

Generalized convolution product for an integral transform on a Wiener space

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Abstract: We introduce a generalized convolution product $(F * G)_{\bar{\alpha}, \bar{\beta}}$ for integral transform $\mathcal{F}_{\gamma, \eta}$ for functionals defined on $K[0, T]$, the space of complex valued continuous functions on $[0, T]$ that vanish at zero. We study some interesting properties of our generalized convolution product and establish various relationships that exist among the generalized convolution product, the integral transform, and the first variation for functionals defined on $K[0, T]$. We also discuss the associativity of the generalized convolution product.

Key words: Wiener integral, integral transform, generalized convolution product, first variation

1. Introduction

In a unifying paper [14], Lee defined an integral transform $\mathcal{F}_{\gamma, \eta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters γ and η and for certain classes of functionals, the Fourier–Wiener transform [1], the Fourier–Feynman transform [2, 8], and the Gauss transform [10] are special cases of his integral transform $\mathcal{F}_{\gamma, \eta}$.

In [5, 11], Chang et al. established interesting relationships that exist among the integral transform, the convolution product, and the first variation. In this paper, we introduce a generalized convolution product and study some interesting properties of this convolution product. Our convolution product generalizes the convolution product defined and studied in [3, 5, 11].

Let $C_0[0, T]$ denote a one-parameter Wiener space, that is, the space of all real valued continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$. Let \mathcal{M} denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let m denote the Wiener measure. Then $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0, T]} F(x) dm(x).$$

Let $K[0, T]$ be the space of complex valued continuous functions defined on $[0, T]$ that vanish at $t = 0$. Let $\{\theta_1, \theta_2, \dots\}$ be a complete orthonormal set of real valued functions in $L_2[0, T]$. Furthermore, assume that

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each θ_i is of bounded variation on $[0, T]$. Then for each $y \in K[0, T]$ and $i = 1, 2, \dots$, the Riemann–Stieltjes integral $\langle \theta_i, y \rangle = \int_0^T \theta_i(t) dy(t)$ exists [11].

Next we describe the class of functionals that we work with in this paper. For $\sigma \in [0, 1)$, let E_σ be the space of all functionals $F : K[0, T] \rightarrow \mathbb{C}$ of the form

$$F(y) = f(\langle \vec{\theta}, y \rangle) = f(\langle \theta_1, y \rangle, \dots, \langle \theta_n, y \rangle) \tag{1.1}$$

for some positive integer n , where $f(\vec{\lambda}) = f(\lambda_1, \dots, \lambda_n)$ is an entire function of the n complex variables $\lambda_1, \dots, \lambda_n$ such that

$$|f(\lambda_1, \dots, \lambda_n)| \leq A_F \exp\left\{B_F \sum_{i=1}^n |\lambda_i|^{1+\sigma}\right\} \tag{1.2}$$

for some positive constants A_F and B_F . Note that if $\sigma = 0$, the space E_σ reduces to E_0 , which was introduced and used in [3, 11, 17]. Moreover, if $0 < \sigma_1 < \sigma_2 < 1$, we have $E_0 \subsetneq E_{\sigma_1} \subsetneq E_{\sigma_2} \subsetneq L_2(C_0[0, T])$.

For any F and G in E_σ , we can always express F by (1.1) and G by

$$G(y) = g(\langle \vec{\theta}, y \rangle) = g(\langle \theta_1, y \rangle, \dots, \langle \theta_n, y \rangle) \tag{1.3}$$

using the same positive integer n , where f and g are entire functions of exponential type. For details, see Remark 1.4 of [11].

E_σ is a very natural class of functionals in which to study the relationships that exist among the integral transform, the generalized convolution product, and the first variation. In addition, E_σ is a very rich class of functionals. For appropriate θ and ϕ , the functionals

$$\exp\left\{\int_0^T \theta(t, x(t)) dt\right\}, \quad \exp\left\{\int_0^T \theta(t, x(t)) dt\right\}\phi(x(T))$$

are elements of E_σ . These functionals are of interest in Feynman integration theories and quantum mechanics. For details on the usefulness of the class E_σ , see Sections 1 and 5 of [11].

In this paper we introduce a generalized convolution product $(F * G)_{\vec{\alpha}, \vec{\beta}}$ for integral transform $\mathcal{F}_{\gamma, \eta}$ for functionals defined on $K[0, T]$. In Section 2, we study some interesting properties of the generalized convolution product. Theorem 2.5 gives necessary and sufficient conditions for the generalized convolution product to be commutative, from which we conclude that the convolution product studied in [3, 11] is also commutative.

In Section 3, we establish various relationships between the generalized convolution product and the integral transform. We also study an associativity result of the generalized convolution product in Theorem 3.9. Although the associativity of the convolution product for the Fourier–Feynman transform was studied in [4, 9], the associativity for the convolution product has not yet been established. In Section 4, we establish various relationships between the generalized convolution product and the first variation, while in Section 5, we obtain relationships involving the integral transform, the generalized convolution product, and the first variation where each concept is used exactly once.

Relevant studies to this paper are [6, 10]. Im et al. [10] introduced a convolution product $F_1 *_{A, B, C, D} F_2$ for Fourier–Gauss transforms, and studied relationships among the first variation, the convolutions, and the Fourier–Gauss transforms. Chang et al. [6] introduced and studied a modified convolution product $(\Psi * \Phi)_{ABCD}$

for generalized exponential type functionals of the form

$$\Psi(x) = \exp\left\{(w, x)^\sim - \frac{1}{2}\|w\|_{C'_{a,b}}^2 - (w, a)_{C'_{a,b}}\right\}$$

for $w \in C'_{a,b}[0, T]$ and $x \in C_{a,b}[0, T]$. For the function space $C_{a,b}[0, T]$ induced by generalized Brownian motion, see [6] and the references therein.

2. Generalized convolution product of functionals in E_σ

In this section we introduce a generalized convolution product and study necessary and sufficient conditions for the generalized convolution product to be commutative. Let $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$, where α_i and β_i for $i = 1, 2$ are complex numbers. We begin with defining a generalized convolution product for functionals defined on $K[0, T]$.

Definition 2.1 *Let F and G be functionals defined on $K[0, T]$. Then a generalized convolution product $(F * G)_{\vec{\alpha}, \vec{\beta}}$ of F and G is defined by*

$$(F * G)_{\vec{\alpha}, \vec{\beta}}(y) = \int_{C_0[0, T]} F(\alpha_1 x + \beta_1 y) G(\alpha_2 x + \beta_2 y) dm(x), \quad y \in K[0, T] \tag{2.1}$$

if it exists.

