N_{j,r}<+\infty\text{ for all }s\in\mathbb{Z},$$

$$\lim_{s\to+\infty}\frac{N_{s,r}}{\int_{t_s}^{t_{s+1}}\frac{du}{\varrho(u)}}=\lim_{s\to+\infty}\frac{N_{s,r}}{\int_{t_{s-1}}^{t_s}\frac{du}{\varrho(u)}}=0$$

such that $|u|,|v|\le r$ *implies*

$$|H\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)|\le N_{k,r}\text{ for all }s\in\mathbb{Z}.$$

Definition 2.4 *[11] Let* E *be Banach spaces. An operator* $T:E\mapsto E$ *is completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Lemma 2.5 *[Schauder 11] Let X be a Banach space and $\Omega \subset X$ a nonempty, bounded, open, and convex subset of X centered at zero point. Let $T : \overline{\Omega} \mapsto X$ be a completely continuous operator with $T(\overline{\Omega}) \subset \overline{\Omega}$. Then T has a fixed point in $\overline{\Omega}$.*

Define

$$X = \left\{ x : \begin{array}{c} x_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}], s \in \mathbb{Z}, \\ \text{the limit } \lim_{t\to t_s^+} x(t) \text{ exists } (s \in \mathbb{Z}), \\ \text{the limits } \lim_{t\to\pm\infty} \frac{x(t)}{\sigma(t)} \text{ exist} \end{array} \right\}$$

and

$$Y = \left\{ y : \begin{array}{c} y_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}], s \in \mathbb{Z}, \\ \text{the limit } \lim_{t\to t_s^+} y(t) \text{ exists } (s \in \mathbb{Z}), \\ \text{the limits } \lim_{t\to\pm\infty} \frac{y(t)}{\tau(t)} \text{ exist} \end{array} \right\}.$$

For $x \in X$, define $||x|| = ||x||_X = \sup_{t\in\mathbb{R}} \frac{|x(t)|}{\sigma(t)}$. For $y \in Y$, define $||y|| = ||y||_Y = \sup_{t\in\mathbb{R}} \frac{|y(t)|}{\tau(t)}$.

Lemma 2.6 *X is a Banach space with the norm $||\cdot||_X$ and Y a Banach space with the norm $||\cdot||_Y$. $E = X\times Y$ is also a Banach space with the norm $||(x, y)|| = \max\{||x||_X, ||y||_Y\}$ for $(x, y) \in X \times Y$.*

Proof In fact, it is easy to see that X is a normed linear space. Let $\{x_n\}$ be a Cauchy sequence in X. Then $||x_m - x_n|| \to 0,\ m, n \to +\infty$. It follows that

$$\sup_{t\in R} \frac{|x_m(t)-x_n(t)|}{\sigma(t)} \to 0, m, n \to +\infty.$$

So

$$\sup_{t\in(t_s,t_{s+1}]} \frac{|x_m(t)-x_n(t)|}{\sigma(t)} \to 0, m, n \to +\infty,\ s \in \mathbb{Z}.$$

For $s \in \mathbb{Z}$, define

$$\overline{x}|_{[t_s,t_{s+1}]}(t) = \begin{cases} \lim_{t\to t_s^+} x(t), t = t_s, \\ x(t), \quad t \in (t_s,t_{s+1}]. \end{cases}$$

We know that $t \mapsto \frac{\overline{x}|_{[t_s,t_{s+1}]}(t)}{\sigma(t)}$ is continuous on $[t_s, t_{s+1}](s \in \mathbb{Z})$. From $\sup_{t\in(t_s,t_{s+1}]} \frac{|x_m(t)-x_n(t)|}{\sigma(t)} \to 0, m, n \to +\infty$, we know that $t \mapsto \frac{\overline{x_n}|_{[t_s,t_{s+1}]}(t)}{\sigma(t)}$ is a Cauchy sequence in $C[t_s, t_{s+1}]$. Then $\frac{\overline{x_n}|_{[t_s,t_{s+1}]}}{\sigma(t)}$ uniformly converges to some $\overline{x_0}_s$ in $C[t_s, t_{s+1}]$ as $n \to +\infty$. Define

$$x_0(t) = \overline{x_0}_s(t),\ t \in (t_s, t_{s+1}],\ s \in \mathbb{Z}.$$

Then x_0 is defined on $\mathbb{R}$ and is continuous on $(t_s, t_{s+1}]$ and the limit $\lim_{t\to t_s^+} x_0(t)(s \in \mathbb{Z})$ exists. Furthermore, we have $\lim_{n\to+\infty} \frac{x_n(t)}{\sigma(t)} = x_0(t)$ for every $t \in \mathbb{R}$. We need to do the following two steps:

Step 1. Prove that $\sigma x_0 \in X$.

In fact, it is easy to see that $\sigma x_0 \in C^0(t_s, t_{s+1}](s \in \mathbb{Z})$ and the limit $\lim_{t \to t_s^+} \sigma(t)x_0(t)$ exists for all $s \in \mathbb{Z}$.

For each $\epsilon > 0$, there exists N such that $\frac{|x_m(t)-x_n(t)|}{\sigma(t)} < \epsilon$ for $\forall m, n > N$ and $\forall t \in \mathbb{R}$. Let $m \to \infty$. We get $\left| x_0(t) - \frac{x_n(t)}{\sigma(t)} \right| \leq \epsilon$ for $\forall n > N$ and $\forall t \in \mathbb{R}$.

Since $\lim_{t \to \pm\infty} \frac{x_n(t)}{\sigma(t)}$ exists, we know

$$\left| \lim_{t \to -\infty} \frac{x_m(t)}{\sigma(t)} - \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} \right| \leq \epsilon, m, n > N,$$

$$\left| \lim_{t \to +\infty} \frac{x_m(t)}{\sigma(t)} - \lim_{t \to +\infty} \frac{x_n(t)}{\sigma(t)} \right| \leq \epsilon, m, n > N.$$

So both $\lim_{n \to \infty} \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)}$ and $\lim_{n \to \infty} \lim_{t \to +\infty} \frac{x_n(t)}{\sigma(t)}$ exist.

For $\forall \epsilon > 0$, choose n sufficiently large such that

$$\left| x_0(t) - \frac{x_n(t)}{\sigma(t)} \right| < \epsilon, \quad \left| \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} - \lim_{n \to \infty} \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} \right| < \epsilon.$$

Fixed this n, there exists $T > 0$ such that $\left| \frac{x_n(t)}{\sigma(t)} - \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} \right| < \epsilon$ for $\forall t < -T$. Then we have

$$\left| x_0(t) - \lim_{n \to \infty} \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} \right| \leq \left| x_0(t) - \frac{x_n(t)}{\sigma(t)} \right|$$

$$+ \left| \frac{x_n(t)}{\sigma(t)} - \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} \right| + \left| \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} - \lim_{n \to \infty} \lim_{t \to -\infty} \frac{x_n(t)}{\sigma(t)} \right| < 3\epsilon, \forall t < -T.$$

Hence $\lim_{t \to -\infty} x_0(t)$ exists. Similarly we have $\lim_{t \to +\infty} x_0(t)$ exists. Thus $t \to \sigma(t)x_0(t)$ is an element in X.

Step 2. Prove that $x_n \to \sigma x_0$ as $n \to \infty$ in X.

From $\sup_{t \in \mathbb{R}} \frac{|x_m(t)-x_n(t)|}{\sigma(t)} \to 0, m, n \to +\infty$, for $\forall \epsilon > 0$, there exists N such that $\frac{|x_m(t)-x_n(t)|}{\sigma(t)} < \frac{\epsilon}{2}$ for $\forall m, n > N$ and $\forall t \in \mathbb{R}$. Fixed t, let $m \to \infty$, we get $\left| x_0(t) - \frac{x_n(t)}{\sigma(t)} \right| \leq \frac{\epsilon}{2} < \epsilon$ for $\forall n > N$ and $\forall t \in \mathbb{R}$. Thus $\sup_{t \in \mathbb{R}} \left| x_0(t) - \frac{x_n(t)}{\sigma(t)} \right| \leq \frac{\epsilon}{2} < \epsilon$ for $\forall n > N$. Then $x_n \to \sigma x_0$ as $n \to \infty$ in X.

From the above discussion, we know that X is a Banach space.

Similarly we can prove that Y is a Banach space. Therefore, $E = X \times Y$ is a Banach space. The proof is complete. □

Lemma 2.7 *$M \subset X$ is relatively compact if and only if the following items are valid:*

(i) $\left\{ t \mapsto \frac{x(t)}{\sigma(t)} : x \in M \right\}$ *is uniformly bounded.*

(ii) $\left\{ t \mapsto \frac{x(t)}{\sigma(t)} : x \in M \right\}$ *is equi-continuous on* $(t_s, t_{s+1}](s \in \mathbb{Z})$.

