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# Positive periodic solutions to impulsive delay differential equations

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**Abstract:** In this paper we discuss the existence of positive periodic solutions for nonautonomous second order delay differential equations with singular nonlinearities in the presence of impulsive effects. Simple sufficient conditions are provided that enable us to obtain positive periodic solutions. Our approach is based on a variational method.

**Key words:** Delay differential equation, periodic solution, singular nonlinearities, impulses, mountain-pass theorem

## 1. Introduction

The impulsive differential equations characterize various processes of the real world, described by models that are subject to sudden changes in their states. Essentially, impulsive differential equations correspond to a smooth evolution that may change instantaneously or even abruptly; this type of equation allows the study of models in physics, population dynamics, ecology, industrial robotics, economics, biotechnology, optimal control, and chaos theory. Due to its significance, a great deal of work has been done in the theory of impulsive differential equations; see for example [3, 13, 28], and for an introduction of the basic theory of impulsive differential equations in $\mathbb{R}^n$ we refer to [5, 12, 17].

Recently, variational methods and critical point theory have been successfully employed to investigate impulsive differential equations when the nonlinearity is regular; the existence and multiplicity of solutions for impulsive boundary value problems have been considered in [9, 16, 19–28].

However, few papers have investigated the case of impulsive boundary value problems with singular nonlinearity [10, 18, 20]. In fact, it seems that the work [20] is the first paper along this line. We must emphasize that singular boundary value problems without impulses have attracted the attention of many researchers [1, 2, 6, 14].

For delay differential equations, the variational approach has been little used (see [4, 8, 11, 23]).

The aim of this paper is to study the existence of positive 2r-periodic solutions of the following nonlinear impulsive problem:

$$\begin{cases} -u''(t)+\lambda(t)u(t)=f(t,u(t-r)),\ a.e\ t\in\mathbb{R}, \\ u(t)-u(t+2r)=u'(t)-u'(t+2r)=0, \\ \Delta u'(t_j)=I_j(u(t_j)), \qquad j\in\mathbb{Z}, \end{cases} \tag{1}$$

where $r\in\mathbb{R}^{+*}$ is a given constant, $f:\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ is $2r-$periodic in $t$, and $f(t,.)$ is singular at 0;

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$\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ with $u'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'(t); t_j,\ j \in \mathbb{Z}$ are the instants where the impulses occur; there exists an $p \in \mathbb{N}$ such that $0 = t_0 < t_1 < t_2 < ... < t_p < t_{p+1} = 2r,\ t_{j+p+1} = t_j + 2r;\ I_j,\ j \in \mathbb{Z}$ are continuous and $I_{j+p+1} \equiv I_j$, for all $j \in \mathbb{Z},\ \lambda$ is a $L^{\infty}$ function $2r-$periodic in $t$ such that $\alpha := \underset{t}{essinf}\lambda(t) > 0$.

We are motivated by the paper by Chen and Dai [8] in which, using the mountain pass theorem, the authors ensured the existence of at least one $2\pi-$periodic solution for the impulsive delay differential system

$$\begin{cases} u''(t) - u(t) = -f(t, u(t-\pi)), & t \in (t_{j-1}, t_j) \\ \Delta u'(t_j) = g_j(u(t_j - \pi)). \end{cases},$$

where $j \in \mathbb{Z},\ f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is $\pi-$periodic in $t$ and satisfying some technical conditions. $t_j,\ j \in \mathbb{Z}$ are the instants where the impulses occur; there exists a $p \in \mathbb{N}$ such that $0 = t_0 < t_1 < t_2 < ... < t_p < t_{p+1} = \pi$, $t_{j+p+1} = t_j + \pi;\ g_j,\ j \in \mathbb{Z}$, are continuous; and $g_{j+p+1} \equiv g_j$ for all $j \in \mathbb{Z}$.

Our goal in this paper is to obtain some simple sufficient conditions to guarantee that problem (1) has at least a *positive 2r*-periodic solution when the nonlinearity $f$ is singular and the impulses are independent of the delay.

This paper is organized as follows, in section 2 we give some necessary preliminaries, in section 3 we show the existence of at least one positive $2r$-periodic solution of problem (1), and lastly we present an example to illustrate our result.

## 2. Preliminaries

We begin this preliminary section with the following theorem, applied in the proof of our main result.