Remark 2.2 (i) *If $\vec{\alpha} = (\alpha/\sqrt{2}, -\alpha/\sqrt{2})$ for a nonzero complex number α and $\vec{\beta} = (1/\sqrt{2}, 1/\sqrt{2})$, then the generalized convolution product (2.1) reduces to the convolution product $(F * G)_\alpha$ in the integral transform defined and studied in [3, 11].*

(ii) *If $\vec{\alpha} = (1/\sqrt{2}, 1/\sqrt{2})$ and $\vec{\beta} = (1/\sqrt{2}, -1/\sqrt{2})$, then the generalized convolution product (2.1) reduces to the convolution product $F * G$ in the Fourier–Wiener transform studied in [17, 18].*

(iii) *Our generalized convolution product satisfies the additive distribution properties, that is,*

$$((F + G) * H)_{\vec{\alpha}, \vec{\beta}}(y) = (F * H)_{\vec{\alpha}, \vec{\beta}}(y) + (G * H)_{\vec{\alpha}, \vec{\beta}}(y) \tag{2.2}$$

and

$$(F * (G + H))_{\vec{\alpha}, \vec{\beta}}(y) = (F * G)_{\vec{\alpha}, \vec{\beta}}(y) + (F * H)_{\vec{\alpha}, \vec{\beta}}(y) \tag{2.3}$$

for $y \in K[0, T]$.

(iv) *If $\vec{\alpha} = (0, 0)$ and $\vec{\beta} = (1, 1)$, then from the definition (2.1), it is easy to see that*

$$(F * G)_{\vec{\alpha}, \vec{\beta}}(y) = F(y)G(y) \tag{2.4}$$

for $y \in K[0, T]$.

After Lee defined the integral transform $\mathcal{F}_{\gamma, \eta}$ as a generalization of the Fourier–Wiener transform [14], Chang and the authors defined the convolution product $(F * G)_\gamma$ so that the relationship (3.8) in Section 3

is satisfied [3]. That is, to satisfy (3.8), the convolution product $(F * G)_\gamma$ should be defined using the same parameter γ in $\mathcal{F}_{\gamma,\eta}$. Our generalized convolution product $(F * G)_{\bar{\alpha},\bar{\beta}}$ does not depend on the parameter γ and η in $\mathcal{F}_{\gamma,\eta}$. Moreover, we find a sufficient condition to satisfy (3.5) in Theorem 3.3 below, which is a generalization of (3.8).

Now we discuss the existence and commutativity of our generalized convolution product for functional in E_σ . In our first theorem, we show that if F and G are elements of E_σ , then the generalized convolution product $(F * G)_{\bar{\alpha},\bar{\beta}}$ exists and is an element of E_σ .

Theorem 2.3 *Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then the generalized convolution product $(F * G)_{\bar{\alpha},\bar{\beta}}$ exists, belongs to E_σ , and is given by the formula*

$$(F * G)_{\bar{\alpha},\bar{\beta}}(y) = k(\langle \vec{\theta}, y \rangle) \tag{2.5}$$

for $y \in K[0, T]$, where

$$k(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha_1 \vec{u} + \beta_1 \vec{\lambda}) g(\alpha_2 \vec{u} + \beta_2 \vec{\lambda}) \exp\left\{-\frac{1}{2} \|\vec{u}\|^2\right\} d\vec{u}, \tag{2.6}$$

where $\|\vec{u}\|^2 = \sum_{i=1}^n u_i^2$ and $d\vec{u} = du_1 \cdots du_n$.

Proof For each $y \in K[0, T]$, using the well-known Wiener integration formula we obtain

$$\begin{aligned} (F * G)_{\bar{\alpha},\bar{\beta}}(y) &= \int_{C_0[0,T]} f(\alpha_1 \langle \vec{\theta}, x \rangle + \beta_1 \langle \vec{\theta}, y \rangle) g(\alpha_2 \langle \vec{\theta}, x \rangle + \beta_2 \langle \vec{\theta}, y \rangle) dm(x) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha_1 \vec{u} + \beta_1 \langle \vec{\theta}, y \rangle) g(\alpha_2 \vec{u} + \beta_2 \langle \vec{\theta}, y \rangle) \exp\left\{-\frac{1}{2} \|\vec{u}\|^2\right\} d\vec{u} \\ &= k(\langle \vec{\theta}, y \rangle), \end{aligned}$$

where k is given by (2.6). By Theorem 3.15 of [7], $k(\vec{\lambda})$ is an entire function. Since

$$|\alpha_1 u_i + \beta_1 \lambda_i|^{1+\sigma} \leq |2\alpha_1 u_i|^{1+\sigma} + |2\beta_1 \lambda_i|^{1+\sigma},$$

we have

$$|f(\alpha_1 \vec{u} + \beta_1 \vec{\lambda})| \leq A_F \exp\left\{B_F |2\alpha_1|^{1+\sigma} \sum_{i=1}^n |u_i|^{1+\sigma} + B_F |2\beta_1|^{1+\sigma} \sum_{i=1}^n |\lambda_i|^{1+\sigma}\right\}.$$

Of course we have a similar inequality for g . Hence we have

$$|k(\vec{\lambda})| \leq A_{(F * G)_{\bar{\alpha},\bar{\beta}}} \exp\left\{B_{(F * G)_{\bar{\alpha},\bar{\beta}}} \sum_{i=1}^n |\lambda_i|^{1+\sigma}\right\},$$

where

$$A_{(F * G)_{\bar{\alpha},\bar{\beta}}} = A_F A_G \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{(B_F |2\alpha_1|^{1+\sigma} + B_G |2\alpha_2|^{1+\sigma})|u|^{1+\sigma} - \frac{1}{2} u^2\right\} du\right)^n$$

which is finite since $0 \leq \sigma < 1$, and

$$B_{(F * G)_{\bar{\alpha},\bar{\beta}}} = B_F |2\beta_1|^{1+\sigma} + B_G |2\beta_2|^{1+\sigma} < \infty.$$

Hence $(F * G)_{\bar{\alpha},\bar{\beta}}$ belongs to E_σ . □

Chang et al. [6] and Im et al. [10] studied convolution products defined using bounded linear operators. Hence our convolution product defined using complex numbers can be viewed, in some sense, as a special case of the convolution products in [6, 10]. However, they considered functionals in a dense subset of a fundamental set in $L_2(C_{a,b}[0, T])$ (Theorem 4.4 in [6]), or studied the existence of the convolution product and some relationships (Theorem 4.6 and Corollaries 4.9, 4.10 in [10]). Our results show not only the existence but also the explicit expressions for our convolution product (Theorem 2.3) and some relationships (2nd displayed equation in Theorem 3.3 etc.).

The convolution product $(F * G)_\alpha$ in the integral transform defined and studied in [3, 11] is commutative, while the convolution product $F * G$ in the Fourier–Wiener transform defined and studied in [17, 18] is not commutative. In general, our generalized convolution product is not commutative, that is, it is not always true that $(F * G)_{\bar{\alpha}, \bar{\beta}}(y) = (G * F)_{\bar{\alpha}, \bar{\beta}}(y)$ as in the following example.