(iii) $\left\{ t \mapsto \frac{x(t)}{\sigma(t)} : x \in M \right\}$ *is equi-convergent as* $t \to \pm\infty$.

Proof "$\Leftarrow$". From Lemma 2.2, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is for all $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by **(i)–(iii)**, there exist constants $A > 0, \delta > 0$, and positive integer s_0 such that

$$\left|\frac{x(u_1)}{\sigma(u_1)} - \frac{x(u_2)}{\sigma(u_2)}\right| \leq \frac{\epsilon}{3},\ u_1, u_2 \leq t_{-s_0} \text{ or } u_1, u_2 \geq t_{s_0},\ x \in M,$$

$$||x|| = \sup_{t\in R} \frac{|x(t)|}{\sigma(t)} \leq A,\ x \in M,$$

$$\left|\frac{x(u_1)}{\sigma(u_1)} - \frac{x(u_2)}{\sigma(u_2)}\right| \leq \frac{\epsilon}{3},\ u_1, u_2 \in [t_s, t_{s+1}],\ |u_1 - u_2| < \delta,\ s = -s_0, -s_0 + 1, \cdots, s_0 - 1,\ x \in M.$$

Define $X|_{[t_{-s_0}, t_{s_0}]} = \left\{x|_{[t_{-s_0}, t_{s_0}]} : x \in X\right\}$. For $x \in X|_{[t_{-s_0}, t_{s_0}]}$, define

$$||x||_{s_0} = \sup_{t\in[t_{-s_0}, t_{s_0}]} \frac{|x(t)|}{\sigma(t)}.$$

Similarly to Lemma 2.2, we can prove that $X_{[t_{-s_0}, t_{s_0}]}$ is a Banach space with the norm $||\cdot||_{s_0}$.

Let $M|_{[t_{-s_0}, t_{s_0}]} = \{t \mapsto x(t), t \in [t_{-s_0}, t_{s_0}] : x \in M\}$. Then $M|_{[t_{-s_0}, t_{s_0}]}$ is a subset of $X|_{[t_{-s_0}, t_{s_0}]}$. By the Ascoli–Arzela theorem, we know that $M|_{[t_{-s_0}, t_{s_0}]}$ is relatively compact in $X|_{[t_{-s_0}, t_{s_0}]}$. Thus, there exist $x_1, x_2, \cdots, x_k \in M$ such that, for any $x \in M$, we have that there exists some $i = 1, 2, \cdots, k$ such that

$$||x - x_i||_{s_0} = \sup_{t\in[t_{-s_0}, t_{s_0}]} \frac{|x(t) - x_i(t)|}{\sigma(t)} \leq \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we have that

$$\begin{aligned} ||x - x_i||_X &= \sup_{t\in R} \frac{|x(t) - x_i(t)|}{\sigma(t)} \\ &\leq \max\left\{\sup_{t\leq t_{-s_0}} \frac{|x(t) - x_i(t)|}{\sigma(t)},\ \sup_{t\in[t_{-s_0}, t_{s_0}]} \frac{|x(t) - x_i(t)|}{\sigma(t)},\ \sup_{t\geq t_{s_0}} \frac{|x(t) - x_i(t)|}{\sigma(t)}\right\}. \end{aligned}$$

We have that

$$\begin{aligned} \sup_{t\leq t_{-s_0}} \frac{|x(t) - x_i(t)|}{\sigma(t)} &\leq \sup_{t\leq t_{-s_0}} \left|\frac{x(t)}{\sigma(t)} - \frac{x(t_{-s_0})}{\sigma(t_{-s_0})}\right| \\ &+ \left|\frac{x(t_{-s_0})}{\sigma(t_{-s_0})} - \frac{x_i(t_{-s_0})}{\sigma(t_{-s_0})}\right| + \sup_{t\leq t_{-s_0}} \left|\frac{x_i(t_{-s_0})}{\sigma(t_{-s_0})} - \frac{x_i(t)}{\sigma(t)}\right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Similarly we can prove that . $\sup_{t\geq t_{s_0}} \frac{|x(t) - x_i(t)|}{\sigma(t)} \leq \epsilon$. Then $||x - x_i||_X < \epsilon$.

Thus, for any $\epsilon > 0$, M has a finite ϵ-net $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

$\Rightarrow$. Assume that M is relatively compact; then for any $\epsilon > 0$ there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$ with $x_i \subset M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and

$$||x|| \leq ||x - x_i|| + ||x_i|| \leq \epsilon + \max\{||x_i|| : i = 1, 2, \cdots, k\}.$$

It follows that $\{||x|| : x \in M\}$ is uniformly bounded. Then **(i)** holds.

Since the limit $\lim\limits_{t\to t_s^+} \frac{x(t)}{\sigma(t)}$ exists, then

$$\frac{\overline{x}(t)}{\sigma(t)} = \begin{cases} \lim\limits_{t\to t_s^+} \frac{x(t)}{\sigma(t)}, t = t_s, \\ \frac{x(t)}{\sigma(t)}, t \in (t_s, t_{s+1}] \end{cases}$$

is continuous on $[t_s, t_{s+1}]$. Thus for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{\overline{x}_i(u_1)}{\sigma(u_1)} - \frac{\overline{x}_i(u_2)}{\sigma(u_2)} \right| < \epsilon$$

for all $u_1, u_2 \in [t_s, t_{s+1}]$ with $|u_1 - u_2| < \delta$ and $i = 1, 2, \cdots, k$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{x_i(u_1)}{\sigma(u_1)} - \frac{x_i(u_2)}{\sigma(u_2)} \right| < \epsilon$$

for all $u_1, u_2 \in (t_s, t_{s+1}]$ with $|u_1 - u_2| < \delta$ and $i = 1, 2, \cdots, k$.

For $x \in M$, there exists i such that $x \in U_{x_i}$. Then we have for $u_1, u_2 \in (t_s, t_{s+1}]$ with $|u_1 - u_2| < \delta$ that

$$\left| \frac{x(u_1)}{\sigma(u_1)} - \frac{x(u_2)}{\sigma(u_2)} \right| \leq \left| \frac{x(u_1)}{\sigma(u_1)} - \frac{x_i(u_1)}{\sigma(u_1)} \right| + \left| \frac{x_i(u_1)}{\sigma(u_1)} - \frac{x_i(u_2)}{\sigma(u_2)} \right| + \left| \frac{x_i(u_2)}{\sigma(u_2)} - \frac{x(u_2)}{\sigma(u_2)} \right| \leq 3\epsilon.$$

$\left\{ t \mapsto \frac{x(t)}{\sigma(t)} : x \in M \right\}$ is equi-continuous in $(t_s, t_{s+1}](s \in Z)$. It follows that **(ii)** holds.

Now we prove that **(iii)** holds. It is easily seen that there exists a positive integer s_0 such that

$$\left| \frac{x_i(u_1)}{\sigma(u_1)} - \frac{x_i(u_2)}{\sigma(u_2)} \right| < \epsilon$$

for all $u_1, u_2 \leq t_{-s_0}$, $i = 1, 2, \cdots, k$. For $x \in M$, there exists i such that $x \in U_{x_i}$. Thus

$$\left| \frac{x(u_1)}{\sigma(u_1)} - \frac{x(u_2)}{\sigma(u_2)} \right| \leq \left| \frac{x(u_1)}{\sigma(u_1)} - \frac{x_i(u_1)}{\sigma(u_1)} \right| + \left| \frac{x_i(u_1)}{\sigma(u_1)} - \frac{x_i(u_2)}{\sigma(u_2)} \right|$$

$$+ \left| \frac{x_i(u_2)}{\sigma(u_2)} - \frac{x(u_2)}{\sigma(u_2)} \right| \leq 3\epsilon, \ u_1, u_2 \leq t_{-s_0}.$$

Then $\lim\limits_{t\to-\infty} \frac{x(t)}{\sigma(t)}$ exists. Similarly we can prove that $\lim\limits_{t\to+\infty} \frac{x(t)}{\sigma(t)}$ exists. Hence **(iii)** holds. Consequently, the lemma is proved. □

Lemma 2.8 *$M \subset Y$ is relatively compact if and only if the following items are valid:*

(i) $\left\{ t \mapsto \frac{x(t)}{\tau(t)} : x \in M \right\}$ *is uniformly bounded.*

(ii) $\left\{ t \mapsto \frac{x(t)}{\tau(t)} : x \in M \right\}$ *is equi-continuous on* $(t_s, t_{s+1}](s \in \mathbb{Z})$.