**Theorem 1** *(Mountain Pass Theorem; Th. 4.10 in [15]). Let $X$ be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. Assume that there exist $u_0 \in X, u_1 \in X$, and a bounded open neighborhood $\Omega$ of $u_0$ such that $u_1 \in X/\Omega$ and*

$$\inf_{\partial\Omega} \varphi > \max\left(\varphi\left(u_0\right), \varphi\left(u_1\right)\right).$$

*Let*

$$\Gamma = \left\{g \in C\left([0,1]; X\right);\ g(0) = u_0,\ g(1) = u_0\right\},$$

*and*

$$c = \inf_{g \in \Gamma} \max_{0 \le s \le 1} \varphi(g(s)).$$

*If $\varphi$ satisfies the Palais–Smale condition, then $c$ is a critical value of $\varphi$ and*

$$c > \max\left(\varphi\left(u_0\right), \varphi\left(u_1\right)\right).$$

In this study we are concerned with the existence of 2r-periodic solutions; for this, the problem (1) is equivalent to the following one

$$\begin{cases} -u''(t) + \lambda(t)u(t) = f(t, u(t-r)),\ a.e\ t \in [0, 2r] \\ u(0) - u(2r) = u'(0) - u'(2r) = 0, \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, ..., p \end{cases} \tag{2}$$

Throughout this work we shall use the following notations: $I = [0, 2r]$; for $1 \leq q < \infty$, $L^q(I)$ is the classical Lebesgue space of measurable functions $u : I \to \mathbb{R}$ such that $|u(.)|^q$ is integrable, and for $u \in L^q(I)$ we define its norm by

$$\|u\|_{L^q} = \left( \int\limits_0^{2r} |u(t)|^q \, dt \right)^{\frac{1}{q}}.$$

$L^\infty(I)$ is the classical Lebesgue space of measurable functions $u : I \to \mathbb{R}$ such that there exists a constant $C > 0$ such that $|u(t)| \leq C \quad a.e. \; t \in I$, and for $u \in L^\infty(I)$ we define its norm by

$$\|u\|_{L^\infty} = \inf \{C; |u(t)| \leq C \quad a.e \; t \in I\}.$$

Let $\|u\|_\infty = \sup \{|u(t)|; t \in I\}$ denote the norm of $u \in C(I)$, the space of real-valued continuous functions. $W^{1;2}(I)$ is the classical Sobolev space of functions $u \in L^2(I)$ with their distributional derivatives $u' \in L^2(I)$. We set $H^1_{2r} = \{u \in W^{1,2}(I); u(0) = u(2r))\}$, and for $u, v \in H^1_{2r}$ we define the inner product $(.,.)$ and the norm $\|.\|$ by

$$(u, v) = \int\limits_0^{2r} u'(t)v'(t)dt + \int\limits_0^{2r} \lambda(t)u(t)v(t)dt,$$

and

$$\|u\| = \left( \|u'\|^2_{L^2} + \left\|\sqrt{\lambda} u\right\|^2_{L^2} \right)^{\frac{1}{2}}.$$

$H^1_{2r}$ endowed with the norm $\|.\|$ is a reflexive Hilbert space.

By comparing the norm $\|u\|$ with the norm $\|u\|_{L^2}$, we find

$$\|u\|_{L^2} \leq \frac{1}{\alpha} \|u\| \quad \text{where, } \alpha = \underset{t}{essinf} \lambda(t). \tag{3}$$

For all $u \in H^1_{2r}$, we denote by $u_r(t) := u(t - r)$ for $t \in \mathbb{R}$.

From an elementary result in analysis, we have that if $u$ is a $T$-periodic function, then $\int\limits_0^T u(t)dt = \int\limits_a^{T+a} u(t)dt$, for all $a \in \mathbb{R}$. Hence, for all $u \in H^1_{2r}$,

$$\|u_r\| = \|u\|. \tag{4}$$

For all $u \in H^1_{2r}$, we have (see [7])

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|u\|_{L^2} + \frac{1}{2} \|u'\|_{L^2}. \tag{5}$$

**Definition 1** *$f : I \times (0, +\infty) \to \mathbb{R}$ is an $L^1$-Carathéodory function if*
*– the mapping $t \longmapsto f(t, x)$ is measurable for every $x \in (0, +\infty)$;*

– *the mapping* $x \longmapsto f(t,x)$ *is continuous for almost every* $t \in I$;
– *for every* $\rho > 0$ *there exists a function* $l_\rho \in L^1(I)$ *such that, for almost every* $t \in I$.

$$\sup_{|x| \le \rho} |f(t,x)| \le l_\rho(t).$$

Now we introduce the concept of solution for problem (2).

Let $H^2(a,b) = \{x : (a,b) \to \mathbb{R};\ x,\ x' \text{ are absolutely continuous},\ x'' \in L^2(a,b)\}$.

For $x \in H^2(0,2r)$ we have that $x$ and $x'$ are absolutely continuous and $x'' \in L^2(0,2r)$. Hence $\Delta x'(t_j) = x'(t_j^+) - x'(t_j^-) = 0$ for every $t \in I$. If $x \in H^1_{2r}$, then $x$ is absolutely continuous and $x' \in L^2(0,2r)$. In this case, the one-sided derivatives $x'(t_j^+)$, $x'(t_j^-)$ may not exist. As a consequence we need to introduce a different concept of solution.