Example 2.4 Let $F(x) = \langle \theta_1, x \rangle$ and let $G \equiv 1$ be the constant function for $x \in K[0, T]$. Then

$$(F * G)_{\bar{\alpha}, \bar{\beta}}(y) = \int_{C_0[0, T]} (\alpha_1 \langle \theta_1, x \rangle + \beta_1 \langle \theta_1, y \rangle) dm(x) = \beta_1 \langle \theta_1, y \rangle$$

and

$$(G * F)_{\bar{\alpha}, \bar{\beta}}(y) = \int_{C_0[0, T]} (\alpha_2 \langle \theta_1, x \rangle + \beta_2 \langle \theta_1, y \rangle) dm(x) = \beta_2 \langle \theta_1, y \rangle$$

for all $y \in K[0, T]$. Hence $(F * G)_{\bar{\alpha}, \bar{\beta}}(y) \neq (G * F)_{\bar{\alpha}, \bar{\beta}}(y)$ unless $\beta_1 = \beta_2$.

The following theorem gives necessary and sufficient conditions for the generalized convolution product on E_σ to be commutative.

Theorem 2.5 $(F * G)_{\bar{\alpha}, \bar{\beta}} = (G * F)_{\bar{\alpha}, \bar{\beta}}$ for all F and G in E_σ if and only if $\alpha_1^2 = \alpha_2^2$ and $\beta_1 = \beta_2$.

Proof Considering the definition (2.1) of the generalized convolution product, it is easy to see that $(F * G)_{\bar{\alpha}, \bar{\beta}} = (G * F)_{\bar{\alpha}, \bar{\beta}}$ for all F and G in E_σ if $\alpha_1^2 = \alpha_2^2$ and $\beta_1 = \beta_2$. Now let us prove the converse, that is, we shall show that if $\alpha_1^2 \neq \alpha_2^2$ or $\beta_1 \neq \beta_2$, then $(F * G)_{\bar{\alpha}, \bar{\beta}} \neq (G * F)_{\bar{\alpha}, \bar{\beta}}$ for some F and G in E_σ . First assume that $\beta_1 \neq \beta_2$. If we take $F(x) = \langle \theta_1, x \rangle$ and $G(x) = 1$ as in Example 2.4, then we know that $(F * G)_{\bar{\alpha}, \bar{\beta}}$ does not equal $(G * F)_{\bar{\alpha}, \bar{\beta}}$ in this case. Next assume that $\beta_1 = \beta_2$ and $\alpha_1^2 \neq \alpha_2^2$. If we take $F(x) = \langle \theta_1, x \rangle^2$ and $G(x) = 1$, then

$$(F * G)_{\bar{\alpha}, \bar{\beta}}(y) = \int_{C_0[0, T]} (\alpha_1 \langle \theta_1, x \rangle + \beta_1 \langle \theta_1, y \rangle)^2 dm(x) = \alpha_1^2 + \beta_1^2 \langle \theta_1, y \rangle^2$$

and

$$(G * F)_{\bar{\alpha}, \bar{\beta}}(y) = \int_{C_0[0, T]} (\alpha_2 \langle \theta_1, x \rangle + \beta_2 \langle \theta_1, y \rangle)^2 dm(x) = \alpha_2^2 + \beta_2^2 \langle \theta_1, y \rangle^2.$$

Since $\beta_1 = \beta_2$ and $\alpha_1^2 \neq \alpha_2^2$, $(F * G)_{\bar{\alpha}, \bar{\beta}}$ does not equal $(G * F)_{\bar{\alpha}, \bar{\beta}}$ in this case, and this completes the proof. □

If $\vec{\alpha} = (\alpha/\sqrt{2}, -\alpha/\sqrt{2})$ and $\vec{\beta} = (1/\sqrt{2}, 1/\sqrt{2})$ for some nonzero complex number α , then $\alpha_1^2 = \alpha_2^2$ and $\beta_1 = \beta_2$. In this case our generalized convolution product $(F * G)_{\vec{\alpha}, \vec{\beta}}$ reduces to the convolution product $(F * G)_\alpha$ in the integral transform as we commented in Remark 2.2. Hence we obtain the following corollary.

Corollary 2.6 *The convolution product $(F * G)_\alpha$ is commutative for any nonzero complex number α .*

3. Generalized convolution product and integral transform

In this section we establish several results involving our generalized convolution product and integral transform. We begin with introducing the definition of integral transform of functionals defined on $K[0, T]$. Let γ and η be complex numbers in the rest of this paper.

Definition 3.1 *Let F be a functional defined on $K[0, T]$. Then the integral transform $\mathcal{F}_{\gamma, \eta}F$ of F is defined by*

$$\mathcal{F}_{\gamma, \eta}F(y) = \int_{C_0[0, T]} F(\gamma x + \eta y) dm(x), \quad y \in K[0, T] \tag{3.1}$$

if it exists [3, 5, 11, 12, 14].

The following theorem shows that if F is an element of E_σ , then the integral transform of F exists and is an element of E_σ . For the proof, see Theorem 2.1 and Remark 5.6 of [11].

Theorem 3.2 *Let $F \in E_\sigma$ be given by (1.1). Then the integral transform $\mathcal{F}_{\gamma, \eta}F$ exists, belongs to E_σ , and is given by the formula*

$$\mathcal{F}_{\gamma, \eta}F(y) = f_t(\langle \vec{\theta}, y \rangle) \tag{3.2}$$

for $y \in K[0, T]$, where

$$f_t(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\gamma \vec{u} + \eta \vec{\lambda}) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u}. \tag{3.3}$$

Now let us investigate relationships involving an integral transform and our generalized convolution product.

The most interesting feature of the generalized convolution product is that if $\vec{\alpha}$ and $\vec{\beta}$ satisfy some conditions, then the integral transform of the generalized convolution product of two functionals is equal to the product of integral transforms of each functional.

Our formula (3.5) below is useful because it permits one to calculate $\mathcal{F}_{\gamma, \eta}(F * G)_{\vec{\alpha}, \vec{\beta}}(y)$ without ever actually calculating $(F * G)_{\vec{\alpha}, \vec{\beta}}$. In general, a convolution product is more complicated to calculate than the integral transforms.

Theorem 3.3 *Suppose that $\vec{\alpha}$ and $\vec{\beta}$ satisfy the condition*

$$\gamma = \frac{\alpha_1}{\beta_1} = -\frac{\alpha_2}{\beta_2}. \tag{3.4}$$

Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then we have

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) &= \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1}F(y)\mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta\beta_2}G(y) \\ &= \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta}F(\beta_1y)\mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta}G(\beta_2y) \end{aligned} \tag{3.5}$$

for $y \in K[0, T]$.