(iii) $\left\{ t \mapsto \frac{x(t)}{\tau(t)} : x \in M \right\}$ *is equi-convergent as* $t \to \pm\infty$.

Proof The proof is similar to Lemma 2.3 and is omitted. □

Lemma 2.9 *Suppose that $(x,y)\in E$. Then $u\in X$ is a solution of BVP*

$$\begin{aligned}&[\Phi(\rho(t)u'(t))]'+f(t,x(t),y(t))=0,\quad a.e.\ t\in\mathbb{R},\\&\lim_{t\to-\infty}u(s)=0,\ \ \lim_{t\to+\infty}\rho(t)u'(s)=0,\\&\Delta u(t_s)=I_0(t_k,x(t_k),y(t_k)),\quad k\in\mathbb{Z},\\&\Delta\Phi(\rho(t_s)u'(t_s))=I_1(t_k,x(t_k),y(t_k)),\quad k\in\mathbb{Z}\end{aligned}\tag{2.1}$$

if and only if

$$\begin{aligned}u(t)=&\sum_{t_s\le t}I_0(t_s,x(t_s),y(t_s))\\&+\int_{-\infty}^t\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}f(u,x(u),y(u))du-\sum_{t_j>s}I_1(t_j,x(t_j),y(t_j))\right)ds.\end{aligned}\tag{2.2}$$

Proof Fix $x,y\in X$; then

$$||(x,y)||=\max\left\{\sup_{t\in\mathbb{R}}\frac{|x(t)|}{\sigma(t)},\ \sup_{t\in\mathbb{R}}\frac{|y(t)|}{\tau(t)}\right\}=r<+\infty.$$

Since f is a sub-Carathédory function, we know that there exists nonnegative function $\phi_r\in L^1(\mathbb{R})$ such that

$$|f\left(t,x(t),y(t)\right)|=\left|f\left(t,\sigma(t)\frac{x(t)}{\sigma(t)},\tau(t)\frac{y(t)}{\tau(t)}\right)\right|\le\phi_r(t),t\in R.\tag{2.3}$$

Furthermore, $I_1,J_1:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}^2\mapsto\mathbb{R}$ are discrete Carathéodory functions, $I_0:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}^2\mapsto\mathbb{R}$ is a discrete $\rho-$Carathéodory function, $J_0:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}^2\mapsto\mathbb{R}$ is a discrete $\varrho-$Carathéodory function. Then there exist nonnegative constants $M_{s,r},N_{s,r}\ge0$ such that

$$\begin{aligned}&|I_0\left(t_s,x(t_s),y(t_s)\right)|\le M_{s,r},s\in\mathbb{Z},\sum_{s=-\infty}^{+\infty}M_{s,r}<+\infty,\\&\sum_{j=-\infty}^{s}N_{j,r}<+\infty\text{ for all }s\in\mathbb{Z},\\&\lim_{s\to+\infty}\frac{N_{s,r}}{\int_{t_s}^{t_{s+1}}\frac{du}{\varrho(u)}}=\lim_{s\to+\infty}\frac{N_{s,r}}{\int_{t_{s-1}}^{t_s}\frac{du}{\varrho(u)}}=0,\\&|I_1\left(t_s,x(t_s),y(t_s)\right)|\le N_{s,r},s\in\mathbb{Z},\sum_{s=-\infty}^{+\infty}M_{s,r}<+\infty.\end{aligned}\tag{2.4}$$

Suppose that $u\in X$ is a solution of (2.1). Then from the boundary conditions we get that

$$\Phi(\rho(t)u'(t))=\int_{-\infty}^t f(r,x(r),y(r))dr-\sum_{t_j>ts}I_1(t_j,x(t_j),y(t_j)),\ t\in\mathbb{R}.$$

Thus

$$u'(t)=\frac{1}{\rho(t)}\Phi^{-1}\left(\int_{-\infty}^t f(r,x(r),y(r))dr-\sum_{t_j>t}I_1(t_j,x(t_j),y(t_j))\right).$$

It follows that

$$u(t)=\sum_{t_s\le t}I_0(t_s,x(t_s),y(t_s))$$

$$+\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}f(u,x(u),y(u))du-\sum_{t_j>s}I_1(t_j,x(t_j),y(t_j))\right)ds.$$

Hence u satisfies (2.2). We now prove that $u\in X$. In fact, we see that $u|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]$ and $\lim_{t\to t_s^+}u(t)(s\in\mathbb{Z})$ exists, $\lim_{t\to-\infty}\frac{u(t)}{\sigma(t)}=0$.

Furthermore, we have $\left|\sum_{t_s\le t}I_0(t_s,x(t_s),y(t_s))\right|\le\sum_{t_s\le t}M_{s,r}$. For $t\in[t_s,t_{s+1})$, we have

$$\frac{\sum_{j=-\infty}^{s}M_{j,r}}{\int_{-\infty}^{t_{s+1}}\frac{du}{\rho(u)}}\le\frac{\sum_{t_s\le t}M_{s,r}}{\sigma(t)}\le\frac{\sum_{j=-\infty}^{s}M_{j,r}}{\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}$$

Then

$$\lim_{s\to+\infty}\frac{\sum_{j=-\infty}^{s}M_{j,r}}{\int_{-\infty}^{t_{s+1}}\frac{du}{\rho(u)}}=\lim_{s\to+\infty}\frac{M_{s,r}}{\int_{t_s}^{t_{s+1}}\frac{du}{\rho(u)}}=0,$$

$$\lim_{s\to+\infty}\frac{\sum_{j=-\infty}^{s}M_{j,r}}{\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}=\lim_{s\to+\infty}\frac{M_{s,r}}{\int_{t_{s-1}}^{t_s}\frac{du}{\rho(u)}}=0$$

imply that $\lim_{t\to+\infty}\frac{\sum_{t_s\le t}M_{s,r}}{\sigma(t)}=0$. Thus $\lim_{t\to+\infty}\frac{\sum_{t_s\le t}I_0(t_s,x(t_s),y(t_s))}{\sigma(t)}=0$. We have

$$\lim_{t\to+\infty}\frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}f(u,x(u),y(u))du-\sum_{t_j>s}I_1(t_j,x(t_j),y(t_j))\right)ds}{\sigma(t)}$$

$$=\lim_{t\to+\infty}\Phi^{-1}\left(\int_t^{+\infty}f(u,x(u),y(u))du-\sum_{t_j>t}I_1(t_j,x(t_j),y(t_j))\right)=0.$$

It follows that $\lim_{t\to+\infty}\frac{u(t)}{\sigma(t)}=0$. Then $u\in X$.

On the other hand, if u satisfies (2.2), we can prove that $u\in X$ is a solution of BVP (2.1) easily. The proof is completed. □

Lemma 2.10 *Suppose that $(x,y)\in E$. Then $v\in Y$ is a solution of BVP*

$$\begin{aligned}&[\Psi(\varrho(t)v'(t))]'+g(t,x(t),y(t))=0,\quad a.e.\ t\in\mathbb{R},\\&\lim_{t\to-\infty}v(s)=0,\ \ \lim_{t\to+\infty}\varrho(t)v'(t)=0,\\&\Delta v(t_s)=J_0(t_s,x(t_s),y(t_s)),\quad s\in\mathbb{Z},\\&\Delta\Psi(\varrho(t_s)v'(t_s))=J_1(t_s,x(t_s),y(t_s)),\quad s\in\mathbb{Z}\end{aligned}\tag{2.5}$$

if and only if

$$v(t) = \sum_{t_s \le t} J_0(t_s, x(t_s), y(t_s)) \\ + \int_{-\infty}^{t} \frac{1}{\varrho(s)} \Psi^{-1}\left(\int_s^{+\infty} g(u, x(u), y(u))du - \sum_{t_j > s} J_1(t_j, x(t_j), y(t_j))\right) ds. \tag{2.6}$$

Proof The proof is similar to that of Lemma 2.2 and is omitted. □

Define the operator T on E by $T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t))$ with

$$T_1(x,y)(t) = \sum_{t_s \le t} I_0(t_s, x(t_s), y(t_s)) \\ + \int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_s^{+\infty} f(u, x(u), y(u))du - \sum_{t_j > s} I_1(t_j, x(t_j), y(t_j))\right) ds, \tag{2.7}$$

and

$$T_2(x,y)(t) = \sum_{t_s \le t} J_0(t_s, x(t_s), y(t_s)) \\ + \int_{-\infty}^{t} \frac{1}{\varrho(s)} \Psi^{-1}\left(\int_s^{+\infty} g(u, x(u), y(u))du - \sum_{t_j > s} J_1(t_j, x(t_j), y(t_j))\right) ds. \tag{2.8}$$