**Definition 2** *We say that* $u \in H^1_{2r}$ *is a solution of* (2), *if* $u \in C(I)$, *for every* $j = 1,2,...,p$, $u_j := u|_{(t_j,t_{j+1})} \in H^2(t_j,t_{j+1})$ *and it satisfies the differential equation of* (2), *for* $t \neq t_j$, *the limits* $u'(t_j^-), u'(t_j^+)$ $j = 1,2,...,p$ *exist, and impulsive conditions and boundary 2r-periodic conditions of* (2) *hold.*

## 3. Main result

In this section, by variational method we show the existence of at least one positive solution for problem (2), which is considered under the following assumptions:

$(H1)$ $(i)$ $f : I \times (0,+\infty) \to \mathbb{R}$ is $2r-$periodic in the first argument $t$, and is a $L^1$-Carathéodory function,

$(ii)$ $\lim_{s \to 0^+} f(t,s) = -\infty$, for almost every $t$ in $I$,

$(iii)$ $K := \sup_{s \in ]0,+\infty[} f(.,s)$ is a function in $L^1(I,\mathbb{R})$,

$(iv)$ $\lim_{s \to 0^+} F(t,s) = +\infty$ and $\lim_{s \to +\infty} F(t,s) = +\infty$, for almost every $t$ in $I$,where $F(t,s) := \int_1^s f(t,\xi)d\xi$, the antiderivative of $f$,

$(v)$ $D_1F(t,s) := \frac{\partial F}{\partial t}(t,s)$ exists and is nonnegative, for almost every $t$ in $I$.

$(H2)$ $(i)$ $I_j$, $j = 1,2,...,p.$ are continuous and there exist two constants $m, M \in \mathbb{R}$ such that, for any $s \in \mathbb{R}$,

$$m \le I_j(s) \le M < 0, \text{for every } j = 1,2,...,p,$$

$(ii)$ $\lambda \in L^\infty(I)$, $2r-$periodic with $\alpha := \underset{t \in I}{essinf}\, \lambda(t) > 0$.

**Theorem 2** *Assume* $(H_1)$, *and* $(H_2)$ *are satisfied. Then for* $\frac{\sqrt{r}(\alpha+\sqrt{2})}{\sqrt{2}\alpha^2}\|\lambda\|_{L^\infty} < 1$, *the problem* (2) *has at least a positive solution.*

**Proof** We use a variational approach based on mountain pass theorem 1 to prove this result and we proceed in five steps.

**Step1:** Modification of the problem.

To avoid the singularity point 0, we introduce the truncation function $f_\beta$ defined for $\beta \in (0,1)$, by $f_\beta : [0,2r] \times \mathbb{R} \to \mathbb{R}$,

$$f_\beta(t,s) = \begin{cases} f(t,s) & if \quad s \geq \beta \\ f(t,\beta) & if \quad s < \beta \end{cases}, \tag{6}$$

and we consider the following modified problem:

$$\begin{cases} -u''(t) + \lambda(t)u(t) = f_\beta(t, u(t-r)), \quad a.e\ t \in I, \\ u(0) - u(2r) = u'(0) - u'(2r) = 0, \\ \Delta u'(t_j) = I_j(u(t_j)), \qquad j = 1, 2, ..., p. \end{cases} \tag{7}$$

Let $F_\beta(t,s) = \int_1^s f_\beta(t,\xi)d\xi$ be the antiderivative of $f_\beta$; we define the functional $\Phi_\beta :\ H^1_{2r} \to \mathbb{R}$, by

$$\Phi_\beta(u) = \frac{1}{2}\ \|u\|^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds - \int_0^{2r} F_\beta(t, u(t-r))dt. \tag{8}$$

$(H_1)$ and $(H_2)$ imply that $\Phi_\beta$ is well defined, weakly lower semicontinuous on $H^1_{2r}$ and it is continuously differentiable functional, whose derivative is the functional $\Phi'_\beta(u)$ given by

$$\begin{aligned} \Phi'_\beta(u).v \quad &= \quad \int_0^{2r} u'(t)v'(t)dt + \int_0^{2r} \lambda(t)u(t)v(t)dt + \sum_{j=1}^{p} I_j(u(t_j))v(t_j) \\ &\quad - \int_0^{2r} f_\beta(t, u(t-r))v(t)dt. \end{aligned} \tag{9}$$

The critical points of $\Phi_\beta$ are weak solutions of (7). Thus, to prove the existence of a solution for problem (2), we show the existence of critical points for $\Phi_\beta$ that are greater than some $\beta$.