Proof By (2.5), (3.2), and (3.3), we have

$$\mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} k(\gamma\vec{u} + \eta\langle\vec{\theta}, y\rangle) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u},$$

where k is given by (2.6). Then

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} f(\alpha_1\vec{v} + \beta_1\gamma\vec{u} + \beta_1\eta\langle\vec{\theta}, y\rangle) \\ &\quad g(\alpha_2\vec{v} + \beta_2\gamma\vec{u} + \beta_2\eta\langle\vec{\theta}, y\rangle) \exp\left\{-\frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2)\right\} d\vec{u} d\vec{v}. \end{aligned}$$

Since $\alpha_1 = \gamma\beta_1$ and $\alpha_2 = -\gamma\beta_2$, we have

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} f\left(\sqrt{2}\gamma\beta_1\frac{\vec{v} + \vec{u}}{\sqrt{2}} + \eta\beta_1\langle\vec{\theta}, y\rangle\right) \\ &\quad g\left(\sqrt{2}\gamma\beta_2\frac{\vec{u} - \vec{v}}{\sqrt{2}} + \eta\beta_2\langle\vec{\theta}, y\rangle\right) \exp\left\{-\frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2)\right\} d\vec{u} d\vec{v}. \end{aligned}$$

Now letting $\vec{p} = \frac{\vec{v} + \vec{u}}{\sqrt{2}}$ and $\vec{q} = \frac{\vec{u} - \vec{v}}{\sqrt{2}}$, we obtain

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} f(\sqrt{2}\gamma\beta_1\vec{p} + \eta\beta_1\langle\vec{\theta}, y\rangle) \\ &\quad g(\sqrt{2}\gamma\beta_2\vec{q} + \eta\beta_2\langle\vec{\theta}, y\rangle) \exp\left\{-\frac{1}{2}(\|\vec{p}\|^2 + \|\vec{q}\|^2)\right\} d\vec{p} d\vec{q}. \end{aligned}$$

Finally by Theorem 3.2, we know that the last expression is equal to the second expression of (3.5) as we wished to show. The second equality follows directly from the fact that

$$\eta\beta_i\langle\vec{\theta}, y\rangle = \eta\langle\vec{\theta}, \beta_iy\rangle,$$

for $i = 1, 2$ and for all $y \in K[0, T]$. □

Remark 3.4 Letting $\vec{p} = \frac{\vec{v} + \vec{u}}{\sqrt{2}}$ and $\vec{q} = \frac{\vec{v} - \vec{u}}{\sqrt{2}}$ in the proof of Theorem 3.3, we obtain

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} f(\sqrt{2}\gamma\beta_1\vec{p} + \eta\beta_1\langle\vec{\theta}, y\rangle) \\ &\quad g(-\sqrt{2}\gamma\beta_2\vec{q} + \eta\beta_2\langle\vec{\theta}, y\rangle) \exp\left\{-\frac{1}{2}(\|\vec{p}\|^2 + \|\vec{q}\|^2)\right\} d\vec{p} d\vec{q}. \end{aligned}$$

Then we have an alternative relationship

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) &= \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1}F(y)\mathcal{F}_{-\sqrt{2}\gamma\beta_2,\eta\beta_2}G(y) \\ &= \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta}F(\beta_1y)\mathcal{F}_{-\sqrt{2}\gamma\beta_2,\eta}G(\beta_2y) \end{aligned} \tag{3.6}$$

for $y \in K[0, T]$.

The following example shows that the condition (3.4) in Theorem 3.3 is not necessary to satisfy the relationship (3.5).

Example 3.5 Let $F(x) = \langle \theta_1, x \rangle$ and let $G \equiv 1$ be the constant function as in Example 2.4. Then

$$\mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1}F(y) = \int_{C_0[0,T]} (\sqrt{2}\gamma\beta_1\langle \theta_1, x \rangle + \eta\beta_1\langle \theta_1, y \rangle) dm(x) = \eta\beta_1\langle \theta_1, y \rangle$$

and

$$\mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta\beta_2}G(y) = 1$$

for all $y \in K[0, T]$. On the other hand, since we obtained

$$(F * G)_{\vec{\alpha},\vec{\beta}}(y) = \beta_1\langle \theta_1, y \rangle$$

in Example 2.4, we have

$$\mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) = \beta_1 \int_{C_0[0,T]} (\gamma\langle \theta_1, x \rangle + \eta\langle \theta_1, y \rangle) dm(x) = \eta\beta_1\langle \theta_1, y \rangle$$

for all $y \in K[0, T]$. Hence we may have

$$\mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) = \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1}F(y)\mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta\beta_2}G(y)$$

even if the condition (3.4) is not satisfied.

Corollary 3.6 Suppose that $\vec{\alpha}$ and $\vec{\beta}$ satisfy one of the following conditions.

(i) $\alpha_1 = \alpha_2 = \frac{\gamma}{\sqrt{2}}, \beta_1 = -\beta_2 = \frac{1}{\sqrt{2}}$

(ii) $\alpha_1 = -\alpha_2 = \frac{\gamma}{\sqrt{2}}, \beta_1 = \beta_2 = \frac{1}{\sqrt{2}}$.

Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then we have

$$\mathcal{F}_{\gamma,\eta}(F * G)_{\vec{\alpha},\vec{\beta}}(y) = \mathcal{F}_{\gamma,\beta_1\eta}F(y)\mathcal{F}_{\pm\gamma,\beta_2\eta}G(y) = \mathcal{F}_{\gamma,\eta}F(\beta_1y)\mathcal{F}_{\pm\gamma,\eta}G(\beta_2y) \tag{3.7}$$

for $y \in K[0, T]$.

Proof In either case, $\vec{\alpha}$ and $\vec{\beta}$ satisfy the condition (3.4). Hence we obtain the relationships by (3.5) and (3.6). □

In particular, if $\vec{\alpha} = (\alpha/\sqrt{2}, -\alpha/\sqrt{2})$ and $\vec{\beta} = (1/\sqrt{2}, 1/\sqrt{2})$ for some nonzero complex number α , then we have the following corollary proved in Formula 3.1 of [11].

Corollary 3.7 *Let α and β be complex numbers. Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then*

$$\mathcal{F}_{\alpha,\beta}(F * G)_\alpha(y) = \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y) = \mathcal{F}_{\alpha,\beta}F\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{\alpha,\beta}G\left(\frac{y}{\sqrt{2}}\right) \quad (3.8)$$

for $y \in K[0, T]$.

We close this section by considering the associativity of our generalized convolution product.

Huffman et al. [9] and Chang et al. [4] studied associativity of the convolution product for the Fourier–Feynman transform. However, Example 3.8 below shows that the generalized convolution product is not associative, that is, it is not always true that $((F * G)_{\vec{\alpha}, \vec{\beta}} * H)_{\vec{\alpha}, \vec{\beta}}(y) = (F * (G * H)_{\vec{\alpha}, \vec{\beta}})_{\vec{\alpha}, \vec{\beta}}(y)$.