Lemma 2.11 *The following results hold:*

(i) $T : E \mapsto E$ *is well defined.*

(ii) $(x,y) \in E$ *is a solution of BVP (1.4) if and only if* (x,y) *is a fixed point of* T *in* E*.*

(iii) $T : E \mapsto E$ *is completely continuous.*

Proof From Lemma 2.2, E is a Banach space. From Lemma 2.5 and Lemma 2.6, we know that (i) and (ii) hold. Note Lemma 2.3 and Lemma 2.4, the proof of (iii) is similar to that of Lemma 2.4 in [22] and is omitted. □

3. Main theorems

In this section, the main results on the existence of solutions of BVP (1.4) are established. The inverse of Φ is $\Phi^{-1}(x) = \Phi_{q_1}(x) = |x|^{q_1-2}x$ with $\frac{1}{p_1} + \frac{1}{q_1} = 1$. The inverse of Ψ is $\Psi^{-1}(x) = \Phi_{q_2}(x) = |x|^{q_2-2}x$ with $\frac{1}{p_2} + \frac{1}{q_2} = 1$. We need the following assumptions:

(A) there exist nonnegative functions $\phi, \psi \in L^1(\mathbb{R})$, nondecreasing functions $F, G, \overline{I_0}, \overline{J_0}, \overline{I_1}, \overline{J_1} : [0,\infty) \times [0,\infty) \to [0,\infty)$, nonnegative constants $\phi_{i,s}, \psi_{i,s} \ge 0$, $(i = 0, 1, s \in \mathbf{Z})$ such that

$$\sum_{s\in\mathbf{Z}} \phi_{1,s} < +\infty, \ \sum_{s\in\mathbf{Z}} \psi_{1,s} < +\infty, \ \sum_{j=-\infty}^{s} \phi_{0,s} < +\infty, \ \sum_{j=-\infty}^{s} \psi_{0,s} < +\infty,$$

$$\lim_{s\to+\infty} \frac{\phi_{0,s}}{\int_{t_{s-1}}^{t_s} \frac{du}{\rho(u)}} = \lim_{s\to+\infty} \frac{\phi_{0,s}}{\int_{t_s}^{t_{s+1}} \frac{du}{\rho(u)}} = \lim_{s\to+\infty} \frac{\psi_{0,s}}{\int_{t_{s-1}}^{t_s} \frac{du}{\varrho(u)}} = \lim_{s\to+\infty} \frac{\psi_{0,s}}{\int_{t_s}^{t_{s+1}} \frac{du}{\varrho(u)}} = 0$$

and

$$|f\left(t,\sigma(t)u,\tau(t)v\right)| \leq \phi(t)\Phi\left(F(|u|,|v|)\right), u,v \in \mathbb{R}, a.e.\ t \in \mathbb{R},$$

$$|g\left(t,\sigma(t)u,\tau(t)v\right)| \leq \psi(t)\Psi\left(G(|u|,|v|)\right), u,v \in \mathbb{R}, a.e.\ t \in \mathbb{R},$$

$$|I_0\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)| \leq \phi_{0,s}\overline{I_0}(|u|,|v|), u,v \in \mathbb{R}, s \in \mathbb{Z},$$

$$|I_1\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)| \leq \phi_{1,s}\Phi\left(\overline{I_1}(|u|,|v|)\right), u,v \in \mathbb{R}, s \in \mathbb{Z},$$

$$|J_0\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)| \leq \psi_{0,s}\overline{J_0}(|u|,|v|), u,v \in \mathbb{R}, s \in \mathbb{Z}$$

$$|J_1\left(t_s,\sigma(t_s)u,\tau(t_s)v\right)| \leq \psi_{1,s}\Psi\left(\overline{J_1}(|u|,|v|)\right), u,v \in \mathbb{R}, s \in \mathbb{Z}.$$

Theorem 3.1 *Suppose that (A) holds. Then BVP (1.4) has at least one solution if*

$$\sup_{s\in\mathbb{Z}} \frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\overline{I_0}(r,r) + \sup_{s\in\mathbb{Z}} \frac{\sum\limits_{j=-\infty}^{s}\psi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\varrho(u)}}\overline{J_0}(r,r)$$

$$+\Phi^{-1}\left(||\phi||_1\Phi\left(F(r,r)\right) + \sum_{s\in\mathbb{Z}}\phi_{1,s}\Phi\left(\overline{I_1}(r,r)\right)\right)$$

$$+\Psi^{-1}\left(||\psi||_1\Psi\left(G(r,r)\right) + \sum_{s\in\mathbb{Z}}\psi_{1,s}\Psi\left(\overline{J_1}(r,r)\right)\right) \leq r$$

has a positive solution r_0*.*

Proof We will apply Lemma 2.1 to show the results. Let X,Y,E, and T be defined in section 2. From Lemma 2.7, $T: E \mapsto E$ is a completely continuous operator and $(x,y) \in E$ is a solution of BVP (1.4) if and only if (x,y) is a fixed point of T in E.

For $(x,y) \in E$, we have $||(x,y)|| = r < +\infty$. Then (B) implies that

$$|f(t,x(t),y(t))| = \left|f\left(t,\sigma(t)\frac{x(t)}{\sigma(t)},\tau(t)\frac{y(t)}{\tau(t)}\right)\right| \leq \phi(t)\Phi\left(F\left(\left|\frac{x(t)}{\sigma(t)}\right|,\left|\frac{y(t)}{\tau(t)}\right|\right)\right)$$

$$\leq \phi(t)\Phi\left(F(||x||,||y||)\right), a.e.\ t \in \mathbb{R},$$

$$|g(t,x(t),y(t))| \leq \psi(t)\Psi\left(G(||x||,||y||)\right), a.e.\ t \in \mathbb{R},$$

$$|I_0\left(t_s,x(t_s),y(t_s)\right)| \leq \phi_{0,s}\overline{I_0}(||x||,||y||), s \in \mathbb{Z},$$

$$|I_1\left(t_s,x(t_s),y(t_s)\right)| \leq \phi_{1,s}\Phi\left(\overline{I_1}(||x||,||y||)\right), s \in \mathbb{Z},$$

and

$$|J_0\left(t_s,x(t_s),y(t_s)\right)| \leq \psi_{0,s}\overline{J_0}(||x||,||y||), s \in \mathbb{Z},$$

$$|J_1\left(t_s,x(t_s),y(t_s)\right)| \leq \psi_{1,s}\Psi\left(\overline{J_1}(||x||,||y||)\right), s \in \mathbb{Z}.$$

Then (2.7) implies that

$$\frac{|T_1(x,y)(t)|}{\sigma(t)} \leq \frac{\sum\limits_{t_s\leq t}|I_0(t_s,x(t_s),y(t_s))|}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}} + \frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}|f(u,x(u),y(u))|du+\sum\limits_{t_j>s}|I_1(t_j,x(t_j),y(t_j))|\right)ds}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}$$

$$\leq \frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\overline{I_0}(||x||,||y||) + \frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}\phi(u)du\Phi(F(||x||,||y||))+\sum\limits_{t_j>s}\phi_{1,j}\phi_{1,s}\Phi\left(\overline{I_1}(||x||,||y||)\right)\right)ds}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}$$

$$\leq \sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\overline{I_0}(||x||,||y||) + \Phi^{-1}\left(||\phi||_1\Phi\left(F(||x||,||y||)\right)+\sum_{s\in\mathbf{Z}}\phi_{1,s}\Phi\left(\overline{I_1}(||x||,||y||)\right)\right)$$

$$\leq \sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\overline{I_0}(r,r) + \Phi^{-1}\left(||\phi||_1\Phi\left(F(r,r)\right)+\sum_{s\in\mathbf{Z}}\phi_{1,s}\Phi\left(\overline{I_1}(r,r)\right)\right).$$

Similarly from (2.8), we have

$$\frac{|T_2(x,y)(t)|}{\tau(t)} \leq \sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\psi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\varrho(u)}}\overline{J_0}(r,r) + \Psi^{-1}\left(||\psi||_1\Psi\left(G(r,r)\right)+\sum_{s\in\mathbf{Z}}\psi_{1,s}\Psi\left(\overline{J_1}(r,r)\right)\right).$$