**Step2:** The functional $\Phi_\beta$ satisfies the Palais–Smale condition.

Indeed, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1_{2r}$ such that $\{\Phi_\beta(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'_\beta(u_n) \to 0$ *as* $n \to +\infty$; i.e. there exist a constant $c_1 > 0$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$ *as* $n \to +\infty$ such that, for all $n$ large enough,

$$\left| \frac{1}{2}\ \|u_n\|^2 - \int_0^{2r} F_\beta(t, u_n(t-r))dt + \sum_{j=1}^{p} \int_0^{u_n(t_j)} I_j(s)ds \right| \leq c_1,$$

and for every $v \in\ H^1_{2r}$

$$\left| \int_0^{2r} u'_n(t)v'(t)dt + \int_0^{2r} \lambda(t)u_n(t)v(t)dt + \sum_{j=1}^{p} I_j(u_n(t_j))v(t_j) - \int_0^{2r} f_\beta(t, u_n(t-r))v(t)dt \right| \leq \varepsilon_n \|v\|. \tag{10}$$

Now we show that $\{u_n\}$ is bounded in $H^1_{2r}$. Taking $v(t) = 1$ in (10), we obtain, for all $n$ large enough,

$$\left| \int_0^{2r} [f_\beta(t, u_n(t-r)) - \lambda(t)u_n(t)]\, dt - \sum_{j=1}^{p} I_j(u_n(t_j)) \right| \leq \varepsilon_n \sqrt{2r}.$$

So that

$$\begin{aligned}
\left| \int_0^{2r} f_\beta(t, u_n(t-r))dt \right| &\leq \varepsilon_n \sqrt{2r} + \int_0^{2r} |\lambda(t)u_n(t)|\, dt + \sum_{j=1}^{p} |I_j(u_n(t_j))| \\
&\leq \varepsilon_n \sqrt{2r} + \|\lambda u_n\|_{L^1} + p\,|m| \\
&\leq c_2 + \|\lambda u_n\|_{L^1}, \qquad (11)
\end{aligned}$$

where $c_2 := \varepsilon_n \sqrt{2r} + p\,|m|$.
Let

$$I_{1,n} := \{t \in [0, 2r]\,;\ f_\beta(t, u_n(t-r)) \geq 0\},$$

and

$$I_{2,n} := \{t \in [0, 2r]\,;\ f_\beta(t, u_n(t-r)) < 0\}.$$

It follows from (11) that

$$\begin{aligned}
\left| \int_{I_{2,n}} f_\beta(t, u_n(t-r))dt \right| &\leq c_2 + \|\lambda u_n\|_{L^1} + \int_{I_{1,n}} f_\beta(t, u_n(t-r))dt \\
&\leq c_2 + \|\lambda u_n\|_{L^1} + \|K\|_{L^1}. \qquad (12)
\end{aligned}$$

Then by (12) we have for all $n$:

$$\begin{aligned}
\int_0^{2r} |f_\beta(t, u_n(t-r))|\, dt &= \left| \int_{I_{2,n}} f_\beta(t, u_n(t-r))dt \right| + \int_{I_{1,n}} f_\beta(t, u_n(t-r))dt \\
&\leq c_2 + \|\lambda u_n\|_{L^1} + 2 \int_{I_{1,n}} f_\beta(t, u_n(t-r))dt \\
&\leq c_2 + \|\lambda u_n\|_{L^1} + 2\,\|K\|_{L^1} \\
&\leq c_3 + \|\lambda u_n\|_{L^1}, \qquad (13)
\end{aligned}$$

where $c_3 := c_2 + 2\,\|K\|_{L^1}$.
On the other hand, if we take, in (10), $v(t) = u_n(t)$, taking into account $(H_2)(ii)$, (13), (5) and $\frac{\sqrt{r}(\alpha+\sqrt{2})}{\sqrt{2}\alpha^2} \|\lambda\|_{L^\infty} <$

1, we get for all $n$ large enough,

$$\begin{aligned} c_4 \|u_n\| &\geq \|u_n\|^2 - \int_0^{2r} f_\beta(t, u_n(t-r))u_n(t)dt + \sum_{j=1}^{p} I_j(u_n(t_j))u_n(t_j) \\ &\geq \|u_n\|^2 - (c_3 + \|\lambda u_n\|_{L^1})\|u_n\|_{L^\infty} + pm\|u_n\|_{L^\infty} \\ &\geq \|u_n\|^2 - \sqrt{2r}\|\lambda\|_{L^\infty}\|u_n\|_{L^2}\left(\frac{1}{\sqrt{2}}\|u_n\|_{L^2} + \frac{1}{2}\|u_n'\|_{L^2}\right) + (c_3 - pm)\|u_n\|_{L^\infty} \\ &= \|u_n\|^2 - \sqrt{r}\|\lambda\|_{L^\infty}\|u_n\|_{L^2}^2 - \frac{\sqrt{r}}{\sqrt{2}}\|\lambda\|_{L^\infty}\|u_n\|_{L^2}\|u_n'\|_{L^2} + (c_3 - pm)\|u_n\|_{L^\infty} \\ &\geq \|u_n\|^2 - \frac{\sqrt{r}(\alpha+\sqrt{2})}{\sqrt{2}\alpha^2}\|\lambda\|_{L^\infty}\|u_n\|^2 + (c_3 - pm)\|u_n\|_{L^\infty} \\ &\geq \left(1 - \frac{\sqrt{r}(\alpha+\sqrt{2})}{\sqrt{2}\alpha^2}\|\lambda\|_{L^\infty}\right)\|u_n\|^2 - c_4\|u_n\|, \end{aligned}$$

