Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2017) 41: 983 – 997
© TÜBİTAK
doi:10.3906/mat-1508-66

Permutation groups with cyclic-block property and $MNFC$-groups

Ali Osman ASAR*
Yargıç Sokak 11/6 Cebeci, Ankara, Turkey

Received: 20.08.2015 • **Accepted/Published Online:** 07.10.2016 • **Final Version:** 25.07.2017

Abstract: This work continues the investigation of perfect locally finite minimal non-FC-groups in totally imprimitive permutation p-groups. At present, the class of totally imprimitive permutation p-groups satisfying the cyclic-block property is known to be the only class of p-groups having common properties with a hypothetical minimal non-FC-group, because a totally imprimitive permutation p-group satisfying the cyclic-block property cannot be generated by a subset of finite exponent and every non-FC-subgroup of it is transitive, which are the properties satisfied by a minimal non-FC-group. Here a sufficient condition is given for the nonexistence of minimal non-FC-groups in this class of permutation groups. In particular, it is shown that the totally imprimitive permutation p-group satisfying the cyclic-block property that was constructed earlier and its commutator subgroup cannot be minimal non-FC-groups. Furthermore, some properties of a maximal p-subgroup of the finitary symmetric group on $\mathbb{N}^*$ are obtained.

Key words: Finitary permutation, totally imprimitive, cyclic-block property, homogeneous permutation, FC-group

1. Introduction

Let Ω be a nonempty (infinite) set. A permutation g on Ω is called finitary if its support $supp(g)$ is finite. The set of all the finitary permutations on Ω forms a normal subgroup of the symmetric group $Sym(\Omega)$ and is called the restricted symmetric group on Ω. It is denoted by $FSym(\Omega)$. A subgroup of $FSym(\Omega)$ is called a finitary permutation group on Ω. Let G be a transitive finitary permutation group on Ω, where Ω is infinite. If G has no proper blocks or has a maximal proper block, then G is called primitive or almost primitive, respectively, and then G has a homomorphic image that is isomorphic to one of $Alt(\Omega)$ or $Fsym(\Omega)$ by [10, p.261] (see also [9, Corollary 6.9]). Note that if Δ is a proper block for G, then there exists a $g \in G$ with $g(\Delta) \cap \Delta = \emptyset$ since two blocks are either equal or disjoint and then Δ must be finite since $supp(g)$ is finite. In the remaining case G is called totally imprimitive. In this case, G has an infinite ascending chain of proper blocks and their union is an infinite block for G, which must be equal to Ω since G is transitive. Thus, Ω and G are countably infinite. It is well known that a finitary permutation group G has only finite orbits if and only if one of the following holds:

G is solvable, hypercentral, an FC-group, or residually finite by [23, Theorems 1,2] or [10, Lemma 8.3D]. If G is locally solvable, then G is totally imprimitive and hyperabelian of height at most ω by [18, Theorem 2].

Let G be a totally imprimitive subgroup of $FSym(\Omega)$, where Ω is infinite. It is well known that set-wise stabilizers of finite sets are FC-groups and they are hypercentral when G is a p-group by [23, Theorem 1] or

*Correspondence: aliasar@gazi.edu.tr
2010 *AMS Mathematics Subject Classification:* 20B 07 20B 35 20E 25 20F 15.

[10, Lemma 8.3D]. Subgroups having infinite orbits of G are non-FC-subgroups (NFC-subgroups for short). In general an NFC-group is called a minimal NFC-group ($MNFC$-group for short) if every proper subgroup of this group is an FC-group.

The structure of an imperfect $MNFC$-group was determined in [6, 7] (see also [22, Theorem 8.13]). In this group the commutator subgroup is a divisible abelian q-group of finite rank (Chernikov q-group) and the commutator quotient is a finite p-group, where p, q are primes. On the other hand, it is still unknown whether or not a perfect $MNFC$-group exists. If a perfect MNF-group exists, then it is a p-group for a prime p by [7, Theorem 2] and [14, Theorem], and it has a nontrivial representation in the group of finitary permutations on some infinite set by the characterizations given in [8, 15]. (Some partial results in this direction are contained in [1–5].)

Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. An element g of G is said to satisfy the cyclic-block property if the support of each cycle in the cycle decomposition of g is a block for G, and a subset Y of G satisfies the cyclic-block property if every element of Y satisfies this property. Now suppose in addition that G satisfies the cyclic-block property. By [4, Lemma 2.2] two blocks for G are either disjoint or one is contained in the other one. This implies that G must be a p-group for a prime p. Furthermore, every NFC-subgroup of G is transitive and a subset of finite exponent of G generates a subgroup of finite exponent and so cannot be equal to G (see Lemma 3.1(b) below). These are properties satisfied by a perfect $MNFC$-group. (A perfect $MNFC$-group cannot be generated by a subset of finite exponent (see [1, Remark 1.10]).) There are no known other types of p-groups that share common properties with a perfect $MNFC$-p-group. For this reason it is a rather crucial step to settle the existence problem of $MNFC$-groups in the class of permutation groups satisfying the cyclic-block property. In this work a new result (Theorem 1.1) is obtained in this direction. This result is a considerable generalization of [1, Theorem 1.5] (see below). In particular, if a group in this class is generated by homogeneous elements and satisfies $(*)$ (see below), then the group cannot be $MNFC$ (Corollary 1.2). Furthermore, Theorem 1.1 provides a short proof for [4, Theorem 1.2] (Corollary 1.3). (Another proof of [4, Theorem 1.2] is contained in [5, Theorem 1.6].) The group given in [4, Theorem 1.1] satisfies the cyclic-block property, it has an easily defined generating set, and all of its blocks of p-power size are easily described, but it is not known whether or not it contains an $MNFC$-subgroup. (This group satisfying the cyclic-block property is a transitive subgroup of the maximal p-subgroup, denoted by W here, of $FSym(\mathbb{N}^*)$ constructed in [22]; see Proposition 2.1 for some properties of W.) [5] contains new properties of NFC-subgroups of a perfect totally imprimitive p-subgroup of $FSym(\Omega)$. Among other things it is shown there that the normalizer of an NFC-subgroup is self-normalizing and a self-normalizing subgroup is closed in the topology of point-wise convergence (see also [15]). It follows from [5] that a group of finitary permutations contains an $MNFC$-subgroup if and only if the set of self-normalizing subgroups contains minimal elements.

Let G be a subgroup of $FSym(\Omega)$ and let $g \in G$. The minimum of the lengths of the cycles in the cycle decomposition of g is denoted by $m(g)$. g is called homogeneous if every cycle of g has equal length. An infinite subset Y of G is called ascending if Y has an infinite exponent and is not contained in a set stabilizer of a finite set. We say that Y satisfies the property $(*)$ if for every $y, z \in Y$, and for all cycles c_y, c_z in the cycle decompositions of y and z, respectively, the following holds. Put $supp(c_y) = \Delta$ and suppose that $\Delta \subseteq supp(c_z)$. Then

(*)

$$[c_z^{s(c_z,\Delta)}|_\Delta, c_y] = 1$$

where $s(c_z,\Delta)$ is the smallest positive integer such that $c_z^{s(c_z,\Delta)} \in G_{\{\Delta\}}$.

It is well known that this condition is equivalent to

$$c_z^{s(c_z,\Delta)}|_\Delta = c_y^k$$

for a $k \geq 1$ by [13, Lemma 1]. (The centralizer of a cycle is generated by the cycle itself and permutations disjoint with it.)

Let Δ be a block for G and put $\Sigma = \{x(\Delta) : x \in G\}$. Then the kernel of the natural permutation representation of G into $Sym(\Sigma)$ is denoted by $Ker_G(\Delta)$ and is called the kernel subgroup of G with respect to Δ. Since $Ker_G(\Delta)$ fixes $x(\Delta)$ for every $x \in G$ it follows that $Ker_G(\Delta)$ is isomorphic to a subgroup of the direct product of copies of a finite group, and so $Ker_G(\Delta)$ is an FC-group of finite exponent.