It follows that

$$||(T_1(x,y),T_2(x,y))|| = \max\left\{\sup_{t\in\mathbb{R}}\frac{|T_1(x,y)(t)|}{\sigma(t)},\sup_{t\in\mathbb{R}}\frac{|T_2(x,y)(t)|}{\tau(t)}\right\}$$

$$\leq \sup_{t\in\mathbb{R}}\frac{|T_1(x,y)(t)|}{\sigma(t)} + \sup_{t\in\mathbb{R}}\frac{|T_2(x,y)(t)|}{\tau(t)}$$

$$\leq \sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\overline{I_0}(r,r) + \sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\psi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\varrho(u)}}\overline{J_0}(r,r)$$

$$+\Phi^{-1}\left(||\phi||_1\Phi\left(F(r,r)\right)+\sum_{s\in\mathbf{Z}}\phi_{1,s}\Phi\left(\overline{I_1}(r,r)\right)\right)$$

$$+\Psi^{-1}\left(||\psi||_1\Psi\left(G(r,r)\right)+\sum_{s\in\mathbf{Z}}\psi_{1,s}\Psi\left(\overline{J_1}(r,r)\right)\right)$$

By the assumption in Theorem 3.1, we know that $\Omega_0 = \{(x,y)\in E : ||(x,y)|| \leq r_0\}$ satisfies $T\Omega_0 \subseteq \Omega_0$. Thus T has at least one fixed point $(x,y)\in\Omega_0$. Then (x,y) is a solution of BVP (1.4). The proof is completed. □

(B) there exist nonnegative constants

$$A_{i,0},B_{i,j}\geq 0,\ a_{i,0},b_{i,j}\geq 0,\ c_{i,0},d_{i,j}\geq 0,\ (i=0,1,j=1,2,\cdots,m),$$

$$\phi_{i,s},\psi_{i,s}\geq 0,\ (i=0,1,s\in\mathbb{Z}),\ \tau_{i,j},\sigma_{i,j}\geq 0(i=1,2,j=1,2,\cdots,m)$$

such that

$$\sum_{s\in\mathbf{Z}}\phi_{1,s}<+\infty,\ \sum_{s\in\mathbf{Z}}\psi_{1,s}<+\infty,\ \sum_{j=-\infty}^{s}\phi_{0,s}<+\infty,\ \sum_{j=-\infty}^{s}\psi_{0,s}<+\infty,$$

$$\lim_{s\to+\infty}\frac{\phi_{0,s}}{\int_{t_{s-1}}^{t_s}\frac{du}{\rho(u)}} = \lim_{s\to+\infty}\frac{\phi_{0,s}}{\int_{t_s}^{t_{s+1}}\frac{du}{\rho(u)}} = \lim_{s\to+\infty}\frac{\psi_{0,s}}{\int_{t_{s-1}}^{t_s}\frac{du}{\varrho(u)}} = \lim_{s\to+\infty}\frac{\psi_{0,s}}{\int_{t_s}^{t_{s+1}}\frac{du}{\varrho(u)}} = 0$$

and

$$|f(t,\sigma(t)u,\tau(t)v)| \leq e^{-t^2}\Phi\left(A_{1,0}+\sum_{j=1}^{m}B_{1,j}|u|^{\tau_{1,j}}|v|^{\sigma_{1,j}}\right), u,v\in\mathbb{R}, a.e.\ t\in\mathbb{R},$$

$$|g(t,\sigma(t)u,\tau(t)v)| \leq e^{-t^2}\Psi\left(A_{2,0}+\sum_{j=1}^{m}B_{2,j}|u|^{\tau_{2,j}}|v|^{\sigma_{2,j}}\right), u,v\in\mathbb{R}, a.e.\ t\in\mathbb{R},$$

$$|I_0(t_s,\sigma(t_s)u,\tau(t_s)v)| \leq \phi_{0,s}\left[a_{0,0}+\sum_{j=1}^{m}b_{0,j}|u|^{\tau_{1,j}}|v|^{\sigma_{1,j}}\right], u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$|I_1(t_s,\sigma(t_s)u,\tau(t_s)v)| \leq \phi_{1,s}\Phi\left(a_{1,0}+\sum_{j=1}^{m}b_{1,j}|u|^{\tau_{1,j}}|v|^{\sigma_{1,j}}\right), u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$|J_0(t_s,\sigma(t_s)u,\tau(t_s)v)| \leq \psi_{0,s}\left[c_{0,0}+\sum_{j=1}^{m}d_{0,j}|u|^{\tau_{2,j}}|v|^{\sigma_{2,j}}\right], u,v\in\mathbb{R}, s\in\mathbb{Z}$$

$$|J_1(t_s,\sigma(t_s)u,\tau(t_s)v)| \leq \psi_{1,s}\Psi\left(c_{1,0}+\sum_{j=1}^{m}d_{1,j}|u|^{\tau_{2,j}}|v|^{\sigma_{2,j}}\right), u,v\in\mathbb{R}, s\in\mathbb{Z}.$$

We denote

$$\prod\nolimits_1 = \sup_{s\in\mathbf{Z}}\frac{\sum_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}},\quad \prod\nolimits_2 = \sup_{s\in\mathbf{Z}}\frac{\sum_{j=-\infty}^{s}\psi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\varrho(u)}},$$

$$\sigma = \max\{\tau_{1,j}+\sigma_{1,j},\ \tau_{2,j}+\sigma_{2,j}:\ j=1,2,\cdots,m\},$$

$$P_1 = \prod\nolimits_1 a_{0,0} + \sigma_{q_1}\Phi^{-1}(\sqrt{\pi})A_{1,0} + \sigma_{q_1}\Phi^{-1}\left(\sum_{s\in\mathbf{Z}}\phi_{1,s}\right)a_{1,0},$$

$$Q_j^1 = \prod\nolimits_1 b_{0,j} + \sigma_{q_1}\Phi^{-1}(\sqrt{\pi})B_{1,j} + \sigma_{q_1}\Phi^{-1}\left(\sum_{s\in\mathbf{Z}}\phi_{1,s}\right)b_{1,j},$$

and

$$P_2 = \prod\nolimits_2 c_{0,0} + \sigma_{q_2}\Psi^{-1}(\sqrt{\pi})A_{2,0} + \sigma_{q_2}\Psi^{-1}\left(\sum_{s\in\mathbf{Z}}\psi_{1,s}\right)c_{1,0},$$

$$Q_j^2 = \prod\nolimits_2 d_{0,j} + \sigma_{q_2}\Psi^{-1}(\sqrt{\pi})B_{2,j} + \sigma_{q_2}\Psi^{-1}\left(\sum_{s\in\mathbf{Z}}\psi_{1,s}\right)d_{1,j},$$

$$A = \max\{P_1,\ P_2\},\quad B = \max\left\{\sum_{j=1}^{m}Q_j^1,\ \sum_{j=1}^{m}Q_j^2\right\}.$$

Theorem 3.2 *Suppose that (B) holds. Then BVP (1.4) has at least one solution if*

- **(i)** $\sigma\in(0,1)$ *or*
- **(ii)** $\sigma=1$ *and* $B<1$ *or*
- **(iii)** $\sigma>1$ *and* $B(A+B)^{\sigma-1}\leq\frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof We will apply Lemma 2.1 to show the results. Let X, Y, E, and T be defined in section 2. From Lemma 2.7, $T : E \mapsto E$ is a completely continuous operator and $(x, y) \in E$ is a solution of BVP (1.4) if and only if (x, y) is a fixed point of T in E.