for some $c_4 > 0$. It follows that $\{u_k\}$ is bounded in $H^1_{2r}$. From the reflexivity of $H^1_{2r}$, we may extract a weakly convergent subsequence that, for simplicity, we label the same, and so there exists $u$ in $H^1_{2r}$, such that $u_k \rightharpoonup u$. Next, we will verify that $\{u_k\}$ is strongly convergent to $u$ in $H^1_{2r}$. By (9) we have

$$\begin{aligned} &\left(\Phi'_\beta(u_k) - \Phi'_\beta(u)\right)(u_k - u) \qquad (14) \\ = &\|u_k - u\|^2 - \int_0^{2r}\left[f_\beta(t, u_k(t-r)) - f_\beta(t, u(t-r))\right](u_k(t) - u(t))\,dt \\ &+ \sum_{j=1}^{p}\left[I_j(u_k(t_j)) - I_j(u(t_j))\right](u_k(t_j) - u(t_j)). \end{aligned}$$

By $u_k \rightharpoonup u$ in $H^1_{2r}$, and the Sobolev embedding theorem, we get $u_k \to u$ in $C(I)$ and $u_k \to u$ in $L^2(I)$. Hence,

$$\begin{cases} \int\limits_0^{2r}\left[f_\beta(t, u_k(t-r)) - f_\beta(t, u(t-r))\right](u_k(t) - u(t))\,dt \to 0, \\ \sum_{j=1}^{p}\left[I_j(u_k(t_j)) - I_j(u(t_j))\right](u_k(t_j) - u(t_j)) \to 0, \text{ as } k \to \infty. \end{cases} \qquad (15)$$

By $\lim_{k\to\infty}\Phi_\beta(u_k) = 0$ and $u_k \rightharpoonup u$, we have

$$\left(\Phi'_\beta(u_k) - \Phi'_\beta(u)\right)(u_k - u) \to 0, \text{ as } k \to \infty. \qquad (16)$$

By (14), (15), (16), and $u_k \to u$ in $L^2(I)$ we obtain $\|u_k - u\| \to 0$, as $k \to \infty$. That is, $\{u_k\}$ is strongly convergent to $u$ in $H^1_{2r}$, which means that $\Phi_\beta$ satisfies the Palais–Smale condition.

Now we proceed to show that $\Phi_\beta$ has a mountain pass geometry.

Let

$$\Omega := \left\{u \in H^1_{2r}; \min u > 1\right\},$$

and

$$\partial\Omega = \left\{u \in H^1_{2r}; u(t) \geq 1 \text{ for every } t \in (0, 2r), \exists t_u \in (0, 2r) : u(t_u) = 1\right\}.$$

**Step3:** There exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\beta(u) \geq -d$.

Indeed, for $u \in \partial\Omega$, there exists $t_u \in (0, 2r)$ such that $\inf_{t\in I} u(t) = u(t_u) = 1$, and so $t_u \in ]t_{i-1}, t_i[$ for some i, $1 \leq i \leq p$. By the $2r$-periodicity of $u, u', F_\beta$, and $I_j$, taking into account (4) we have

$$\begin{aligned}
\Phi_\beta(u) &= \frac{1}{2}\int_{t_u}^{t_u+2r} \left[(u'(t))^2 + \lambda(t)(u(t))^2\right] dt - \int_{t_u}^{t_u+2r} F_\beta(t, u(t-r))dt \\
&\quad + \sum_{j=i}^{p+i-1} \int_0^{u(t_j)} I_j(s)ds \\
&= \frac{1}{2}\int_{t_u}^{t_u+2r} \left[(u'(t))^2 + \lambda(t)(u(t))^2\right] dt - \int_{t_u}^{t_u+2r} F_\beta(t, u(t-r))dt \\
&\quad + \sum_{j=i}^{p+i-1} \int_0^{1} I_j(s)ds + \sum_{j=i}^{p+i-1} \int_1^{u(t_j)} I_j(s)ds \\
&\geq \frac{1}{2}\|u\|^2 - \|K_\beta\|_{L^2} \|u_r - 1\|_{L^2} + pm + pm\|u-1\|_\infty \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{\alpha}\|K_\beta\|_{L^2} \|u_r - 1\| + pm + c_6 pm\|u-1\| \\
&\geq \frac{1}{2}\|u\|^2 - \left(\frac{1}{\alpha}\|K_\beta\|_{L^2} - c_6 pm\right)\|u-1\| + pm,
\end{aligned}$$