For a nonempty subset X of G, $exp(X)$ denotes the maximum of the set $\{|x| : x \in X\}$ if it exists; otherwise, it is equal to ∞.

Theorem 1.1 *Let G be a perfect totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that G contains an ascending subset X satisfying the cyclic-block property such that the following properties hold.*

(a) X satisfies $(*)$. Thus for all $x, y \in X$ and for all cycles c_x, c_y in the cycle decompositions of x and y, respectively, the following holds. If $supp(c_x) \subseteq supp(c_y)$, then

$$[c_y^{s(c_y,supp(c_x))}|_{supp(c_x)}, c_x] = 1,$$

where $supp(c_x)$ and $supp(c_y)$ are blocks for G, which is equivalent to

$$c_y^{s(c_y,supp(c_x))}|_{supp(c_x)} = c_x^{q(c_x)}$$

for a $q(c_x) \geq 1$.

(b) For every $x \in X$ there exists a $y \in X$ so that $m(x) < m(y)$.

Then G cannot be an $MNFC$-group.

Theorem 1.1 is a considerable generalization of [1, Theorem 1.5]. In [1, Theorem 1.5] if F is a finite subgroup of G and $supp(F) \subseteq \Delta$ for a finite block Δ, then there exists $y \in G \setminus G_{\{\Delta\}}$ so that $y^{s(y,\Delta)} \in C_G(F)$. In particular $[F^y, F] = 1$ since $supp(F^y) \cap \Delta = 1$. This leads to the existence of an ascending subgroup H of G for a given $a \in G$ with $\langle a^G \rangle$ nonabelian so that $\langle a^H \rangle$ is abelian, which gives a contradiction. On the other hand, in Theorem 1.1, there is information only about the centralizer of a cycle, namely c_x of $x \in X$, but X is required to satisfy the additional property called the cyclic-block property. (Also in the proof of Theorem 1.1 $\langle c_x^G \rangle$ is not abelian, but there will exist an ascending subgroup, say X^* of G, so that $\langle c_x^{X^*} \rangle$ is abelian, which gives a contradiction.) It is not known yet whether condition (b) of Theorem 1.1 is indispensable.

Corollary 1.2 *Let G be a perfect totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that G contains an ascending subset X of homogeneous elements satisfying the cyclic-block property and the $(*)$ condition. Then G cannot be an $MNFC$-group.*

Corollary 1.3 *The totally imprimitive p-subgroup of $FSym(\mathbb{N}^*)$ given in [4, Theorem 1.1] and its commutator subgroup cannot be $MNFC$-groups.*

For definitions, notations, and basic properties the reader is referred to [9, 10, 21, 22].

Question. Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$ satisfying the cyclic-block property where Ω is infinite. Does G contain a minimal non-FC subgroup?

2. A finitary permutation group with cyclic-block property

In this section the finitary permutation p-group given in [4] and satisfying the cyclic-block property is described briefly for the convenience of the reader. This group is a subgroup of the example given in [23] by Wiegold.

For each $k, n \geq 1$ define

$$x_{k,n} = \prod_{i=1}^{p^{k-1}} (i + (n-1)p^k, i + (n-1)p^k + p^{k-1}, \ldots, i + (n-1)p^k + (p-1)p^{k-1}).$$

Each $x_{k,n}$ is a disjoint product of p^{k-1} cycles, each of which has length p.

For each $k \geq 1$ define

$$T_k = \{x_{k,n}; n \geq 1\} \text{ and } T_k^* = \langle T_i : 1 \leq i \leq k \rangle.$$

Wiegold's group, denoted here by W, is defined as $W = \langle T_k : k \geq 1 \rangle$. T_k is a set of pairwise disjoint permutations of order p and it is easy to check that $T_k^* \lhd W$ and T_{k+1}^*/T_k^* is elementary abelian for every $k \geq 0$, where $T_0 = 1$. W is a totally imprimitive p-subgroup of $FSym(\mathbb{N}^*)$ since every element of every T_k^* has finite support.

For $p = 2$,

$$T_1 = \{(1,2), (3,4), (5,6), \ldots\}, T_2 = \{(1,3)(2,4), (5,7)(6,8), (9,11)(10,12), \ldots\}$$

$$T_3 = \{(1,5)(2,6)(3,7)(4,8), (9,13)(10,14)(11,15)(12,16), \ldots\}.$$

For all $k, n \geq 1$ the sets

$$\Delta_{k,n} = \{1 + (n-1)p^k, 2 + (n-1)p^k, \ldots, p^k + (n-1)p^k\}$$

are blocks for W and $|\Delta_{k,n}| = p^k$. We may show that each $\Delta_{k,1}$ is a block. We may put $\Delta_k = \Delta_{k,1}$ when no confusion arises. Thus, $\Delta_k = \{1, 2, \ldots, p^k\}$ for $k \geq 1$. It suffices to show that $T_k^*(1) = \Delta_k$ for all $k \geq 1$ by [10, Theorem 1.6A(i)]. For $k = 1$ $T_1(1) = \{1, 2, \ldots, p\} = \Delta_1$. Assume that $T_k^*(1) = \Delta_k$. Now

$$x_{k+1,1} = (1, 1 + p^k, 1 + 2p^k, \ldots, 1 + (p-1)p^k) \cdots (p^k, p^k + p^k, p^k + 2p^k, \ldots, p^k + (p-1)p^k)$$

$$= (1, 1 + p^k, 1 + 2p^k, \ldots, 1 + (p-1)p^k) \cdots (p^k, 2p^k, 3p^k, \ldots, p^{k+1}).$$

Hence, it is easy to see that

$$\begin{aligned}\langle x_{k+1,1}\rangle(\Delta_k) &= \{1,2,\ldots,p^k\}\cup\{1+p^k,2+p^k,\ldots,2p^k\}\cup\cdots\cup\{1+(p-1)p^k,2+(p-1)p^k,\ldots,p^{k+1}\}\\ &= \Delta_{k+1}\end{aligned}$$

since the sets in the union are pairwise disjoint and are contained in Δ_{k+1}. In particular, it is easy to see that $x_{k+1,1}$ permutes the sets

$$\{1,2,\ldots,p^k\},\{1+p^k,2+p^k,\ldots,2p^k\},\ldots,\{1+(p-1)p^k,2+(p-1)p^k,\ldots,p^{k+1}\}$$

among themselves. Since $\langle x_{k+1,1}\rangle(\Delta_k) = \langle x_{k+1,1}\rangle(T_k^*(1)) = (\langle x_{k+1,1}\rangle T_k^*)(1) = T_{k+1}^*(1)$ it follows that $T_{k+1}^*(1) = \Delta_{k+1}$, which was to be shown. It can be shown that the finite blocks of p-power size for W consist of

$$\Delta_{k,n} = \{1+(n-1)p^k,2+(n-1)p^k,\ldots,p^k+(n-1)p^k\}$$

for $k,n\geq 1$.