For $(x, y) \in E$, we have $||(x, y)|| = r < +\infty$. Then (B) implies that

$$|f(t,x(t),y(t))| = \left|f\left(t,\sigma(t)\frac{x(t)}{\sigma(t)},\tau(t)\frac{y(t)}{\tau(t)}\right)\right| \le e^{-t^2}\Phi\left(A_{1,0}+\sum_{j=1}^{m}B_{1,j}\left|\frac{x(t)}{\sigma(t)}\right|^{\tau_{1,j}}\left|\frac{y(t)}{\tau(t)}\right|^{\sigma_{1,j}}\right)$$

$$\le e^{-t^2}\Phi\left(A_{1,0}+\sum_{j=1}^{m}B_{1,j}||x||^{\tau_{1,j}}||y||^{\sigma_{1,j}}\right) \le e^{-t^2}\Phi\left(A_{1,0}+\sum_{j=1}^{m}B_{1,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right), a.e.\ t\in\mathbb{R},$$

and

$$|g(t,x(t),y(t))| \le e^{-t^2}\Psi\left(A_{2,0}+\sum_{j=1}^{m}B_{2,j}||(x,y)||^{\tau_{2,j}+\sigma_{2,j}}\right), a.e.\ t\in\mathbb{R},$$

$$|I_0(t_s,x(t_s),y(t_s))| \le \phi_{0,s}\left[a_{0,0}+\sum_{j=1}^{m}b_{0,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right], s\in\mathbb{Z},$$

$$|I_1(t_s,x(t_s),y(t_s))| \le \phi_{1,s}\Phi\left(a_{1,0}+\sum_{j=1}^{m}b_{1,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right), s\in\mathbb{Z},$$

$$|J_0(t_s,x(t_s),y(t_s))| \le \psi_{0,s}\left[c_{0,0}+\sum_{j=1}^{m}d_{0,j}||(x,y)||^{\tau_{2,j}+\sigma_{2,j}}\right], s\in\mathbb{Z},$$

$$|J_1(t_s,x(t_s),y(t_s))| \le \psi_{1,s}\Psi\left(c_{1,0}+\sum_{j=1}^{m}d_{1,j}||(x,y)||^{\tau_{2,j}+\sigma_{2,j}}\right), s\in\mathbb{Z}.$$

Note $\int_{-\infty}^{+\infty}e^{-s^2}ds=\sqrt{\pi}$. Then (2.7) implies that

$$\frac{|T_1(x,y)(t)|}{\sigma(t)} \le \frac{\sum\limits_{t_s\le t}|I_0(t_s,x(t_s),y(t_s))|}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}+\frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}|f(u,x(u),y(u))|du+\sum\limits_{t_j>s}|I_1(t_j,x(t_j),y(t_j))|\right)ds}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}$$

$$\le \frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\left[a_{0,0}+\sum_{j=1}^{m}b_{0,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right]$$

$$+\frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}e^{-u^2}du\Phi\left(A_{1,0}+\sum\limits_{j=1}^{m}B_{1,j}||x||^{\tau_{1,j}}||y||^{\sigma_{1,j}}\right)+\sum\limits_{t_j>s}\phi_{1,j}\Phi\left(a_{1,0}+\sum\limits_{j=1}^{m}b_{1,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right)\right)ds}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}$$

$$\le \prod\nolimits_1\left[a_{0,0}+\sum_{j=1}^{m}b_{0,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right]$$

$$+\Phi^{-1}\left(\sqrt{\pi}\Phi\left(A_{1,0}+\sum_{j=1}^{m}B_{1,j}||x||^{\tau_{1,j}}||y||^{\sigma_{1,j}}\right)+\sum_{s\in Z}\phi_{1,s}\Phi\left(a_{1,0}+\sum_{j=1}^{m}b_{1,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right)\right).$$

One knows that

$$\phi_p(u+v) \le \sigma_p[\phi_p(u)+\phi_p(v)], u,v \ge 0 \text{ with } \sigma_p = \begin{cases} 1, \ 1<p<2, \\ 2^{p-1}, \ p \ge 2. \end{cases}$$

Hence

$$\frac{|T_1(x,y)(t)|}{\sigma(t)} \le \prod_1 \left[a_{0,0} + \sum_{j=1}^{m} b_{0,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right]$$

$$+\sigma_{q_1}\Phi^{-1}\left(\sqrt{\pi}\right)\left(A_{1,0}+\sum_{j=1}^{m} B_{1,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right)$$

$$+\sigma_{q_1}\Phi^{-1}\left(\sum_{s\in Z}\phi_{1,s}\right)\left(a_{1,0}+\sum_{j=1}^{m} b_{1,j}||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}\right)$$

$$=\prod_1 a_{0,0}+\sigma_{q_1}\Phi^{-1}\left(\sqrt{\pi}\right)A_{1,0}+\sigma_{q_1}\Phi^{-1}\left(\sum_{s\in Z}\phi_{1,s}\right)a_{1,0}$$

$$+\sum_{j=1}^{m}\left(\prod_1 b_{0,j}+\sigma_{q_1}\Phi^{-1}\left(\sqrt{\pi}\right)B_{1,j}+\sigma_{q_1}\Phi^{-1}\left(\sum_{s\in Z}\phi_{1,s}\right)b_{1,j}\right)||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}.$$

It follows that

$$\sup_{t\in\mathbb{R}}\frac{|T_1(x,y)(t)|}{\sigma(t)} \le P_1+\sum_{j=1}^{m}Q_j^1||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}. \tag{3.1}$$

Similarly from (2.8), we get

$$\sup_{t\in\mathbb{R}}\frac{|T_2(x,y)(t)|}{\tau(t)} \le P_2+\sum_{j=1}^{m}Q_j^2||(x,y)||^{\tau_{1,j}+\sigma_{1,j}}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$||(T_1(x,y),T_2(x,y))|| \le A+B\max\{||(x,y)||^{\sigma},\ 1\} \le A+B+B||(x,y)||^{\sigma}.$$

(i) $\sigma \in (0,1)$.

Since $\sigma\in(0,1)$, change $r_0>0$ such that $A+B+Br_0^{\sigma}\le r_0$. Let $\Omega_0=\{(x,y)\in X\times Y: ||(x,y)||\le r_0\}$. Then we get

$$||(T_1(x,y),T_2(x,y))|| \le A+B+Br_0^{\sigma}\le r_0.$$

Hence $T\overline{\Omega_0}\subset\overline{\Omega_0}$. Thus Lemma 2.1 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$ and so BVP (1.4) has at least one solution.

(ii) $\sigma = 1$.

Let $r_0=\frac{A+B}{1-B}$ such that $A+B+Br_0=r_0$. Let $\Omega_0=\{(x,y)\in X\times Y: ||(x,y)||\le r_0\}$. Then we get

$$||T_1(x,y),T_2(x,y)|| \le A+B+Br_0\le r_0.$$

Hence $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Thus Lemma 2.1 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$ and so BVP (1.4) has at least one solution.

(iii) $\sigma > 1$.

Let $r_0 = \left(\frac{A+B}{B(\sigma-1)}\right)^{\frac{1}{\sigma}}$. It is easy to show from $\frac{(A+B)^{\sigma-1}\sigma^\sigma}{(\sigma-1)^{\sigma-1}} \le \frac{1}{B}$ that $A+B+Br_0^\sigma \le r_0$. Let $\Omega_0 = \{(x,y) \in X \times Y : ||(x,y)|| \le r_0\}$. Then we get

$$||(T_1(x,y), T_2(x,y))|| \le A+B+Br_0^\sigma \le r_0.$$

Hence $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Thus Lemma 2.1 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$ and so BVP (1.4) has at least one solution.

The proof of Theorem 3.2 is completed. □

Remark 3.3 *From Theorem 3.2, BVP (1.4) has at least one solution if* $\sigma \in (0,1)$.

For $\sigma = 1$, *BVP (1.4) has at least one solution for all sufficiently small* $b_{i,j}, d_{i,j}, B_{1,j}, B_{2,j} (i = 0,1, j = 1,2,3,\cdots,m)$ *since* $B < 1$ *for all sufficiently small* $b_{i,j}, d_{i,j}, B_{1,j}, B_{2,j} (i = 0,1, j = 1,2,3,\cdots,m)$.

For $\sigma > 1$, then BVP (1.4) has at least one solution for all sufficiently small

$$A_{i,0}, B_{i,j} \ge 0,\ (i = 1,2, j = 1,2,\cdots,m),$$

$$a_{i,0}, b_{i,j} \ge 0,\ (i = 0,1, j = 1,2,\cdots,m),$$

$$c_{i,0}, c_{i,j} \ge 0,\ (i = 0,1, j = 1,2,\cdots,m),$$

$$\phi_{i,s}, \psi_{i,s} \ge 0,\ (i = 0,1, s \in \mathbf{Z}).$$

(C) there exist nonnegative functions $\phi, \psi \in L^1(\mathbb{R})$, nonnegative numbers $M_f, M_g, M_{I0}, M_{J0}, M_{I1}, M_{J1}$, nonnegative constants $\phi_{i,s}, \psi_{i,s} \ge 0,\ (i = 0,1, s \in \mathbf{Z})$ such that

$$\sum_{s\in\mathbf{Z}} \phi_{1,s} < +\infty,\ \sum_{s\in\mathbf{Z}} \psi_{1,s} < +\infty,\ \sum_{j=-\infty}^{s} \phi_{0,s} < +\infty,\ \sum_{j=-\infty}^{s} \psi_{0,s} < +\infty,$$