for some constant $c_6 > 0$. Thus, applying triangular inequality to $\|u-1\|$,

$$\begin{aligned}
\Phi_\beta(u) &\geq \frac{1}{2}\|u\|^2 - \left(\frac{1}{\alpha}\|K_\beta\|_{L^2} - c_6 pm\right)\left(\|u\| + \sqrt{2r}\right) + pm \\
&= \frac{1}{2}\|u\|^2 - \left(\frac{1}{\alpha}\|K_\beta\|_{L^2} - c_6 pm\right)\|u\| + pm\left(c_6 - \sqrt{2r}\right) - \frac{\sqrt{2r}}{\alpha}\|K_\beta\|_{L^2}.
\end{aligned}$$

The above inequality shows that

$$\Phi_\beta(u) \to +\infty \ as \ \|u\| \to +\infty, \ u \in \partial\Omega.$$

We infer that $\Phi_\beta$ is coercive, and so it has a minimizing sequence; the weak lower semicontinuity of $\Phi_\beta$ yields

$$\inf_{u\in\partial\Omega} \Phi_\beta(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\beta(u) \geq -d$.

**Step4:** There exists $\beta_0 \in (0,1)$ such that for every $\beta \in (0, \beta_0)$, any solution $u$ of (7) with $\Phi_\beta(u) \geq -d$ satisfies $\min u \geq \beta_0$.

Assume on the contrary that there are sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that

$(a)$ $\beta_n \leq \frac{1}{n}$,

$(b)$ $u_n$ is a solution of (7) with $\beta = \beta_n$,

$(c)$ $\Phi_{\beta_n}(u_n) \geq -d$,

$(d)$ $\min u_n < \frac{1}{n}$.

By $(H_1)\,(iii)$, and

$$\begin{aligned}\int_0^{2r}\left[f_{\beta_n}(t,u_n(t-r))-\lambda(t)u_n(t)\right]dt &= -\int_0^{2r}u_n''(t)dt\\ &= -\sum_{j=0}^{p}\int_{t_j}^{t_{j+1}}u_n''(t)dt\\ &= -\sum_{j=0}^{p}\left(u_n'(t_{j+1}^-)-u_n'(t_j^+)\right)\\ &= \sum_{j=1}^{p}\Delta u_n'(t_j)+u_n'(0)-u_n'(2r)\\ &= \sum_{j=1}^{p}I_j(u_n(t_j)).\end{aligned}$$

Thus, if $h$ denotes the function

$$h(t)=f_{\beta_n}(t,u_n(t-r)),$$

$(H_2)$ implies the existence of a constant $c_7>0$, such that

$$\|h\|_{L^1}\leq c_7.$$

Hence,

$$\|u_n'\|_{L^\infty}\leq c_8,\ \ \textit{for some constant}\ c_8>0.$$

Now, since $\Phi_{\beta_n}(u_n)\geq -d$ it follows that there must exist two constants $R_1$ and $R_2$, with $0<R_1<R_2$ such that

$$\max\left\{u_n(t);\quad t\in I\right\}\in\left[R_1,R_2\right].$$

Otherwise, by the assumption $(b)$, $u_n$ is periodic solution of (7) with $\beta=\beta_n$, and so $u_n$ would tend uniformly to $0$, and in this case from $(H_1)\,(iv)$ and $\|u_n'\|_{L^\infty}\leq c_8$, $\Phi_{\beta_n}(u_n)$ would go to $-\infty$, which contradicts $\Phi_{\beta_n}(u_n)\geq -d$.