Define

$$u_k = x_{k,1}x_{k-1,1}\cdots x_{1,1}$$

for all $k\geq 1$. Then $u_k\in T_k^*$ and $u_k=(a_1,a_2,\ldots,a_{p^k})$, where $1\leq a_i\leq p^k$ by [4, Lemma 3.2(a)]. Next define

$$v_k = u_k^{x_{k+1,1}}\cdots u_k^{x_{k+1,1}^{p-1}}.$$

Then $v_k = u_k^{x_{k+1,1}}\times\cdots\times u_k^{x_{k+1,1}^{p-1}}$, i.e. a product of disjoint cycles since $supp(u_k)=\Delta_k$ and $x_{k+1,1}$ sends each $1\leq i\leq p^{k+1}$ to $i+p^k \bmod (p^{k+1})$. (Always $c=a\times b$ means that a,b are disjoint permutations.) Put $g_k=u_k\times v_k$ for every $k\geq 1$ and define $G=\langle g_k : k\geq 1\rangle$. Then G satisfies the cyclic-block property by [4, Theorem 1.1]. We see from the definitions that $\{g_k : k\geq 1\}$ is an ascending set of homogeneous elements of G. Furthermore, it follows from the definition that

$$u_{k+1}^p = (x_{k+1,1}u_k)^p = x_{k+1,1}^p u_k^{x_{k+1,1}^{p-1}}\cdots u_k^{x_{k+1,1}}u_k = u_k\times u_k^{x_{k+1,1}}\times\cdots\times u_k^{x_{k+1,1}^{p-1}} = g_k$$

for every $k\geq 1$. Hence, it follows that the g_k satisfy $(*)$ as can be seen from the proof of Corollary 1.2. It can also be shown easily that $G\leq W'$. Indeed,

$$g_1 = x_{1,1}x_{1,1}^{x_{2,1}}\cdots x_{1,1}^{x_{2,1}^{p-1}} = x_{1,1}^p[x_{1,1},x_{2,1}]\cdots[x_{1,1},x_{2,1}^{p-1}]\in W'$$

since $x_{1,1}^p=1$. Assume that $g_k\in W'$ for a $k\geq 1$. Now

$$g_{k+1} = u_k u_k^{x_{k+1,1}}\cdots u_k^{x_{k+1,1}^{p-1}} = u_k^p[u_k,x_{k+1,1}]\cdots[u_k,x_{2,1}^{p-1}].$$

Since $u_k^p = g_{k-1}\in W'$ it follows that $g_{k+1}\in W'$, which completes the induction, and so $G\leq W'$.

As was indicated above, each u_k is a cycle of length p^k with $supp(u_k)=\Delta_k$ by [4, Lemma 3.2(a)]. Hence, $supp(u_k^{x_{k+1,1}^i}) = x_{k+1,1}^{-i}(\Delta_k)$ for every $i\geq 1$ and hence

$$supp(v_k) = \bigcup_{i=1}^{p^{k-1}} supp(u_k^{x_{k+1,1}^i}) = \Delta_{k+1}\setminus\Delta_k.$$

For $p = 2$

$$u_1 = (1,2); u_2 = (1,4,2,3); u_3 = (1,8,4,6,2,7,3,5)$$

and

$$u_4 = (1,16,8,12,4,14,6,10,2,15,7,11,3,13,5,9).$$

Hence,

$$g_1 = (1,2)(3,4); g_2 = (1,4,2,3)(5,8,6,7); g_3 = (1,8,4,6,2,7,3,5)(9,16,12,14,10,15,11,13).$$

Finally, it follows from [4, Theorem 1.1, Lemmas 2.2 and 3.4] that G satisfies the cyclic-block property, any two blocks for G are either disjoint or one is contained in the other one, and the blocks for G are the blocks for W. Thus, the set of the blocks of the same p-power size for G form a block system for G and hence also for W.

We end this section with a characterization of W.

Proposition 2.1 *W is a transitive maximal p-subgroup of $FSym(\mathbb{N}^*)$, $Z(W) = 1$, self-normalizing, and W/W' is infinite elementary abelian.*

Proof Put $W_k = \langle x_{1,1}, x_{2,1}, \ldots, x_{k,1}\rangle$ for every $k \geq 1$. Then $W = \bigcup_{k=1}^{\infty} W_k$ and also $\mathbb{N}^* = \bigcup_{k=1}^{\infty} \Delta_k$, where $\Delta_k = \{1, 2, \ldots, p^k\}$ for every $k \geq 1$. It is easy to see that each W_k is transitive on Δ_k, which implies that W is transitive on Ω, and then $Z(W) = 1$ by [10, Lemma 8.3C(ii)].

First we show that W_k is a Sylow p-subgroup of $Sym(\Delta_k)$ for every $k \geq 1$. Note that $supp(W_k) = \Delta_k$. Put

$$P_k = (\ldots(\langle x_{1,1}\rangle \wr \langle x_{2,1}\rangle) \wr \cdots \wr \langle x_{k,1}\rangle).$$

Then P_k is isomorphic to a Sylow p-subgroup of $Sym(\Delta_k)$ by [11, Proposition 19.10] since each $\langle x_{k,i}\rangle$ has order p. It will suffice to show that $W_k \cong P_k$. For $k = 1$ the assertion holds since $\langle x_{1,1}\rangle = \langle(1, 2, \ldots, p)\rangle$ is a Sylow p-subgroup of $Sym(\{1, 2, \ldots, p\})$. Suppose that the assertion holds for $k \geq 1$. Then $W_k \cong \langle x_{1,1}\rangle \wr \langle x_{2,1}\rangle \wr \cdots \wr \langle x_{k,1}\rangle$, and by identifying these two groups, W_k becomes a Sylow p-subgroup of $Sym(\Delta_k)$. Thus, we get $P_{k+1} \cong W_k \wr \langle x_{k+1,1}\rangle$. Let B_k be the base subgroup; that is, $B_k = \prod_{b \in \langle x_{k+1,1}\rangle} (W_k)_b$, where each $(W_k)_b$ is equal to W_k. Then P_{k+1} is isomorphic to $B_k\langle x_{k+1,1}\rangle$ and so $|P_{k+1}| = p|B_k| = p|W_k|^p$. However, we have seen above that $x_{k+1,1}$ permutes the sets $\{1, 2, \ldots, p^k\}, \{1+p^k, 2+p^k, \ldots, 2p^k\}, \ldots, \{1+(p-1)p^k, 2+(p-1)p^k, \ldots, p^{k+1}\}$ among themselves and induces a cycle of length p on them. Also, W_k is a Sylow p-subgroup of $Sym(\Delta_k)$. Clearly then $x_{k+1,1}^{-i} W_k x_{k+1}^{i}$ are Sylow p-subgroups on the corresponding sets $x_{k+1,1}^{-i}(\Delta_k)$ for $i = 1, \ldots, p$. Thus, $x_{k+1,1}^{-i} W_k x_{k+1}^{i}$ and $x_{k+1,1}^{-j} W_k x_{k+1}^{j}$ have disjoint supports for $i \neq j$ and so they commute. Therefore, we get

$$W_{k+1} = (W_k \times x_{k+1,1}^{-1} W_k x_{k+1} \times \cdots \times x_{k+1,1}^{-(p-1)} W_k x_{k+1}^{p-1})\langle x_{k+1,1}\rangle.$$

This gives $|W_{k+1}| = |W_k|^p |x_{k+1,1}| = p|W_k|^p$ and hence $|W_{k+1}| = |P_{k+1}|$. This implies that W_{k+1} is a Sylow p-subgroup of $Sym(\Delta_{k+1})$ since $W_{k+1} \leq Sym(\Delta_{k+1})$, which completes the induction. Clearly it follows from this that W is a Sylow p-subgroup of $FSym(\mathbb{N}^*)$ since $FSym(\mathbb{N}) = \bigcup_{k=1}^{\infty} Sym(\Delta_k)$.

Next we show that W is self-normalizing. Assume not. Then there exists a subgroup Y of $FSym(\mathbb{N}^*)$ with $W < Y$ and Y/W is abelian. Also, Y is transitive since W is. Moreover, $Y' \leq W$ and so Y' is a p-group, but then Y is a p-group by [20, Lemma 2.1], which is a contradiction.