$$\lim_{s\to+\infty} \frac{\phi_{0,s}}{\int_{t_{s-1}}^{t_s} \frac{du}{\rho(u)}} = \lim_{s\to+\infty} \frac{\phi_{0,s}}{\int_{t_s}^{t_s+1} \frac{du}{\rho(u)}} = \lim_{s\to+\infty} \frac{\psi_{0,s}}{\int_{t_s}^{t_s+1} \frac{du}{\varrho(u)}} = \lim_{s\to+\infty} \frac{\psi_{0,s}}{\int_{t_s}^{t_s+1} \frac{du}{\varrho(u)}} = 0$$

and

$$|f(t, \sigma(t)u, \tau(t)v)| \le \phi(t)M_f, u, v \in \mathbb{R}, a.e.\ t \in \mathbb{R},$$

$$|g(t, \sigma(t)u, \tau(t)v)| \le \psi(t)M_g, u, v \in \mathbb{R}, a.e.\ t \in \mathbb{R},$$

$$|I_0(t_s, \sigma(t_s)u, \tau(t_s)v)| \le \phi_{0,s}M_{I0}, u, v \in \mathbb{R}, s \in \mathbf{Z},$$

$$|I_1(t_s, \sigma(t_s)u, \tau(t_s)v)| \le \phi_{1,s}M_{I1}, u, v \in \mathbb{R}, s \in \mathbf{Z},$$

$$|J_0(t_s, \sigma(t_s)u, \tau(t_s)v)| \le \psi_{0,s}M_{J0}, u, v \in \mathbb{R}, s \in \mathbf{Z}$$

$$|J_1(t_s, \sigma(t_s)u, \tau(t_s)v)| \le \psi_{1,s}M_{J1}, u, v \in \mathbb{R}, s \in \mathbf{Z}.$$

Theorem 3.4 *Suppose that (C) holds. Then BVP (1.4) has at least one solution.*

Proof In Theorem 3.1, choose $F(u,v)=\Phi^{-1}(M_f), G(u,v)=\Psi^{-1}(M_g)$, $\overline{I_0}(u,v)=M_{I0}$, $\overline{J_0}(u,v)=M_{J0}$, $\overline{I_1}(u,v)=\Phi^{-1}(M_{I1})$, and $\overline{J_1}(u,v)=\Psi^{-1}(M_{J1})$. Then (C) implies that (A) holds. It is easy to see that

$$\sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\phi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\overline{I_0}(r,r)+\sup_{s\in\mathbf{Z}}\frac{\sum\limits_{j=-\infty}^{s}\psi_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\varrho(u)}}\overline{J_0}(r,r)$$
$$+\Phi^{-1}\left(||\phi||_1\Phi\left(F(r,r)\right)+\sum_{s\in\mathbf{Z}}\phi_{1,s}\Phi\left(\overline{I_1}(r,r)\right)\right)$$
$$+\Psi^{-1}\left(||\psi||_1\Psi\left(G(r,r)\right)+\sum_{s\in\mathbf{Z}}\psi_{1,s}\Psi\left(\overline{J_1}(r,r)\right)\right)\le r$$

has positive solution r_0. Hence Theorem 3.1 implies that BVP (1.4) has at least one solution. □

4. Examples

In this section, we present some examples to illustrate the main theorems.

Example 4.1 *Consider the following problem*

$$\begin{aligned}&[(e^{-t}x'(t))^3]'+e^{-t^2}\left[a\sin\left(e^tx(t)\right)+b\cos\left(e^{2t}y(t)\right)\right]^3=0,\quad a.e.\ t\in\mathbb{R},\\&[(e^{-2t}y'(t))^5]'+e^{-t^2}\left[c\sin\left(e^tx(t)\right)+d\cos\left(e^{2t}y(t)\right)\right]^5=0,\quad a.e.\ t\in\mathbb{R}\\&\lim_{t\to-\infty}x(s)=0,\ \ \lim_{t\to+\infty}e^tx'(t)=0,\ \ \lim_{t\to-\infty}y(s)=0,\ \ \lim_{t\to+\infty}e^{2t}y'(t)=0,\\&\Delta x(s)=\phi_{0,s},\ \Delta\Phi(e^{-s}x'(s))=\phi_{1,s},\ \Delta y(s)=\psi_{0,s},\ \Delta\Psi(e^{-2s}y'(s))=\psi_{1,s},s\in\mathbb{Z},\end{aligned}\tag{4.1}$$

where $a,b,c,d\in\mathbb{R}$, $\phi_{0,s},\phi_{1,s},\psi_{0,s},\psi_{1,s}\ge 0$ *satisfy*

$$\sum_{s\in\mathbf{Z}}\phi_{1,s}<+\infty,\ \sum_{s\in\mathbf{Z}}\psi_{1,s}<+\infty,\ \sum_{j=-\infty}^{s}\phi_{0,s}<+\infty,\ \sum_{j=-\infty}^{s}\psi_{0,s}<+\infty,$$
$$\lim_{s\to+\infty}\frac{\phi_{0,s}}{e^s-e^{s-1}}=\lim_{s\to+\infty}\frac{\phi_{0,s}}{e^{s+1}-e^s}=\lim_{s\to+\infty}\frac{\psi_{0,s}}{e^{2s}-e^{2(s-1)}}=\lim_{s\to+\infty}\frac{\psi_{0,s}}{e^{2(s+1)}-e^{2s}}=0.$$

By Theorem 3.3, BVP (4.1) has at least one solution. □

Example 4.2 *Consider the following problem*

$$
\begin{aligned}
&[(e^{-t}x'(t))^3]' + e^{-t^2}\left(A_{1,0} + B_{1,1}\left(\tfrac{x(t)}{\sigma(t)}\right)^{\tau_{1,1}}\left(\tfrac{y(t)}{\tau(t)}\right)^{\sigma_{1,1}}\right)^3 = 0, \quad a.e.\ t\in\mathbb{R},\\
&[(e^{-2t}y'(t))^5]' + e^{-t^2}\left(A_{2,0} + B_{2,1}\left(\tfrac{x(t)}{\sigma(t)}\right)^{\tau_{2,1}}\left(\tfrac{y(t)}{\tau(t)}\right)^{\sigma_{2,1}}\right)^5 = 0, \quad a.e.\ t\in\mathbb{R}\\
&\lim_{t\to-\infty} x(s)=0,\ \ \lim_{t\to+\infty} e^{t}x'(t)=0,\ \ \lim_{t\to-\infty} y(s)=0,\ \ \lim_{t\to+\infty} e^{2t}y'(t)=0,\\
&\Delta x(s) = \phi_{0,s}\left[a_{0,0} + b_{0,1}\left(\tfrac{x(s)}{\sigma(s)}\right)^{\tau_{1,1}}\left(\tfrac{y(s)}{\tau(s)}\right)^{\sigma_{1,1}}\right], s\in\mathbb{Z},\\
&\Delta\Phi(e^{-s}x'(s)) = \phi_{1,s}\left(a_{1,0} + b_{1,1}\left(\tfrac{x(s)}{\sigma(s)}\right)^{\tau_{1,1}}\left(\tfrac{y(s)}{\tau(s)}\right)^{\sigma_{1,1}}\right)^3, s\in\mathbb{Z},\\
&\Delta y(s) = \psi_{0,s}\left[c_{0,0} + d_{0,1}\left(\tfrac{x(s)}{\sigma(s)}\right)^{\tau_{2,1}}\left(\tfrac{y(s)}{\tau(s)}\right)^{\sigma_{2,1}}\right], s\in\mathbb{Z},\\
&\Delta\Psi(e^{-2s}y'(s)) = \psi_{1,s}\left(c_{1,0} + d_{1,1}\left(\tfrac{x(s)}{\sigma(s)}\right)^{\tau_{2,1}}\left(\tfrac{y(s)}{\tau(s)}\right)^{\sigma_{2,1}}\right)^5, s\in\mathbb{Z},
\end{aligned}
\tag{4.2}
$$

where $\Phi(x)=|x|^2x,\ \Psi(x)=|x|^4x,\ \sigma(t)=e^t,\ \tau(t)=\frac{1}{2}e^{2t}$ *and*

$$\phi_{1,s}=2^{-|s|},\ \psi_{1,s}=3^{-|s|},\ s\in\mathbb{Z},\quad \phi_{0,s}=2^s,\ \psi_{0,s}=3^s,\ s\in\mathbb{Z},\quad \mu=\max\{\tau_{1,1}+\sigma_{1,1},\ \tau_{2,1}+\sigma_{2,1}\}.$$

Then (4.2) has at least one solution if

(i) $\mu\in(0,1)$ *or*

(ii) $\mu=1$ *and* $2b_{0,1}+\sqrt[6]{\pi}B_{1,1}+\sqrt[3]{3}b_{1,1}+3d_{0,1}+\sqrt[10]{\pi}B_{2,1}+\sqrt[5]{2}d_{1,1}<1$ *or*

(iii) $\mu>1$ *and* $[2b_{0,1}+\sqrt[6]{\pi}B_{1,1}+\sqrt[3]{3}b_{1,1}+3d_{0,1}+\sqrt[10]{\pi}B_{2,1}+\sqrt[5]{2}d_{1,1}][2a_{0,0}+\sqrt[6]{\pi}A_{1,0}+\sqrt[3]{3}a_{1,0}+3c_{0,0}+\sqrt[10]{\pi}A_{2,0}+\sqrt[5]{2}c_{1,0}+2b_{0,1}+\sqrt[6]{\pi}B_{1,1}+\sqrt[3]{3}b_{1,1}+3d_{0,1}+\sqrt[10]{\pi}B_{2,1}+\sqrt[5]{2}d_{1,1}+]^{\mu-1}\le\frac{(\mu-1)^{\mu-1}}{\mu^\mu}$.