Example 3.8 *Let $F(x) = \langle \theta_1, x \rangle$ and let 1 be the constant function $1(x) = 1$ for $x \in K[0, T]$. Since $(F * 1)_{\vec{\alpha}, \vec{\beta}}(y) = \beta_1 \langle \theta_1, y \rangle$ and $(1 * 1)_{\vec{\alpha}, \vec{\beta}}(y) = 1$, we have*

$$((F * 1)_{\vec{\alpha}, \vec{\beta}} * 1)_{\vec{\alpha}, \vec{\beta}}(y) = \beta_1^2 \langle \theta_1, y \rangle$$

and

$$(F * (1 * 1)_{\vec{\alpha}, \vec{\beta}})_{\vec{\alpha}, \vec{\beta}}(y) = \beta_1 \langle \theta_1, y \rangle$$

for all $y \in K[0, T]$. Hence

$$((F * 1)_{\vec{\alpha}, \vec{\beta}} * 1)_{\vec{\alpha}, \vec{\beta}}(y) \neq (F * (1 * 1)_{\vec{\alpha}, \vec{\beta}})_{\vec{\alpha}, \vec{\beta}}(y)$$

unless $\beta_1^2 = \beta_1$.

However, we have an associativity result of the generalized convolution product for the integral transform in the following sense.

Theorem 3.9 *Let $\vec{\beta}_i = (\beta_{i1}, \beta_{i2})$, $i = 1, 2, 3, 4$, satisfy the following three conditions.*

(i) $\beta_{12}\beta_{21} = \beta_{31}\beta_{42}$

(ii) $\beta_{22} = \sqrt{2}\beta_{32}\beta_{42}$

(iii) $\beta_{41} = \sqrt{2}\beta_{11}\beta_{21}$.

Let

$$\vec{\alpha}_1 = \sqrt{2}\gamma\beta_{21}(\beta_{11}, -\beta_{12}), \quad \vec{\alpha}_2 = \gamma(\beta_{21}, -\beta_{22}),$$

$$\vec{\alpha}_3 = \sqrt{2}\gamma\beta_{42}(\beta_{31}, -\beta_{32}), \quad \vec{\alpha}_4 = \gamma(\beta_{41}, -\beta_{42}).$$

Let F, G , and H in E_σ be given by (1.1) with corresponding entire functions f, g , and h , respectively. Then we have

$$\left[(F * G)_{\vec{\alpha}_1, \vec{\beta}_1}(\cdot) * H\left(\frac{\cdot}{\sqrt{2}}\right) \right]_{\vec{\alpha}_2, \vec{\beta}_2}(y) = \left[F\left(\frac{\cdot}{\sqrt{2}}\right) * (G * H)_{\vec{\alpha}_3, \vec{\beta}_3}(\cdot) \right]_{\vec{\alpha}_4, \vec{\beta}_4}(y) \quad (3.9)$$

for all $y \in K[0, T]$.

Proof For a nonzero complex number η , we know from Section 5 of [11] that $\mathcal{F}_{i\gamma/\eta, 1/\eta}$ is an inverse integral transform of $\mathcal{F}_{\gamma, \eta}$, that is,

$$\mathcal{F}_{i\gamma/\eta, 1/\eta} \mathcal{F}_{\gamma, \eta} F(y) = F(y)$$

for all $y \in K[0, T]$. Hence to establish (3.9) it is enough to show that

$$\mathcal{F}_{\gamma, \eta} \left[(F * G)_{\vec{\alpha}_1, \vec{\beta}_1}(\cdot) * H\left(\frac{\cdot}{\sqrt{2}}\right) \right]_{\vec{\alpha}_2, \vec{\beta}_2}(y) = \mathcal{F}_{\gamma, \eta} \left[F\left(\frac{\cdot}{\sqrt{2}}\right) * (G * H)_{\vec{\alpha}_3, \vec{\beta}_3}(\cdot) \right]_{\vec{\alpha}_4, \vec{\beta}_4}(y)$$

for all $y \in K[0, T]$. First consider the left-hand side of the last expression. Since $(F * G)_{\vec{\alpha}_1, \vec{\beta}_1}$ and $(G * H)_{\vec{\alpha}_3, \vec{\beta}_3}$ belong to E_σ by Theorem 2.3 and since

$$\gamma = \frac{\gamma\beta_{21}}{\beta_{21}} = -\frac{-\gamma\beta_{22}}{\beta_{22}},$$

we can apply Theorem 3.3 to obtain

$$\begin{aligned} L &\equiv \mathcal{F}_{\gamma, \eta} \left[(F * G)_{\vec{\alpha}_1, \vec{\beta}_1}(\cdot) * H\left(\frac{\cdot}{\sqrt{2}}\right) \right]_{\vec{\alpha}_2, \vec{\beta}_2}(y) \\ &= \mathcal{F}_{\sqrt{2}\gamma\beta_{21}, \eta} (F * G)_{\vec{\alpha}_1, \vec{\beta}_1}(\beta_{21}y) \mathcal{F}_{\sqrt{2}\gamma\beta_{22}, \eta} H\left(\frac{1}{\sqrt{2}}\beta_{22}y\right). \end{aligned}$$

Moreover, since

$$\sqrt{2}\gamma\beta_{21} = \frac{\sqrt{2}\gamma\beta_{11}\beta_{21}}{\beta_{11}} = -\frac{-\sqrt{2}\gamma\beta_{12}\beta_{21}}{\beta_{12}},$$

we can apply Theorem 3.3 once more to obtain

$$L = \mathcal{F}_{2\gamma\beta_{11}\beta_{21}, \eta} F(\beta_{11}\beta_{21}y) \mathcal{F}_{2\gamma\beta_{12}\beta_{21}, \eta} G(\beta_{12}\beta_{21}y) \mathcal{F}_{\sqrt{2}\gamma\beta_{22}, \eta} H\left(\frac{1}{\sqrt{2}}\beta_{22}y\right).$$

Now consider the integral transform of the right-hand side of (3.9). Since

$$\gamma = \frac{\gamma\beta_{41}}{\beta_{41}} = -\frac{-\gamma\beta_{42}}{\beta_{42}},$$

by Theorem 3.3, we have

$$\begin{aligned} R &\equiv \mathcal{F}_{\gamma, \eta} \left[F\left(\frac{\cdot}{\sqrt{2}}\right) * (G * H)_{\vec{\alpha}_3, \vec{\beta}_3}(\cdot) \right]_{\vec{\alpha}_4, \vec{\beta}_4}(y) \\ &= \mathcal{F}_{\sqrt{2}\gamma\beta_{41}, \eta} F\left(\frac{1}{\sqrt{2}}\beta_{41}y\right) \mathcal{F}_{\sqrt{2}\gamma\beta_{42}, \eta} (G * H)_{\vec{\alpha}_3, \vec{\beta}_3}(\beta_{42}y). \end{aligned}$$