Finally, we show that $W/W' = \prod_{k=1}^{\infty}\langle x_{k,1}W'\rangle$, as a direct product. This will be the case if we can show that $W_k/W_k' = \prod_{i=1}^{k}\langle x_{i,1}W_k'\rangle$, as a direct product, for every $k \geq 1$. For $k = 1$ this is trivial. Suppose that the assertion holds for $k \geq 1$. We have seen above that

$$W_{k+1} \cong W_k \wr \langle x_{k+1,1}\rangle = (\prod_{b\in\langle x_{k+1,1}\rangle}(W_k)_b)\langle x_{k+1,1}\rangle = B_k\langle x_{k+1,1}\rangle,$$

which implies that $W_{k+1}/W_{k+1}' \cong B_k\langle x_{k+1,1}\rangle/(B_k\langle x_{k+1,1}\rangle)'$. We can now apply [17, Corollary 4.5] to $B_k\langle x_{k+1,1}\rangle$. This gives

$$(B_k\langle x_{k+1,1}\rangle)' = M$$

where $M = \{f \in B_k : \pi(f) \in W_k'\}$ and $\pi(f) = \prod_{b\in\langle x_{k+1,1}\rangle} f(b)$. Next define $x_{i,1}^*(1) = x_{i,1}$ and $x_{i,1}^*(b) = 1$ for $b \neq 1$ for $1 \leq i \leq k$. Each $x_{i,1}^* \in B_k = \prod_{b\in\langle x_{k+1,1}\rangle}(W_k)_b)$. We claim that $x_{1,1}^*M, \ldots, x_{k,1}^*M$ are linearly independent over $\mathbb{Z}_p$, the field of p elements. Assume if possible that there exists an $f = (x_{1,1}^*)^{s_1}\cdots(x_{r,1}^*)^{s_r}$, where $1 \leq r \leq k$ and $1 \leq s_i < p$ so that $f \in M$. Then $f = (x_{1,1}^{s_1}\cdots x_{r,1}^{s_r}, 1, \ldots, 1)$ and $\pi(f) = x_{1,1}^{s_1}\cdots x_{r,1}^{s_r} \in W_k'$, but since $W_k/W_k' = \langle x_{1,1}W_k'\rangle \times \cdots \times \langle x_{k,1}W_k'\rangle$ by the induction hypothesis it follows that $x_{1,1}^{s_1}W_k' = \cdots = x_{r,1}^{s_r}W_k' = 1$, which means that $x_{i,1}^{s_i} \in W_k'$ and then $p|s_i$ since $|x_{i,1}| = p$, which is impossible since $1 \leq s_i < p$ for every $i \geq 1$. Consequently it follows that $x_{1,1}^*M, \ldots, x_{k,1}^*M$ are linearly independent in $B_k\langle x_{k+1,1}/M$. Then also $x_{1,1}^*M, \ldots, x_{k,1}^*M, x_{k+1,1}^*M$ are linearly independent in $B_k\langle x_{k+1,1}/M$ since $\langle x_{k+1,1}^*\rangle \cap B_k = 1$. Therefore,

$$B_k\langle x_{k+1,1}\rangle/M = \langle x_{1,1}^*M\rangle \times \cdots \times \langle x_{k+1,1}^*M\rangle.$$

Hence, using the above isomorphism, we get

$$W_{k+1}/W_{k+1}' = \langle x_{1,1}W_{k+1}'\rangle \times \cdots \times \langle x_{k+1,1}W_{k+1}'\rangle,$$

which completes the induction. Now since $W = \bigcup_{k=1}^{\infty} W_k$ it follows easily that

$$W/W' = \prod_{k=1}^{\infty}\langle x_{k,1}W'\rangle$$

as a direct product. Suppose that $\langle x_{t,1}W'\rangle \cap \langle x_{k,1}W' : k \geq 1, k \neq t\rangle \neq 1$ for a $t \geq 1$. Then $\langle x_{t,1}W'\rangle \leq \langle x_{k,1}W' : k \geq 1, k \neq t\rangle$ since $|x_{t,1}| = p$. Hence, $x_{t,1}$ is a finite product of elements of certain cosets of the right side. Also, $W' = \bigcup_{k=1}^{\infty} W_k'$. Clearly then there exists an $n > t$ so that $x_{t,1} \in \langle x_{k,1}W_n' : 1 \leq k \leq n, k \neq t\rangle$, but since $W_n/W_n' = \langle x_{1,1}W_n'\rangle \times \cdots \times \langle x_{n,1}W_n'\rangle$, as was shown above, this is impossible. Therefore, the assumption is false and so W/W' is a direct product of the $\langle x_{k,1}W'\rangle$ as k ranges over the positive integers. □

Remark. The commutator subgroup W' of W is perfect and transitive by [19, Theorem 1]. Also, W does not satisfy the normalizer condition by [1, Theorem 1.2(b)] since $G \leq W$ and G' is not an $MNFC$-group by Corollary 1.3. The reader may observe that W is exactly the same group that is constructed in [12, 18.2.2 Example], where it is shown also that this group does not satisfy the normalizer condition.

3. Proof of Theorem 1.1

We begin with a known result on the cyclic-block property for the convenience of the reader. (See also [5, Proposition 1.7].)

Lemma 3.1 *3.1 Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$ satisfying the cyclic-block property, where Ω is infinite. Then the following hold:*

(a) Let Δ be a finite block for G and let $\alpha \in \Delta$. Then for every $y \in G \setminus G_{\{\Delta\}}$, $\langle y^{s(y,\Delta)}\rangle(\alpha) = \Delta$.

(b) Let Δ be a finite block for G. Then

$$Ker_G(\Delta) = \{g \in G : |g| \leq |\Delta|\}.$$

Furthermore, $exp(G_{\{\Delta\}})$ is infinite.

(c) Any NFC-subgroup of G is transitive on Ω.

Proof (a) (See [4, Lemma 2.1].) Put $H = G_{\{\Delta\}}$ and let $y \in G \setminus H$. Put $t = s(y, \Delta)$. Then t is the smallest number such that $y^t \in H$. Also, $t = p^r$ for an $r \geq 1$. Next put $\langle y\rangle(\alpha) = \Gamma$ and $\langle y^t\rangle(\alpha) = \Lambda$. Then Γ and Λ are blocks for G by the cyclic-block property. Also, $\Delta \subset \Gamma$ and $\Lambda \subseteq \Delta$ by [4, Lemma 2.2] since $y \notin H$ but $y^t \in H$. Clearly $|\Gamma| = p^r|\Lambda|$. Assume if possible that there exists a $y^j(\alpha) \in \Delta \setminus \Lambda$. Then $j \nmid p^r$ and so $j < p^r$, but since $\alpha \in \Delta \cap y^{-j}(\Delta)$ and since Δ is a block, it follows that $y^j(\Delta) = \Delta$, which is a contradiction since $t = p^r$ is the smallest number with the property that $y^t(\Delta) = \Delta$. Therefore, the assumption is false and so $\langle y^t\rangle(\alpha) = \Delta$.