Proof Corresponding to BVP (1.4), we have $\rho(t)=e^{-t}$ and $\varrho(t)=e^{-2t}$. Hence (1.5) holds. $\Phi(x)=x^3$ and $\Psi(x)=x^5$ with $p_1=4$ and $p_2=6$, then $q_1=\frac{4}{3}$ and $q_2=\frac{6}{5}$, $t_s=s(s\in\mathbb{Z})$ and

$$
\begin{aligned}
f(t,u,v) &= e^{-t^2}\left(A_{1,0}+B_{1,1}\left(\tfrac{u}{\sigma(t)}\right)^{\tau_{1,1}}\left(\tfrac{v}{\tau(t)}\right)^{\sigma_{1,1}}\right)^3,\\
g(t,u,v) &= e^{-t^2}\left(A_{2,0}+B_{2,1}\left(\tfrac{u}{\sigma(t)}\right)^{\tau_{2,1}}\left(\tfrac{v}{\tau(t)}\right)^{\sigma_{2,1}}\right)^5,\\
I_0(t_s,u,v) &= \phi_{0,s}\left[a_{0,0}+b_{0,1}\left(\tfrac{u}{\sigma(t_s)}\right)^{\tau_{1,1}}\left(\tfrac{v}{\tau(t_s)}\right)^{\sigma_{1,1}}\right],\\
J_0(t_s,u,v) &= \psi_{0,s}\left[c_{0,0}+d_{0,1}\left(\tfrac{u}{\sigma(t_s)}\right)^{\tau_{2,1}}\left(\tfrac{v}{\tau(t_s)}\right)^{\sigma_{2,1}}\right],\\
I_1(t_s,u,v) &= \phi_{1,s}\left(a_{1,0}+b_{1,1}\left(\tfrac{u}{\sigma(t_s)}\right)^{\tau_{1,1}}\left(\tfrac{v}{\tau(t_s)}\right)^{\sigma_{1,1}}\right)^3,\\
J_1(t_s,u,v) &= \psi_{1,s}\left(c_{1,0}+d_{1,1}\left(\tfrac{u}{\sigma(t_s)}\right)^{\tau_{2,1}}\left(\tfrac{v}{\tau(t_s)}\right)^{\sigma_{2,1}}\right)^5.
\end{aligned}
$$

It is easy to see that

$$|f(t,\sigma(t)u,\tau(t)v)| \le e^{-t^2}\left(A_{1,0}+B_{1,1}|u|^{\tau_{1,1}}|v|^{\sigma_{1,1}}\right)^3, u,v\in\mathbb{R}, a.e.\ t\in\mathbb{R},$$

$$|g(t,\sigma(t)u,\tau(t)v)| \le e^{-t^2}\left(A_{2,0}+B_{2,1}|u|^{\tau_{2,1}}|v|^{\sigma_{2,1}}\right)^5, u,v\in\mathbb{R}, a.e.\ t\in\mathbb{R},$$

$$|I_0(t_s,\sigma(t_s)u,\tau(t_s)v)| \le \phi_{0,s}\left[a_{0,0}+b_{0,1}|u|^{\tau_{1,1}}|v|^{\sigma_{1,1}}\right], u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$|I_1(t_s,\sigma(t_s)u,\tau(t_s)v)| \le \phi_{1,s}\left(a_{1,0}+b_{1,1}|u|^{\tau_{1,1}}|v|^{\sigma_{1,1}}\right)^3, u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$|J_0(t_s,\sigma(t_s)u,\tau(t_s)v)| \le \psi_{0,s}\left[c_{0,0}+d_{0,1}|u|^{\tau_{2,1}}|v|^{\sigma_{2,1}}\right], u,v\in\mathbb{R}, s\in\mathbb{Z}$$

$$|J_1(t_s,\sigma(t_s)u,\tau(t_s)v)| \le \psi_{1,s}\left(c_{1,0}+d_{1,1}|u|^{\tau_{2,1}}|v|^{\sigma_{2,1}}\right)^5, u,v\in\mathbb{R}, s\in\mathbb{Z}.$$

Then

$$\sum_{s\in\mathbb{Z}}\phi_{1,s}=3,\quad \sum_{s\in\mathbb{Z}}\psi_{1,s}=2,$$

$$\sum_{j=-\infty}^{s}\phi_{0,s}=2^{s+1},\quad \sum_{j=-\infty}^{s}\psi_{0,s}=\frac{3^{s+1}}{2},$$

$$1+\int_{-\infty}^{s}\frac{du}{\rho(u)}=1+e^s,\quad 1+\int_{-\infty}^{s}\frac{du}{\varrho(u)}=1+\frac{1}{2}e^{2s},$$

$$\lim_{s\to+\infty}\frac{\phi_{0,s}}{\int_{t_{s-1}}^{t_s}\frac{du}{\rho(u)}}=\lim_{s\to+\infty}\frac{\phi_{0,s}}{\int_{t_s}^{t_s+1}\frac{du}{\rho(u)}}=\lim_{s\to+\infty}\frac{\psi_{0,s}}{\int_{t_{s-1}}^{t_s}\frac{du}{\varrho(u)}}=\lim_{s\to+\infty}\frac{\psi_{0,s}}{\int_{t_s}^{t_s+1}\frac{du}{\varrho(u)}}=0.$$

By direct computation, we get

$$\prod\nolimits_1=\sup_{s\in\mathbb{Z}}\frac{2^{s+1}}{1+e^s}<2,\quad \prod\nolimits_2=\sup_{s\in\mathbb{Z}}\frac{\frac{3^{s+1}}{2}}{1+\frac{1}{2}e^{2s}}=\sup_{s\in\mathbb{Z}}\frac{3^{s+1}}{2+e^{2s}}<3,$$

$$P_1=\prod\nolimits_1 a_{0,0}+\sqrt[6]{\pi}A_{1,0}+\sqrt[3]{3}a_{1,0}<2a_{0,0}+\sqrt[6]{\pi}A_{1,0}+\sqrt[3]{3}a_{1,0},$$

$$Q_1^1=\prod\nolimits_1 b_{0,1}+\sqrt[6]{\pi}B_{1,1}+\sqrt[3]{3}b_{1,1}<2b_{0,1}+\sqrt[6]{\pi}B_{1,1}+\sqrt[3]{3}b_{1,1},$$

$$P_2=\prod\nolimits_2 c_{0,0}+\sqrt[10]{\pi}A_{2,0}+\sqrt[5]{2}c_{1,0}<3c_{0,0}+\sqrt[10]{\pi}A_{2,0}+\sqrt[5]{2}c_{1,0},$$

$$Q_1^2=\prod\nolimits_2 d_{0,1}+\sqrt[10]{\pi}B_{2,1}+\sqrt[5]{2}d_{1,1}<3d_{0,1}+\sqrt[10]{\pi}B_{2,1}+\sqrt[5]{2}d_{1,1},$$

$$A=\max\{P_1,\ P_2\}<2a_{0,0}+\sqrt[6]{\pi}A_{1,0}+\sqrt[3]{3}a_{1,0}+3c_{0,0}+\sqrt[10]{\pi}A_{2,0}+\sqrt[5]{2}c_{1,0},$$

$$B=\max\left\{Q_1^1,\ Q_1^2\right\}<2b_{0,1}+\sqrt[6]{\pi}B_{1,1}+\sqrt[3]{3}b_{1,1}+3d_{0,1}+\sqrt[10]{\pi}B_{2,1}+\sqrt[5]{2}d_{1,1}.$$

By Theorem 3.2, we know that (4.2) has at least one solution if (i) or (ii) or (iii) in Example 4.2 holds. The proof is complete. □

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