Moreover, since

$$\sqrt{2}\gamma\beta_{42} = \frac{\sqrt{2}\gamma\beta_{31}\beta_{42}}{\beta_{31}} = -\frac{-\sqrt{2}\gamma\beta_{32}\beta_{42}}{\beta_{32}},$$

we have

$$R = \mathcal{F}_{\sqrt{2}\gamma\beta_{41}, \eta} F\left(\frac{1}{\sqrt{2}}\beta_{41}y\right) \mathcal{F}_{2\gamma\beta_{31}\beta_{42}, \eta} G(\beta_{31}\beta_{42}y) \mathcal{F}_{2\gamma\beta_{32}\beta_{42}, \eta} H(\beta_{32}\beta_{42}y).$$

Finally, by the conditions (i), (ii), and (iii) we conclude that $L = R$ and this completes the proof. \square

Remark 3.10 1. If we use the same technique as in the proof of Theorem 3.9 and apply relationship (3.6) instead of (3.5), then we obtain the same associativity result.

2. Some sets of vectors of complex numbers satisfying the conditions in Theorem 3.9 are as follows:

$$\begin{aligned} \vec{\alpha}_1 &= \alpha(\beta, -\sqrt{2}\beta^2), & \vec{\alpha}_2 &= \alpha(\beta, \mp\beta), \\ \vec{\alpha}_3 &= \alpha(\sqrt{2}\beta^2, \mp\beta), & \vec{\alpha}_4 &= \alpha(\beta, -\beta) \end{aligned}$$

and

$$\vec{\beta}_1 = (1/\sqrt{2}, \beta), \quad \vec{\beta}_2 = (\beta, \pm\beta), \quad \vec{\beta}_3 = (\beta, \pm 1/\sqrt{2}), \quad \vec{\beta}_4 = (\beta, \beta),$$

where α and β are any complex numbers.

For any complex number α , $\vec{\alpha}_i = (\alpha/\sqrt{2}, -\alpha/\sqrt{2})$ and $\vec{\beta}_i = (1/\sqrt{2}, 1/\sqrt{2})$ for $i = 1, 2, 3, 4$ satisfy the conditions in Theorem 3.9. Hence we have the following associativity result for the convolution product of the integral transform.

Corollary 3.11 Let F, G , and H in E_σ be given by (1.1) with corresponding entire functions f, g , and h , respectively. Then for any complex number α , we have

$$\left[(F * G)_\alpha(\cdot) * H\left(\frac{\cdot}{\sqrt{2}}\right) \right]_\alpha(y) = \left[F\left(\frac{\cdot}{\sqrt{2}}\right) * (G * H)_\alpha(\cdot) \right]_\alpha(y) \tag{3.10}$$

for all $y \in K[0, T]$.

4. Generalized convolution product and the first variation

In this section we establish several results involving our generalized convolution product and the first variation. We begin by introducing the definition of the first variation of functionals defined on $K[0, T]$.

Definition 4.1 Let F be a functional defined on $K[0, T]$ and let $w \in K[0, T]$. Then the first variation δF of F is defined by

$$\delta F(y|w) = \left. \frac{\partial}{\partial t} F(y + tw) \right|_{t=0}, \quad y \in K[0, T] \tag{4.1}$$

if it exists [11, 13, 15, 16].

The following theorem shows that if F is an element of E_σ , then the first variation $\delta F(y|w)$ of F exists and is an element of E_σ as a function of y and as a function of w . For the proofs, see Theorems 2.3 and 2.4 and Remark 5.6 of [11].

Theorem 4.2 Let $F \in E_\sigma$ be given by (1.1). Then the first variation $\delta F(y|w)$ exists for $y, w \in K[0, T]$ and is given by the formula

$$\delta F(y|w) = \sum_{j=1}^n \langle \theta_j, w \rangle f_j(\langle \theta_j, y \rangle) = \sum_{j=1}^n \langle \theta_j, w \rangle F_j(y), \tag{4.2}$$

where $f_j(\vec{\lambda}) = (\partial/\partial \lambda_j) f(\vec{\lambda})$ and $F_j(y) = f_j(\langle \theta_j, y \rangle)$ for $j = 1, \dots, n$. Furthermore, it belongs to E_σ as a function of $y \in K[0, T]$ and as a function of $w \in K[0, T]$.

In our next theorem, we obtain a formula for the first variation of the generalized convolution product.

Theorem 4.3 *Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then*

$$\delta(F * G)_{\bar{\alpha}, \bar{\beta}}(y|w) = \beta_1(\delta F(\cdot|w) * G(\cdot))_{\bar{\alpha}, \bar{\beta}}(y) + \beta_2(F(\cdot) * \delta G(\cdot|w))_{\bar{\alpha}, \bar{\beta}}(y) \tag{4.3}$$

for $y, w \in K[0, T]$. Moreover, both sides of the expressions in (4.3) are given by the formula

$$\sum_{j=1}^n \langle \theta_j, w \rangle [\beta_1(F_j * G)_{\bar{\alpha}, \bar{\beta}}(y) + \beta_2(F * G_j)_{\bar{\alpha}, \bar{\beta}}(y)] \tag{4.4}$$

for $y, w \in K[0, T]$.

Proof By applying Theorem 4.2 to the expression (2.5), we obtain

$$\delta(F * G)_{\bar{\alpha}, \bar{\beta}}(y|w) = \sum_{j=1}^n \langle \theta_j, w \rangle k_j(\langle \vec{\theta}, y \rangle),$$

where k_j is the j -th partial derivative of k in (2.6). Since $f(\langle \vec{\theta}, \cdot \rangle)$ and $g(\langle \vec{\theta}, \cdot \rangle)$ belong to E_σ , we can pass the partial derivative under the integral sign to obtain

$$\begin{aligned} k_j(\langle \vec{\theta}, y \rangle) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} [\beta_1 f_j(\alpha_1 \vec{u} + \beta_1 \langle \vec{\theta}, y \rangle) g(\alpha_2 \vec{u} + \beta_2 \langle \vec{\theta}, y \rangle) \\ &\quad + \beta_2 f(\alpha_1 \vec{u} + \beta_1 \langle \vec{\theta}, y \rangle) g_j(\alpha_2 \vec{u} + \beta_2 \langle \vec{\theta}, y \rangle)] \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u}. \end{aligned}$$

By Theorem 2.3, the last integral can be expressed as a sum of two generalized convolution products, that is,

$$k_j(\langle \vec{\theta}, y \rangle) = \beta_1(F_j * G)_{\bar{\alpha}, \bar{\beta}}(y) + \beta_2(F * G_j)_{\bar{\alpha}, \bar{\beta}}(y).$$

Since our generalized convolution product is distributive as we see in Remark 2.2, by Theorem 4.2 again, we have

$$\begin{aligned} \delta(F * G)_{\bar{\alpha}, \bar{\beta}}(y|w) &= \sum_{j=1}^n \langle \theta_j, w \rangle [\beta_1(F_j * G)_{\bar{\alpha}, \bar{\beta}}(y) + \beta_2(F * G_j)_{\bar{\alpha}, \bar{\beta}}(y)] \\ &= \beta_1(\delta F(\cdot|w) * G(\cdot))_{\bar{\alpha}, \bar{\beta}}(y) + \beta_2(F(\cdot) * \delta G(\cdot|w))_{\bar{\alpha}, \bar{\beta}}(y) \end{aligned}$$

as we wanted to show. □

Next we obtain two formulas: in Theorem 4.4 we take generalized convolution product with respect to the first argument of the variation, while in Theorem 4.5 we take generalized convolution product with respect to the second argument of the variation.