(b) Put $M = Ker_G(\Delta)$. Then $M < H$ since $H \neq G$ due to the fact that Ω is infinite and G is transitive. Let $y \in G$ and put $|y| = t$. First suppose that $t \leq |\Delta|$. Then we claim that $y \in H$. This is trivial if $supp(y) \cap \Delta = \emptyset$ since then $y(\Delta) = \Delta$. Suppose that $y(\alpha) \neq \alpha$ for an $\alpha \in \Delta$. Put $\Gamma = \langle y\rangle(\alpha)$. Then Γ is a block for G by the hypothesis and $|\Gamma| \leq t \leq |\Delta|$. Also, since $\alpha \in \Gamma \cap \Delta$ applying [4, Lemma 2.2], we get $\Gamma \subseteq \Delta$, which implies that $y \in H$. Thus, $\{g \in G : |g| \leq |\Delta|\} \subseteq M$. Next suppose that $t > |\Delta|$. There exists a $\beta \in \Omega$ so that $t = |\langle y\rangle(\beta)|$. Also, there exists a $g \in G$ so that $g(\beta) = \alpha$. Since $\langle y\rangle(\beta) = \{\beta, y(\beta), \ldots, y^{t-1}(\beta)\}$, it follows that $\langle gyg^{-1}\rangle(\alpha) = \{gy(\beta), \ldots, gy^{t-1}(\beta), g(\beta)\}$. Now if $y \in M$ then also $gyg^{-1} \in M$, but since $\langle gyg^{-1}\rangle(\alpha)$ is a block containing α and has size greater than $|\Delta|$, this is a contradiction. Therefore, $M = \{g \in G : |g| \leq |\Delta|\}$. In particular it follows that any subgroup of finite exponent of G is contained in a kernel subgroup which is nilpotent of finite exponent. It is well-known that a transitive subgroup of $FSym(\Omega)$ has infinite exponent if Ω is infinite by [18, Lemma 3.1] or [10, Theorem 8.3A]). Let $\alpha \in \Omega$. We show that G_α contains a conjugate of every element of G. Let $g \in G$. There exists a $\beta \in \Omega$ so that $g(\beta) = \beta$ and so $g \in G_\beta$. Also, $\beta = x(\alpha)$ for an $x \in G$. Hence, $g \in G_{x(\alpha)} = xG_\alpha x^{-1}$ and so $g^x \in G_\alpha$, which completes the proof of (b).

(c) Let X be a proper NFC- subgroup of G. Then X cannot be contained in the set-wise stabilizer of a finite block for G since X is not an FC-group. However, if $exp(X) \leq |\Delta|$ for a finite block Δ, then $X \leq Ker(\Delta) \leq G_{\{\Delta\}}$ by (b), which is impossible. Therefore, $exp(X) = \infty$. Let $\alpha, \beta \in \Omega$ and let Δ be a finite block for G containing both of them. Then there exists a $g \in X \setminus X_{\{\Delta\}}$ so that $\langle g^p\rangle(\alpha) = \Delta$ by (a), which implies that $\beta = (g^p)^j(\alpha)$ for a $j \geq 1$, and so X is transitive. □

Lemma 3.2 *Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$ and let c, d be two cycles in G such that $supp(c), supp(d)$ are blocks for G and $supp(c) \subseteq supp(d)$. Let $|c| = p^a, |d| = p^b$ and put $t = s(d, supp(c))$.*

Then $t = p^{b-a}$. *If* $d^{p^{b-a}}|_{supp(c)} = c^k$, *then* $(p,k) = 1$ *and*

$$d^{p^{b-a}} = c^k \times (c^k)^d \times \cdots \times (c^k)^{d^{p^{b-a}-1}}.$$

Proof Put $\Delta = supp(c)$, $\Gamma = supp(d)$. Then $|\Delta| = p^a$ and $|\Gamma| = p^b$. Let $\alpha \in \Delta$. Now $\Delta \subseteq \Gamma$. Clearly $\Gamma = \Delta \cup d(\Delta) \cup \cdots \cup d^{t-1}(\Delta)$ as a disjoint union since Δ is a block and d is a cycle. Hence, $p^b = tp^a$ and hence $t = p^{b-a}$.

Put $H = G_{\{\Delta\}}$. Then t is the smallest number with $d^t \in H$. Hence, $\langle d^{p^{b-a}}\rangle(\alpha) \subseteq \Delta$ and $|d^{p^{b-a}}| = |\langle d^{p^{b-a}}\rangle(\alpha)|$ since d is a cycle, which implies that $|\langle d^{p^{b-a}}\rangle(\alpha)| = p^a$. Now suppose that $d^{p^{b-a}}|_{\Delta} = c^k$. Then $p \nmid k$ since $|c|$ is a cycle of length p^a. Thus, $(p,k) = 1$ and c^k is a cycle.

Now $d^{p^{b-a}}|_{d^i(\Delta)} = d^i c^k d^{-i}$ for every $1 \le i \le p^{b-a}$ and $\Gamma = \Delta \cup d(\Delta) \cup \cdots \cup d^{p^{b-a}-1}(\Delta)$. Obviously then

$$d^{p^{b-a}} = d^{p^{b-a}}|_{\Gamma} = c^k \times (c^k)^d \times \cdots \times (c^k)^{p^{b-a}-1}.$$

□

Lemma 3.3 *Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$. Let X be an ascending subset of G satisfying the cyclic-block property. Suppose also that X satisfies $(*)$. Then X contains an ascending subset $Y = \{y_i : i \ge 1\}$ of G such that the following holds. Each y_i can be expressed as a direct product of cycles as*

$$y_i = c_{i,1} \times \cdots \times c_{i,r(i)}$$

so that $supp(y_i) \subset supp(c_{i+1,1})$, $m(y_i) \le m(y_{i+1})$, and the following hold. Let $1 \le j \le r(i)$ and $k \ge i$. Put $|c_{i,j}| = p^a$ and $|c_{k,1}| = p^b$. Then

$$[c_{k,1}^{p^{b-a}}|_{supp(c_{i,j})}, c_{i,j}] = 1.$$

Proof Choose a $y_1 \ne 1$ in X so that $m(y_1) \le m(x)$ for every $x \in X$ and let

$$y_1 = c_{1,1} \times \cdots \times c_{1,r(1)}$$

be the cycle decomposition of y_1. Let Γ_1 be the smallest block containing $supp(y_1)$. Next choose a y_2 in $X \backslash G_{\{\Gamma_1\}}$ so that $m(y_2) \le m(x)$ for every $x \in X \backslash G_{\{\Gamma_1\}}$. Now $\langle y_2\rangle(\alpha)$ is a block by the cyclic-block property and $\Gamma_1 \subset supp(\langle y_2\rangle(\alpha))$ by [4, Lemma 2.2] since $y_2 \notin G_{\{\Gamma_1\}}$. Also, $m(y_1) \le m(y_2)$. Put $c_{2,1} = (\alpha, \ldots, y_2^{t_2-1}(\alpha))$, where t_2 is the smallest number such that $y_2^{t_2}(\alpha) = \alpha$. Thus, $\Gamma_1 \subset supp(c_{2,1})$. Continuing in this way we obtain an infinite subset $Y = \{y_i : i \ge 1\}$ of X such that $m(y_i) \le m(y_{i+1})$ and $supp(y_i) \subset supp(c_{i+1,1})$ for every $i \ge 1$, where

$$y_i = c_{i,1} \times \cdots \times c_{i,r(i)}$$

is the cycle decomposition of y_i. Let $1 \le i < k$ and let $1 \le j \le r(i)$. Then $supp(c_{i,j}) \subset supp(c_{k,1})$. Also, Y satisfies $(*)$ since Y is a subset of X. Therefore,

$$[c_{k,1}^{s(c_{k,1}, supp(c_{i,j}))}|_{supp(c_{i,j})}, c_{i,j}] = 1.$$

Let $|c_{i,j}| = p^a$ and $|c_{k,1}| = p^b$. Then since $s(c_{k,1}, supp(c_{i,j})) = p^{b-a}$ by Lemma 3.2, substituting this value above the desired equality is obtained. Furthermore, Y is ascending since $supp(c_{i,1})$ is a block and $supp(c_{i,1}) \subset supp(c_{i+1,1})$ for every $i \geq 1$. □

Lemma 3.4 *Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$. Let X be an ascending subset of G satisfying the cyclic-block property and $(*)$. Let $Y = \{y_i : i \geq 1\}$ be the subset of X obtained in Lemma 3.3. Thus, for each $i \geq 1$, the cycle decomposition of y_i can be written as*

$$y_i = c_{i,1} \times \cdots \times c_{i,r(i)}$$

such that $supp(y_i) \subset supp(c_{i+1,1})$ and $m_1(y_i) \leq m_1(y_{i+1})$ for every $i \geq 1$. Moreover, if we put $|c_{i,j}| = p^{a(i,j)}$, for every $i \geq 1$ and $1 \leq j \leq r(i)$, then for $1 \leq i \leq k$ the equality

$$c_{k,1}^{p^{a(k,1)-a(i,j)}}|_{supp(c_{i,j})} = c_{i,j}^{q(i,j)} \tag{1}$$

holds for a $q(i,j) \geq 1$ with $(p, q(i,j)) = 1$ by Lemma 3.3.