For $n$ large enough, the continuity of $u_n$ implies that there exist $\tau_n^1,\tau_n^2\in I$ such that

$$u_n\left(\tau_n^1-r\right)=\frac{1}{n}<R_1=u_n\left(\tau_n^2-r\right).$$

Multiplying the equation $u_n''(t)+f_{\beta_n}(t,u_n(t-r))=\lambda(t)u_n(t)$ by $u_n'(t-r)$ and integrating the resulting equation on $\left[\tau_n^1,\tau_n^2\right]$, or on $\left[\tau_n^2,\tau_n^1\right]$, we get

$$\begin{aligned}J\quad:&=\int_{\tau_n^1}^{\tau_n^2}u_n''(t)u_n'(t-r)dt+\int_{\tau_n^1}^{\tau_n^2}f_{\beta_n}(t,u_n(t-r))u_n'(t-r)dt\\ &=\int_{\tau_n^1}^{\tau_n^2}\lambda(t)u_n(t)u_n'(t-r)dt.\end{aligned}$$

Since $u_n$, $u_n' \in L^2(I;\mathbb{R})$, and $\lambda \in L^\infty(I;\mathbb{R})$, $J$ is bounded.
Let us write J as follows:

$$J = J_1 + \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t-r)dt,$$

where

$$J_1 = \int_{\tau_n^1}^{\tau_n^2} f_{\beta_n}(t, u_n(t-r))u_n'(t-r)dt$$

Since $\tau_n^1, \tau_n^2 \in I$, there exists $1 \le k, l \le p$ such that $\tau_n^1 \in ]t_{k-1}, t_k[$ and $\tau_n^2 \in ]t_{l-1}, t_l[$, then, $\|u_n'\|_{L^\infty} \le c_8$ implies

$$\begin{aligned} \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t-r)dt &\le c_8 \int_{\tau_n^1}^{\tau_n^2} u_n''(t)dt \\ &= c_8[u_n'(\tau_n^2) - u_n'(\tau_n^1) - \sum_{j=k}^{l-1} \Delta u_n'(t_j)] \end{aligned}$$

Consequently $J_1$ is bounded.
On the other hand, for a.e. $t \in I$, we have

$$f_{\beta_n}(t, u_n(t-r))u_n'(t-r) = \frac{d}{dt} F_{\beta_n}(t, u_n(t-r)) - D_1 F_{\beta_n}(t, u_n(t-r)).$$

Thus,

$$J_1 = F_{\beta_n}(\tau_n^2, R_1) - F_{\beta_n}(\tau_n^1, \frac{1}{n}) - \int_{\tau_n^1}^{\tau_n^2} D_1 F_{\beta_n}(t, u_n(t-r)).$$

The assumption $(H_1)(v)$ implies that

$$J_1 \le F_{\beta_n}(\tau_n^2, R_1) - F_{\beta_n}(\tau_n^1, \frac{1}{n}).$$

It follows from $(H_1)(ii)$ that $J_1$ is not bounded. This is a contradiction.

Consequently, there exists $\beta_0 \in (0,1)$ such that for every $\beta \in (0, \beta_0)$, any solution $u$ of (7) with $\Phi_\beta(u) \ge -d$ satisfies $\min u \ge \beta_0$ and so by (6) $u$ is a solution of (2).

**Step5**: $\Phi_\beta$ has a mountain-pass geometry, for all $\beta \le \beta_0$, where $\beta_0$ is defined in **step4**.

Indeed, from $(H_1)(ii)$ we can choose $\beta \in (0, \beta_0]$ such that

$$f(t, \beta) < 0, \text{ uniformly in } t \in I.$$

$$\begin{aligned} F_\beta(t,0) &= \int_1^0 f_\beta(t,s)ds = -\int_0^1 f_\beta(t,s)ds \\ &= -\int_0^\beta f_\beta(t,s)ds - \int_\beta^1 f_\beta(t,s)ds \\ &= -\int_0^\beta f(t,\beta)ds - \int_\beta^1 f_\beta(t,s)ds \\ &= -\beta f(t,\beta) - \int_\beta^1 f_\beta(t,s)ds. \end{aligned}$$

This implies that

$$F_\beta(t,0) > -\int_\beta^1 f_\beta(t,s)ds = \int_1^\beta f_\beta(t,s)ds = F_\beta(t,\beta);$$

hence

$$\Phi_\beta(0) = -\int_0^{2r} F_\beta(t,0)dt < -\int_0^{2r} F_\beta(t,\beta)dt.$$

By $(H_1)\,(iv)$, we can consider $\beta \in (0,\beta_0]$ such that

$$F_\beta(t,\beta) > \frac{d}{2r} \text{ for a.e. } t \in I.$$

It follows that $\Phi_\beta(0) < -d$.
Moreover, using $(H_1)$ $(iv)$ we can find $\delta$ sufficiently large ($\delta > 1$) such that for a.e. $t \in I$,

$$F_\beta(t,\delta) > \frac{d + \|\lambda\|_{L^\infty} r\delta^2}{2r}.$$

By $(H_2)\,(i)$ we have

$$\begin{aligned} \Phi_\beta(\delta) &= -\int_0^{2r} F_\beta(t,\delta)dt + \sum_{j=1}^{p} \int_0^\delta I_j(s)ds + \int_0^{2r} \frac{\lambda(t)\delta^2}{2}dt \\ &\leq -\int_0^{2r} F_\beta(t,\delta)dt + \|\lambda\|_{L^\infty} r\delta^2, \end{aligned}$$

and this implies that

$$\Phi_\beta(\delta) < -d.$$

Since $\Omega$ is a neighborhood of $\delta$, $0 \notin \Omega$ and

$$\max\{\Phi_\beta(0), \Phi_\beta(\delta)\} < \inf_{u\in\partial\Omega} \Phi_\beta(u).$$