Theorem 4.4 *Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then*

$$(\delta F(\cdot|w) * \delta G(\cdot|w))_{\bar{\alpha}, \bar{\beta}}(y) = \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle (F_j * G_l)_{\bar{\alpha}, \bar{\beta}}(y) \tag{4.5}$$

for $y, w \in K[0, T]$.

Proof Applying the additive distribution properties of the generalized convolution product in Remark 2.2 to the expressions

$$\delta F(y|w) = \sum_{j=1}^n \langle \theta_j, w \rangle F_j(y), \quad \delta G(y|w) = \sum_{l=1}^n \langle \theta_l, w \rangle G_l(y)$$

yields (4.5) as desired. □

Theorem 4.5 Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then

$$(\delta F(y|\cdot) * \delta G(y|\cdot))_{\bar{\alpha}, \bar{\beta}}(w) = \beta_1 \beta_2 \delta F(y|w) \delta G(y|w) + \alpha_1 \alpha_2 \sum_{j=1}^n F_j(y) G_j(y) \tag{4.6}$$

for $y, w \in K[0, T]$.

Proof By the same method as in the proof of Theorem 4.4, we have

$$(\delta F(y|\cdot) * \delta G(y|\cdot))_{\bar{\alpha}, \bar{\beta}}(w) = \sum_{j=1}^n \sum_{l=1}^n (\langle \theta_j, \cdot \rangle * \langle \theta_l, \cdot \rangle)_{\bar{\alpha}, \bar{\beta}}(w) F_j(y) G_l(y),$$

where

$$(\langle \theta_j, \cdot \rangle * \langle \theta_l, \cdot \rangle)_{\bar{\alpha}, \bar{\beta}}(w) = \int_{C_0[0, T]} (\alpha_1 \langle \theta_j, x \rangle + \beta_1 \langle \theta_j, w \rangle) (\alpha_2 \langle \theta_l, x \rangle + \beta_2 \langle \theta_l, w \rangle) dm(x).$$

Evaluating the last Wiener integral we obtain

$$(\langle \theta_j, \cdot \rangle * \langle \theta_l, \cdot \rangle)_{\bar{\alpha}, \bar{\beta}}(w) = \begin{cases} \alpha_1 \alpha_2 + \beta_1 \beta_2 \langle \theta_j, w \rangle^2, & \text{if } j = l \\ \beta_1 \beta_2 \langle \theta_j, w \rangle \langle \theta_l, w \rangle, & \text{if } j \neq l \end{cases}$$

which completes the proof. □

Letting $G = F$ in (4.3), (4.5), and (4.6) yields the following corollary.

Corollary 4.6 Let F in E_σ be given by (1.1) with corresponding entire function f . Then

$$\delta(F * F)_{\bar{\alpha}, \bar{\beta}}(y|w) = \sum_{j=1}^n \langle \theta_j, w \rangle [\beta_1 (F_j * F)_{\bar{\alpha}, \bar{\beta}}(y) + \beta_2 (F * F_j)_{\bar{\alpha}, \bar{\beta}}(y)], \tag{4.7}$$

$$(\delta F(\cdot|w) * \delta F(\cdot|w))_{\bar{\alpha}, \bar{\beta}}(y) = \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle (F_j * F_l)_{\bar{\alpha}, \bar{\beta}}(y) \tag{4.8}$$

and

$$(\delta F(y|\cdot) * \delta F(y|\cdot))_{\bar{\alpha}, \bar{\beta}}(w) = \beta_1 \beta_2 [\delta F(y|w)]^2 + \alpha_1 \alpha_2 \sum_{j=1}^n [F_j(y)]^2 \tag{4.9}$$

for $y, w \in K[0, T]$. In addition, if $\alpha_1^2 = \alpha_2^2$ and $\beta_1 = \beta_2$, then the generalized convolution product is commutative and so (4.7) reduces to

$$\delta(F * F)_{\bar{\alpha}, \bar{\beta}}(y|w) = (\beta_1 + \beta_2) \sum_{j=1}^n \langle \theta_j, w \rangle (F * F_j)_{\bar{\alpha}, \bar{\beta}}(y) \tag{4.10}$$

for $y, w \in K[0, T]$.

Note that (4.10), (4.8), and (4.9) are generalizations of (3.18), (3.19), and (3.20) in [11], respectively.

For a complex number α , let $\vec{\alpha} = (\alpha/\sqrt{2}, -\alpha/\sqrt{2})$ and $\vec{\beta} = (1/\sqrt{2}, 1/\sqrt{2})$. Then Formulas 3.5, 3.6, and 3.7 in [11] can be obtained as corollaries of our Theorems 4.3, 4.4, and 4.5, respectively, as follows.

Corollary 4.7 *Let F and G in E_σ be given by (1.1) with corresponding entire functions f and g , respectively. Then*

$$\delta(F * G)_\alpha(y) = \sum_{j=1}^n \frac{\langle \theta_j, w \rangle}{\sqrt{2}} [(F_j * G)_\alpha(y) + (F * G_j)_\alpha(y)], \tag{4.11}$$

$$(\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha(y) = \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle (F_j * G_l)_\alpha(y) \tag{4.12}$$

and

$$(\delta F(y|\cdot) * \delta G(y|\cdot))_\alpha(w) = \frac{1}{2} \delta F(y|w) \delta G(y|w) - \frac{\alpha^2}{2} \sum_{j=1}^n F_j(y) G_j(y) \tag{4.13}$$

for $y, w \in K[0, T]$.

5. Further results

Combining properties in Sections 3 and 4, we obtain various interesting relationships involving the integral transform, the generalized convolution product, and the first variation where each concept is used exactly once.

We begin this section by introducing two relationships involving integral transform and the first variation proved in Formulas 3.3 and 3.4, respectively, in [11].