Now let $j, k, t \geq 1$ be integers with $j \leq k, t$ and suppose that $|y_j| \leq min\{m(y_k), m(y_t)\}$. Let $m, n \geq 1$. Then

$$c_{j,1}^{y_k^m y_t^n} = c_{j,1}^{y_r^s}$$

for an $r \in \{k, t\}$ and $s \geq 1$.

Proof Put $c_i = c_{i,1}$ and let $|supp(c_i)| = p^{a(i)}$ for $i = j, k, t$. Then c_i is a factor of the cycle decomposition of y_i for $i = j, k, t$ and $supp(y_u) \subset supp(c_v)$ for every $1 \leq u < v$. We may suppose that $j < k, t$.

Case 1 $j < k < t$. Now

$$c_j^{y_k^m y_t^n} = c_j^{c_k^m y_t^n} \tag{2}$$

since $supp(c_j) \subseteq supp(c_k)$. On the other hand,

$$c_t^{p^{a(t)-a(k)}} = c_k^{q(k,1)} \times \cdots \times (c_k^{q(k,1)})^{c_t^{p^{a(t)-a(k)}-1}} = c_k^{q(k,1)} \times v_k$$

by (1) and Lemma 3.2, where $supp(v_k) \cap supp(c_k) = \emptyset$. Also, $bq(k,1) \equiv 1 \mod (p^{a(k)})$ for an integer b since $(q(k,1), p) = 1$ by Lemma 3.2. Using this above gives

$$c_t^{bp^{a(t)-a(k)}} = c_k \times \cdots \times c_k^{bc_t^{p^{a(t)-a(k)}-1}} = c_k \times v_k^b.$$

Hence, $c_k^m = c_t^{mbp^{a(t)-a(k)}} v_k^{-bm}$. Substituting this in (2) gives

$$c_j^{c_k^m y_t^n} = c_j^{v_k^{-bm} c_t^{mbp^{b-a}} y_t^n} = c_j^{c_t^{mbp^{a(t)-a(k)}} y_t^n} = c_j^{c_t^{mbp^{a(t)-a(k)}+n}} = c_j^{y_t^{mbp^{b-a}+n}}$$

since $supp(c_j) \subset supp(c_k)$.

Case 2 $k > t > j$. We may suppose that $supp(c_j^{c_k^m}) \cap supp(y_t^n) \neq \emptyset$; otherwise, $c_j^{c_k^m y_t^n} = c_j^{c_k^m}$ and we are done. Then there exists a cycle $c_{t,r}$ in the cycle decomposition of y_t so that $supp(c_j^{c_k^m}) \subseteq supp(c_{t,r})$ by the cyclic-block property since $|y_j| < m(y_t)$ by the hypothesis. For simplicity, put $u_t = c_{t,r}$ and $q(t) = q(t,r)$. Clearly now $c_j^{c_k^m y_t^n} = c_j^{c_k^m u_t^n}$. Let $|u_t| = p^z$. Then

$$c_k^{p^{a(k)-z}} = u_t^{q(t)} \times \cdots \times (u_t^{q(t)})^{c_k^{p^{a(k)-z}-1}} = u_t^{q(t)} \times v_t$$

where $(q(t), p) = 1$ by (1). Then, as in Case 1, there exists an integer b so that

$$c_k^{bp^{a(k)-z}} = u_t \times v_t^b$$

where $supp(u_t) \cap supp(v_t) = \emptyset$. Hence $u_t^n = v_t^{-bn} c_k^{nbp^{a(k)-z}}$. Substituting this above gives

$$c_j^{c_k^m u_t^n} = c_j^{c_k^m v_t^{-bn} c_k^{nbp^{a(k)-z}}} = c_j^{c_k^m c_k^{nbp^{a(k)-z}}}$$

$$= c_j^{c_k^{m+nbp^{a(k)-z}}} = c_j^{y_k^{m+nbp^{a(k)-z}}}$$

since $supp(c_j^{c_k^m}) \subseteq supp(u_t)$ and $supp(u_t) \cap supp(v_t) = \emptyset$, which completes the proof of the lemma. □

Lemma 3.5 *Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$. Let X be an ascending subset of G satisfying the cyclic-block property and $(*)$. Let $Y = \{y_i : i \geq 1\}$ be the subset of X obtained in Lemma 3.3 and suppose that $|y_i| < m(y_{i+1})$ for every $i \geq 1$. Let $j \geq 1$ and put $Y_j^* = \langle y_i : i \geq j \rangle$. Let $y = y_{k_1}^{m_1} \cdots y_{k_r}^{m_q} \in Y_j^*$, where $k_i \geq j$, $r \geq 1$, $m_i \geq 1$, and $k_u \neq k_{u+1}$. Then $c_{j,1}^y = c_{j,1}^{y_u^s}$ for a $u \in \{k_1, \ldots, k_r\}$ and $s \geq 1$.*

Proof We may use induction on $r \geq 1$. For $r = 1$ the assertion is obvious. Suppose that $r > 1$ and the assertion holds for numbers less than r. Note that $|y_j| < m(y_{k_i})$ for $i = 1, \ldots, q$ by the hypothesis. Hence, applying Lemma 3.4 we obtain a $k \in \{k_1, k_2\}$ and an $m \geq 1$ so that $c_{j,1}^{y_{k_1}^{m_1} y_{k_2}^{m_2} \cdots y_{k_r}^{m_q}} = c_{j,1}^{y_k^m y_{k_3}^{m_3} \cdots y_{k_r}^{m_q}}$. Then the induction hypothesis applies to the right side of the preceding equality. Therefore, there exist a $u \in \{k, k_3, \ldots, k_r\}$ and an $s \geq 1$ so that $c_{j,1}^{y_k^m y_{k_3}^{m_3} \cdots y_{k_r}^{m_q}} = c_{j,1}^{y_u^s}$. Then since $c_{j,1}^y = c_{j,1}^{y_u^s}$ the induction and the proof of the lemma are complete. □

Lemma 3.6 *Let the hypothesis and the notation be as in Lemma 3.5. Let $j \geq 1$. Then $[c_{j,1}^y, c_{j,1}] = 1$ for every $y \in Y_j^*$.*

Proof Put $c_j = c_{j,1}$. Let $y \in Y_j^*$. We have $c_j^y = c_j^{y_k^s}$ for a $k \geq j$ and an $s \geq 1$ by Lemma 3.5. Let $supp(c_j) = \Gamma_j$ and put $H = G_{\{\Gamma_j\}}$. If $y_k^s \notin H$, then $y_k^s(\Gamma_j) \cap \Gamma_j = \emptyset$ and since $supp(c_j^{y_k^s}) = y_k^{-s}(\Gamma_j)$ it follows that $[c_j^{y_k^s}, c_j] = 1$, and the assertion holds in this case since $c_j^y = c_j^{y_k^s}$.