Hence, we are in the situation of the mountain-pass theorem 1. **Step2** and **step5** imply that $\Phi_\beta$ has a critical point $u_\beta$ such that

$$\Phi_\beta(u_\beta) = \inf_{\eta\in\Gamma} \max_{0\leq s\leq 1} \Phi_\beta(\eta(s)) \geq \inf_{u\in\partial\Omega} \Phi_\beta(u),$$

where

$$\Gamma := \left\{\eta \in C\left([0,1]\,; H^1_{2r}\right);\ \eta(0) = 0,\ \eta(1) = R\right\}.$$

Now since by **step3** $\inf_{u\in\partial\Omega} \Phi_\beta(u) \geq -d$, it follows from **step4** that $u_\beta$ is a solution of (2). This completes the proof of the main result. □

**Example 1** We consider the following problem:

$$\begin{cases} -u''(t) + \lambda(t)u(t) = f(t, , u(t-r)),\ t \neq t_j,\ \ 0 < t < 2r, \\ u(0) - u(2r) = u'(0) - u'(2r) = 0, \\ \Delta u'(t_j) = I_j(u(t_j)), \quad\ j = 1, 2, \end{cases} \tag{17}$$

where $r = \frac{1}{3}$,

$$f(t, u(t-r)) := \mu\,(t)\,\frac{\ln u(t-r)}{u(t-r)},\ 0 \leq t < 2r,\ \text{with}\ \mu(t) = \begin{cases} t \ \ if\ \ 0 \leq t \leq r \\ 1\ \ if\ \ r < t < 2r \end{cases},$$

$$\begin{aligned} I_1(u(t_1)) \ &: \ = \cos(u(t_1)) - 2, \\ I_2(u(t_2)) \ &: \ = \frac{u(t_2)}{(u(t_2))^2 + 1} - 1, \end{aligned}$$

and

$$\lambda(t) := \begin{cases} t + 2 \quad\ if\ 0 \leq t \leq r \\ 2 + \sin t \quad\ if\ r < t < 2r \end{cases}.$$

The conditions $(H_1)$ and $(H_2)$ of theorem 2 are satisfied, indeed,

$(H_1)(i)$ $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$; given by $f(t, s) = \mu(t)\frac{\ln s}{s}$ is a Carathéodory function, $2r-$periodic in $t$.

$(ii)$ $\lim_{s\to 0^+} f(t, s) = -\infty$, for almost every $t$ in $I$,

$(iii)$ $K(t) = \sup\limits_{s\in]0,+\infty[} f(t, s) = \mu(t)e$ is a function in $L^1(I, \mathbb{R})$, where $\max\limits_{s\in]0,+\infty[} \frac{\ln s}{s} = e$.

$(iv)$ Since $\mu(t)$ is $2r-$periodic, then it is bounded, and so we have

$$\lim_{s\to 0^+} F(t, s) = \lim_{s\to 0^+} \mu(t)\frac{(\ln s)^2}{2} = +\infty,\ \text{and}\ \lim_{s\to +\infty} F(t, s) = +\infty,$$

for almost every $t$ in $I$.

$(v)$ For any $(t, s) \in (0, 2r) \times (0, +\infty)$,

$$D_1F(t, s) := \frac{\partial F}{\partial t}(t, s) = \begin{cases} \dfrac{(\ln s)^2}{2}\ \ if\ \ 0 \leq t \leq r \\ 0 \qquad\ \ if\ \ r < t < 2r \end{cases},$$

Thus, $D_1F(t, s) \geq 0$ for all $s \in (0, +\infty)$ and for almost every $t$ in $I$.

$(H_2)$ $(i)$ for $m = -3$ and $M = -\frac{1}{2}$, we have

$$m \leq I_j(s) \leq M < 0, \text{ for any } s \in \mathbb{R}, \text{and } \ j = 1, 2.$$

$(ii)\lambda \in L^\infty(I)$ with $\alpha := \underset{t\in I}{essinf}\lambda(t) = 2 > 0,\ \|\lambda\|_{L^\infty} = 2 + \sin\frac{2}{3}$.

Then, since $\frac{\sqrt{r}(\alpha+\sqrt{2})}{\sqrt{2}\alpha^2}\|\lambda\|_{L^\infty} = 0,912 < 1$, the problem (17) has at least a positive $\frac{2}{3}$-periodic solution.

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