Formula 5.1 *Let $F \in E_\sigma$ be given by (1.1). Then we have*

$$\mathcal{F}_{\gamma,\eta} \delta F(\cdot|w)(y) = \frac{1}{\eta} \delta \mathcal{F}_{\gamma,\eta} F(y|w) = \sum_{j=1}^n \langle \theta_j, w \rangle \mathcal{F}_{\gamma,\eta} F_j(y) \tag{5.1}$$

and

$$\mathcal{F}_{\gamma,\eta} \delta F(y|\cdot)(w) = \eta \delta F(y|w) \tag{5.2}$$

for $y, w \in K[0, T]$.

Because of Theorems 2.3, 3.2, and 4.2, all the functionals that arise in this section are automatically elements of E_σ . As usual, F and G in E_σ are given by (1.1) with corresponding entire functions f and g , respectively.

Formula 5.2 *Let $\vec{\alpha}$ and $\vec{\beta}$ satisfy the condition (3.4). Taking the first variation of the expressions in (3.5) with respect to the first argument of the variation yields the formula*

$$\begin{aligned} & \delta \mathcal{F}_{\gamma,\eta} (F * G)_{\vec{\alpha}, \vec{\beta}}(y|w) \\ &= \mathcal{F}_{\sqrt{2}\gamma\beta_1, \eta\beta_1} F(y) \delta \mathcal{F}_{\sqrt{2}\gamma\beta_2, \eta\beta_2} G(y|w) + \delta \mathcal{F}_{\sqrt{2}\gamma\beta_1, \eta\beta_1} F(y|w) \mathcal{F}_{\sqrt{2}\gamma\beta_2, \eta\beta_2} G(y) \\ &= \mathcal{F}_{\sqrt{2}\gamma\beta_1, \eta} F(\beta_1 y) \delta \mathcal{F}_{\sqrt{2}\gamma\beta_2, \eta} G(\beta_2 y|w) + \delta \mathcal{F}_{\sqrt{2}\gamma\beta_1, \eta} F(\beta_1 y|w) \mathcal{F}_{\sqrt{2}\gamma\beta_2, \eta} G(\beta_2 y) \end{aligned} \tag{5.3}$$

for $y, w \in K[0, T]$.

Formula 5.3 Replacing F with $\mathcal{F}_{\gamma,\eta}F$ and G with $\mathcal{F}_{\gamma,\eta}G$ in (4.3) and (4.4) yields the formula

$$\begin{aligned} & \delta(\mathcal{F}_{\gamma,\eta}F * \mathcal{F}_{\gamma,\eta}G)_{\vec{\alpha},\vec{\beta}}(y|w) \\ &= \sum_{j=1}^n \langle \theta_j, w \rangle [\beta_1(\mathcal{F}_{\gamma,\eta}F_j * \mathcal{F}_{\gamma,\eta}G)_{\vec{\alpha},\vec{\beta}}(y) + \beta_2(\mathcal{F}_{\gamma,\eta}F * \mathcal{F}_{\gamma,\eta}G_j)_{\vec{\alpha},\vec{\beta}}(y)] \end{aligned} \tag{5.4}$$

for $y, w \in K[0, T]$.

Formula 5.4 Taking the integral transform of the expressions in (4.3) and (4.4) with respect to the second argument of the variation or using (5.2) yields the formula

$$\begin{aligned} \mathcal{F}_{\gamma,\eta} \delta(F * G)_{\vec{\alpha},\vec{\beta}}(y|\cdot)(w) &= \eta \delta(F * G)_{\vec{\alpha},\vec{\beta}}(y|w) \\ &= \eta \sum_{j=1}^n \langle \theta_j, w \rangle [\beta_1(F_j * G)_{\vec{\alpha},\vec{\beta}}(y) + \beta_2(F * G_j)_{\vec{\alpha},\vec{\beta}}(y)] \end{aligned} \tag{5.5}$$

for $y, w \in K[0, T]$.

Formula 5.5 Let $\vec{\alpha}$ and $\vec{\beta}$ satisfy the condition (3.4). Taking the integral transform of the expressions in (4.5) with respect to the first argument of the variation and then using (3.5) and (5.1) yields the formula

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(\delta F(\cdot|w) * \delta G(\cdot|w))_{\vec{\alpha},\vec{\beta}}(y) &= \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1}[\delta F(\cdot|w)](y) \mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta\beta_2}[\delta G(\cdot|w)](y) \\ &= \frac{1}{\eta^2\beta_1\beta_2} \delta \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1} F(y|w) \delta \mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta\beta_2} G(y|w) \end{aligned} \tag{5.6}$$

for $y, w \in K[0, T]$.

Formula 5.6 Let $\vec{\alpha}$ and $\vec{\beta}$ satisfy the condition (3.4). Taking the integral transform of the expressions in (4.6) with respect to the second argument of the variation, and then using (5.2) yields the formula

$$\begin{aligned} \mathcal{F}_{\gamma,\eta}(\delta F(y|\cdot) * \delta G(y|\cdot))_{\vec{\alpha},\vec{\beta}}(w) &= \mathcal{F}_{\sqrt{2}\gamma\beta_1,\eta\beta_1} \delta F(y|\cdot)(w) \mathcal{F}_{\sqrt{2}\gamma\beta_2,\eta\beta_2} \delta G(y|\cdot)(w) \\ &= \eta^2 \beta_1 \beta_2 \delta F(y|w) \delta G(y|w) \end{aligned} \tag{5.7}$$

for $y, w \in K[0, T]$.

Formula 5.7 Taking the convolution product of the expressions in (5.1) with respect to the first argument of the variation yields the formula

$$(\mathcal{F}_{\gamma,\eta} \delta F(\cdot|w) * \mathcal{F}_{\gamma,\eta} \delta G(\cdot|w))_{\vec{\alpha},\vec{\beta}}(y) = \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle (\mathcal{F}_{\gamma,\eta} F_j * \mathcal{F}_{\gamma,\eta} G_l)_{\vec{\alpha},\vec{\beta}}(y) \tag{5.8}$$

for $y, w \in K[0, T]$.

Formula 5.8 Taking the convolution product of the expressions in (5.2) with respect to the second argument of the variation, and then using (4.6) yields the formula

$$\begin{aligned}
 & (\mathcal{F}_{\gamma,\eta}\delta F(y|\cdot) * \mathcal{F}_{\gamma,\eta}\delta G(y|\cdot))_{\bar{\alpha},\bar{\beta}}(w) \\
 &= \eta^2(\delta F(y|\cdot) * \delta G(y|\cdot))_{\bar{\alpha},\bar{\beta}}(w) \\
 &= \eta^2 \left[\beta_1\beta_2\delta F(y|w)\delta G(y|w) + \alpha_1\alpha_2 \sum_{j=1}^n F_j(y)G_l(y) \right]
 \end{aligned} \tag{5.9}$$

for $y, w \in K[0, T]$.

Note that the left-hand side of each of the formulas (5.3)–(5.9) involves all three of the operations of integral transform, convolution product, and first variation, while each right-hand side involves at most two.

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