Next suppose that $y_k^s \in H$. Let $c_{k,1} = c_k$, $supp(c_k) = \Gamma_k$, $|\Gamma_j| = p^{a(j)}$, and $|\Gamma_k| = p^{a(k)}$. Then $p^{a(k)-a(j)}|s$ by Lemma 3.2 and hence $s = p^{a(k)-a(j)}t$ for a $t \geq 1$. Also, $supp(y_j) \subseteq \Gamma_k$ and $c_k^{p^{a(k)-a(j)}} = c_j^{q(j)} \times v_k$ for a $v_k \in FSym(\Omega)$ and $q(j) \geq 1$ by (1) in Lemma 3.4. Hence, $c_k^s = c_j^{tq(j)} v_k^t$, but also $y_k = c_k \times z_k$ for a

$z_k \in FSym(\Omega)$. Combining these values we get $y_k^s = c_j^{tq(j)}(v_k^t z_k^s)$, where $supp(v_k z_k) \cap \Gamma_k = \emptyset$. Using this last equality we get

$$c_j^{y_k^s} = c_j^{c_j^t q(j)(v_k^t z_k^s)} = c_j^{v_k^t z_k^s} = c_j$$

and hence

$$[c_j^{y_k^s}, c_j] = [c_j, c_j] = 1,$$

which was to be shown. □

Lemma 3.7 *Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. Let $y \in G$ and let $j > 1$ so that $\alpha \in supp(y) \subset \langle c_{j,1}^{p^2} \rangle(\alpha)$ and $|c_{j,1}| = p^t$ for a $t \geq 4$. Then $[c_{j,1}^y, c_{j,1}] \neq 1$.*

Proof Put $c = c_{j,1}$. Then $supp(y) \subseteq \{\alpha, c^{p^2}(\alpha), \ldots, (c^{p^2})^{p^{t-2}-1}(\alpha)\}$ by the hypothesis. This means that if y moves an element of Ω, then it must be of the form $(c^{p^2})^k(\alpha)$ for a $0 \leq k \leq p^{t-2} - 1$.

Assume if possible that $c^y c = cc^y$. Then

$$ycy^{-1}c(\alpha) = cycy^{-1}(\alpha). \qquad (1)$$

Now

$$ycy^{-1}c(\alpha) = ycc(\alpha) = y(c^2(\alpha))$$

since y cannot move $c(\alpha)$ and

$$cycy^{-1}(\alpha) = cyc(c^{kp^2}(\alpha)) = cy(c^{kp^2+1}(\alpha)) = c^{kp^2+2}(\alpha)$$

since y cannot move $c^{kp^2+1}(\alpha)$, where $y^{-1}(\alpha) = (c^{p^2})^k(\alpha)$ and $1 \leq k \leq p^{t-2} - 1$ since $y(\alpha) \neq \alpha$. Thus the equality (1) takes the form

$$y(c^2(\alpha)) = c^{kp^2+2}(\alpha).$$

Now if $p > 2$, then $y(c^2(\alpha)) = c^2(\alpha)$ since $c^2(\alpha)$ is not of the form $(c^{p^2})^k(\alpha)$. Indeed, if $c^2(\alpha) = (c^{p^2})^k(\alpha)$, then $c^{kp^2-2}(\alpha) = \alpha$, which implies that $p^t | kp^2 - 2$ since $|c| = p^t$, which is impossible. Therefore, $c^2(\alpha) = c^{kp^2+2}(\alpha)$ and hence $\alpha = c^{kp^2}(\alpha)$, which is a contradiction since $1 \leq k \leq p^{t-2} - 1$, c is a cycle, $|c| = p^t$, and $t \geq 4$. Next suppose that $p = 2$. Again since y can move only elements of the form $(c^{p^2})^k(\alpha) = c^{4k}(\alpha)$ and since $c^2(\alpha)$ is not of this form, we get $y(c^2(\alpha)) = c^2(\alpha)$ and hence $c^2(\alpha) = c^{4k+2}(\alpha)$. Hence, $c^{4k}(\alpha) = \alpha$, which is another contradiction since $|c| = 2^t$, $t \geq 4$, and $1 \leq k \leq 2^{t-2} - 1$. □

Lemma 3.8 *Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. Let X be the ascending subset of G satisfying the cyclic-block property such that for every $x \in X$ there exists a $y \in X$ such that $m(x) < m(y)$. Then there exists an ascending subset $Z = \{z_i : i \geq 1\}$ of G so that $m(z_i) < m(z_{i+1})$ for every $i \geq 1$.*

Proof By the hypothesis we can obtain easily an infinite subset $X^* = \{x_i : i \geq 1\}$ of X so that $m(x_i) < m(x_{i+1})$ for every $i \geq 1$. We may suppose that $x_1 \neq 1$. Let d_i be a cycle of x_i of the smallest length, that is, of length $m(x_i)$ for every $i \geq 1$. Then $|d_i| < |d_{i+1}|$ for every $i \geq 1$. Choose an $\alpha \in supp(d_1)$. By the transitivity of G for every $i \geq 1$ there exists an $a_i \in G$ so that $\alpha \in supp(d_i^{a_i})$. Then $supp(d_i^{a_i}) \subset supp(d_{i+1}^{a_{i+1}})$ by [4, Lemma 2.2 since $\alpha \in supp(d_i^{a_i}) \cap supp(d_{i+1}^{a_{i+1}})$ for every $i \geq 1$. Put $z_i = x_i^{a_i}$ for every $i \geq 1$ and define $Z = \{z_i : i \geq 1\}$. Since $supp(d_i^{a_i}) \subset supp(d_{i+1}^{a_{i+1}})$ and since each $supp(d_i^{a_i})$ is a block for G it follows that $\bigcup_{i=1}^{\infty} supp(d_i^{a_i}) = \Omega$ due to the fact that every proper block is finite by the transitivity of G on Ω. Hence, it follows that Z cannot be contained in the set stabilizer of a finite subset of G and also $exp(Z)$ is infinite. Therefore, Z is an ascending subset of G. □

Proof of Theorem 1.1 Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. Let X be an ascending subset of G satisfying the cyclic-block property so that conditions (a) and (b) are satisfied. Then applying Lemma 3.8 we obtain an ascending subset $Z = \{z_i : i \geq 1\}$ of G so that $m(z_i) < m(z_{i+1})$ for every $i \geq 1$. Next we can choose an infinite subset $U = \{u_i : i \geq 1\}$ of Z so that $|u_i| < m(u_{i+1})$ for every $i \geq 1$ since the numbers $m(z_i)$ are increasing without bound. We now substitute U in place of X in Lemma 3.3. This gives an ascending subset $Y = \{y_i : i \geq 1\}$ of G so that the following hold. The cycle decomposition of each y_i can be expressed as

$$y_i = c_{i,1} \times \cdots \times c_{i,r(i)}$$

so that $supp(y_i) \subset supp(c_{i+1,1})$ and if $1 \leq j \leq r(i)$, $k \geq i$, $|c_{i,j}| = p^a$, $|c_{k,1}| = p^b$, then

$$c_{k,1}^{p^{b-a}}|_{supp(c_{i,j})} = c_{i,j}^{q(i,j)}$$

for a $q(i,j) \geq 1$. Furthermore, for each $i \geq 1$, the inequality $|y_i| < m(y_{i+1})$ is satisfied by definition of U. Thus, Lemmas 3.4, 3.5, and 3.6 can be applied to Y.

Next we may suppose that $y_1 \neq 1$. Choose an $\alpha \in supp(y_1)$. Let Δ be the smallest block such that $supp(y_1) \subseteq \Delta$ and let $|\Delta| \leq p^t$, for a $t \geq 4$. There exists a $j > 1$ so that $|c_{j,1}| \geq p^{2t}$ and $\Delta \subsetneq supp(c_{j,1})$. Put $c_j = c_{j,1}$. Then $c_j = (\alpha, c_j(\alpha), \ldots, c_j^{|c_j|-1}(\alpha))$. Now $c_j^{p^2}$ is a product of p^2 cycles each of length $\geq p^{2t-2} = p^{2(t-1)} \geq p^t$ since $t \geq 4$. Then it is easy to see that $\Delta \subseteq \langle c_j^{p^2} \rangle(\alpha)$ by the cyclic-block property since $\langle c_j^{p^2} \rangle(\alpha)$ is a block and $\alpha \in \Delta \cap \langle c_j^{p^2} \rangle(\alpha)$.

Put $Y_j^* = \langle y_i : i \geq j \rangle$. Then the application of Lemmas 3.4, 3.5, and 3.6 gives $[c_{j,1}^{y}, c_{j,1}] = 1$ for every $y \in Y^*$, but application of Lemma 3.7 gives $[c_j^{y_1}, c_j] \neq 1$, which implies that $y_1 \notin Y^*$ and so $Y^* \neq G$. However, since $\{y_i : i \geq j\}$ is ascending by definition of Y, the subgroup Y^* cannot be an FC-subgroup of G. Therefore, G cannot be an $MNFC$-group and so the proof of the theorem is complete. □

Proof of Corollary 1.2 Let G be a totally imprimitive p-subgroup of $FSym(\Omega)$, where Ω is infinite. Let X be an ascending subset of homogeneous elements of G satisfying the cyclic-block property so that X satisfies the $(*)$ condition. Then condition (a) of Theorem 1.1 is satisfied. Therefore, we need only show that condition (b) of Theorem 1.1 is satisfied. Since X is ascending by the hypothesis, $exp(X)$ is infinite and $\langle X \rangle$ is a non-FC-subgroup of G. Also, since G is locally finite, it follows that for every $x \in X$ there exists a $y \in X$ so that

$|x| < |y|$. Now the homogeneity of the elements of X shows that (b) is satisfied by X. Therefore, G cannot be $MNFC$ by Theorem 1.1. □

Proof of Corollary 1.3 Let G be the p-subgroup of $FSym(\mathbb{N}^*)$ described in Section 2. Then G satisfies the cyclic-block property by [4, Theorem 1.1]. We have $G = \langle g_k : k \geq 1\rangle$, where $g_k = u_k \times v_k$, $u_k = (a_1, \ldots, a_{p^k})$, $v_k = u_k \times \cdots \times u_k^{x_{k+1,1}^{p-1}}$, $supp(u_k) = \Delta_k$, and $supp(v_k) = \Delta_{k+1} \setminus \Delta_k$. Hence, it follows that each g_k is homogeneous; that is, $|g_k| = m(g_k) = p^k$ for every $k \geq 1$. Furthermore,

$$g_{k+1}^p|_{\Delta_k} = g_k$$

since $u_{k+1}^p = g_k$ as was shown in Section 2. Thus, G satisfies the hypothesis of Corollary 1.2 and therefore G cannot be an $MNFC$-group.

Next we show that G' cannot be $MNFC$. For each $s \geq 2$ let $Y_s = \{g_k^{-1}g_k^{g_s} : 1 \leq k < s\}$ and put $Y = \bigcup_{s\geq 2} Y_s$. Then Y is an ascending subset of homogeneous elements of G'. To see this let $1 \leq k < s$. Then $g_k^{-1}g_k^{g_s} = g_k^{-1}g_k^{u_s}$ since $supp(g_k) = \Delta_{k+1} = supp(u_{k+1}) \subseteq supp(u_s)$. Also $u_{k+1}^p = g_k$ (see Section 2). Hence $g_k^{-1}g_k^{g_{k+1}} = g_k^{-1}g_k^{u_{k+1}} = 1$. So suppose that $s > k+1$. Then $u_s(\Delta_{k+1}) \cap \Delta_{k+1} = \emptyset$. Also, $supp(g_k^{u_s}) = u_s^{-1}(supp(g_k)) = u_s^{-1}(\Delta_{k+1})$. Clearly it follows from this that $g_k^{-1}g_k^{g_s} = g_k^{-1} \times g_k^{g_s}$ and so $g_k^{-1}g_k^{g_s}$ is homogeneous since g_k is homogeneous. Furthermore, $g_k \notin G_{\{\Delta_{k-1}\}}$ since $g_k = u_k \times v_k$, $supp(u_k) = \Delta_k$ and $\Delta_{k-1} \subset \Delta_k$. Now suppose that $s > k+1$. Then also $g_k^{-1}g_k^{u_s} \notin G_{\{\Delta_{k-1}\}}$ since $\Delta_{k-1} \subset supp(g_k)$ and $g_k^{u_s} \in G_{\Delta_{k-1}}$ due to the fact that $supp(g_k) \cap supp(g_k^{u_s}) = \emptyset$. Therefore, Y is an ascending subset of homogeneous elements of G'. In particular, (b) of Theorem 1.1 is satisfied.

Finally, let $1 \leq k+1 < s$. Then

$$(g_{k+1}^{-1}g_{k+1}^{g_s})^p|_{\Delta_k} = g_{k+1}^{-p}|_{\Delta_k} = u_{k+1}^{-p}|_{\Delta_k} = g_k^{-1}|_{\Delta_k} = g_k^{-1} \times g_k^{g_s}|_{\Delta_k}$$

and so (a) of Theorem 1.1 is satisfied. Therefore, G' cannot be $MNFC$ by Theorem 1.1. (A different proof of this result is given in [5,Theorem 1.6].) □

Acknowledgment

The author is very grateful to the editor and the referee(s) for accepting this work for publication in Turkish Journal of Mathematics.

References

[1] Asar AO. On finitary permutation groups. Turk J Math 2006; 30: 101-116.

[2] Asar AO. Subgroups of finitary permutation groups. J Group Theory 2008; 11: 229-234.

[3] Asar AO. Corrigendum: Subgroups of finitary permutation groups. J Group Theory 2009; 12: 487-489.

[4] Asar AO. Totally imprimitive permutation groups with the cyclic-block property. J Group Theory 2011; 14: 127-141.

[5] Asar AO. Subgroups of totally imprimitive permutation groups. Comm Algebra 2017; 6: 2690-2707.

[6] Belyaev VV, Sesekin, NF. On infinite groups of Miller-Moreno type. Acta Math Acad Sci Hungar 1975; 26: 369-376.

[7] Belyaev VV. Minimal non-FC-groups. All Union Symposium Kiev 1980: 97-108.

[8] Belyaev VV. On the question of existence of minimal non-FC-groups. Siberian Math J 1998; 39: 1093-1095.

[9] Bhattacharjee, M, Machperson, DR, Möller, G, Neumann, PM. Notes on Infinite Permutation Groups. Lecture Notes in Mathematics, Vol. 1698. Berlin, Germany: Springer-Verlag, 1998.)

[10] Dixon JD, Moretimer, B. Permutation Groups. New York, NY, USA: Springer, 1996.

[11] Humphreys JH. A Course in Group Theory. Oxford, UK: Oxford University Press, 1996.

[12] Kargapolov MI, Merzljakov, JI. Fundamentals of the Theory of Groups. 2nd ed. Translated from the Russian by RG Burns. New York, NY, USA: Springer, 1979.

[13] Kezlan TP, Rhee NH. A characterization of the centralizer of a permutation. Missouri J Math Sci 1999; 11: 158-163.

[14] Kuzucuoglu M, Phillips R. Locally finite minimal non-FC-groups. Math Proc Cambridge 1989; 105: 417-420.

[15] Leinen F. A reduction theorem for perfect locally finite minimal non-FC-groups. Glasgow Math J 1999; 41: 81-83.

[16] Leinen F, Puglisi O. Finitary representations and images of transitive finitary permutation groups. J Algebra 1999; 222: 524-549.

[17] Neumann PM. On the structure of standard wreath products of groups. Math Z 1964; 84: 343-373.

[18] Neumann PM. The lawlessness of groups of finitary permutations. Arch Math 1975; 26: 561-566.

[19] Neumann PM. The structure of finitary permutation groups. Arch Math 1976; 27: 3-17.

[20] Pinnock CJE. Irreducible and transitive locally- nilpotent by abelian groups. Arch Math 2000; 74: 168-172.

[21] Robinson DJS. A Course in the Theory of Groups. New York, NY, USA: Springer, 1980.

[22] Tomkinson MJ. FC-Groups. Boston, MA, USA: Pitman Advanced Publishing Program, 1984.

[23] Wiegold J. Groups of finitary permutations. Arch Math 1974; 25: 466-469.