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Cardinal Hermite interpolant multiscaling functions for solving a parabolic inverse problem

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Abstract: An effective method based upon cardinal Hermite interpolant multiscaling functions is proposed for the solution of the one-dimensional parabolic partial differential equation with given initial condition and known boundary conditions and subject to overspecification at a point in the spatial domain. The properties of multiscaling functions are first presented. These properties together with a collocation method are then utilized to reduce the parabolic inverse problem to the solution of algebraic equations. The scheme described is efficient. The numerical results obtained using the present algorithms for test problems show that this method can solve the model effectively.

Key words: Parabolic inverse problem, Hermite interpolant, multiscaling functions, operational matrix, collocation method

1. Introduction

In this paper, we consider the inverse problem of finding a source parameter in the parabolic partial differential equation

$$v_t = \alpha v_{xx} + \beta v_x + p(t)v + q(x,t), \qquad 0 \le x \le \mathtt{L}, \quad 0 < t \le T_1, \tag{1}$$

with initial condition

$$v(x,0) = f(x), \qquad 0 \le x \le \mathtt{L}, \tag{2}$$

and boundary conditions

$$v(0,t) = g_0(t), \qquad 0 < t \le T_1, \tag{3}$$

$$v(1,t) = g_1(t), \qquad 0 < t \le T_1, \tag{4}$$

subject to the overspecification at a point in the spatial domain:

$$v(x^*,t) = E(t), \qquad 0 < t \le T_1, \tag{5}$$

where f, g_0, g_1, q, and $E(t)$ are known functions, α and β are known constants, while the functions v and p are unknown. Certain types of physical problems can be modeled by Eqs. (1)–(5), and have been investigated

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by many authors (see for example [1,3–6,8], and the references therein). Eq. (1) can be used to describe a heat transfer process with a source parameter present. Eq. (5) represents the temperature at a given point x^*, in a spatial domain at time t. Thus, the purpose of solving this inverse problem is to identify the source parameter that will produce at each time t a desired temperature at a given point x^* in a spatial domain [2, 15, 22, 35, 37].

The existence and uniqueness of the problem and also other applications are discussed in Cannon and Yin [6], Cannon and Lin [3], and Dehghan [9, 13]. This inverse problem as well as some other similar inverse parabolic problems has recently attracted much attention, and various numerical methods are developed for these problems (see for example, [1,8–12,14,15,20,22,23,29,30,32–38]).

In this paper, we use the cardinal Hermite interpolant multiscaling functions for solving the parabolic inverse problem. These multiscaling functions are constructed in [21] and have several advantages in applications, such as smoothness, short support, symmetry, and interpolation properties. The principal advantage of using these functions is the simplicity with explicit expressions and therefore they can be implemented efficiently. Our method consists of reducing the parabolic inverse problem to a set of algebraic equations by expanding the unknown function as multiscaling functions with unknown coefficients. The operational matrices of derivative, integration, and product are given. The idea of using operational matrices was used in the literature by several authors [19,23–28,31]. These matrices together with the Hermite scaling functions are then utilized to evaluate the unknown coefficients.

The paper is organized as follows: Section 2 is devoted to the basic formulation of the cardinal Hermite interpolant multiscaling functions required for our subsequent development. In Section 3, the proposed method is used to approximate the solution of a parabolic inverse problem. In Section 4, we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering numerical examples. Finally, Section 5 completes this paper with a brief conclusion.

2. Cardinal Hermite interpolant multiscaling functions

The cardinal Hermite interpolant scaling functions $\phi = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x))^T$ are defined in [21] by

$$\begin{aligned}\phi_1(x) &= (x+1)^4(1-4x+10x^2-20x^3)\chi_{[-1,0]}(x)\\ &\quad +(x-1)^4(1+4x+10x^2+20x^3)\chi_{[0,1]}(x),\end{aligned}$$

$$\begin{aligned}\phi_2(x) &= (x+1)^4(x-4x+10x^3)\chi_{[-1,0]}(x)\\ &\quad +(x-1)^4(x+4x+10x^3)\chi_{[0,1]}(x),\end{aligned}$$

$$\phi_3(x) = (x+1)^4(x^2/2-2x^3)\chi_{[-1,0]}(x) + (x-1)^4(x^2/2+2x^3)\chi_{[0,1]}(x),$$

$$\phi_4(x) = (x+1)^4x^3/6\chi_{[-1,0]}(x) + (x-1)^4x^3/6\chi_{[0,1]}(x),$$

where T stands for transpose and

$$\chi_{[x_0,x_1]}(x) = \begin{cases} 1, & x \in [x_0, x_1], \\ 0, & Otherwise. \end{cases}$$

It is seen that ϕ is a piecewise polynomial of degree 7, three times continuously differentiable and symmetric, supported on $[-1,1]$, has accuracy of order 8, and belongs to $W^{4.5}$. Therefore [21], $\phi \in C^{4-\epsilon}$ for any $\epsilon > 0$.

Moreover, the vector ϕ satisfies the following properties (the cardinal Hermite interpolant properties):

$$\phi(k) = \delta_k[1,0,0,0]^T, \qquad \phi'(k) = \delta_k[0,1,0,0]^T,$$
$$\phi''(k) = \delta_k[0,0,1,0]^T, \qquad \phi'''(k) = \delta_k[0,0,0,1]^T, \qquad \forall k \in \mathbb{Z}, \tag{6}$$

where δ is the Dirac sequence such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$.

Suppose

$$\phi_i^{J,k}(x) = \phi_i(2^J x - k), \quad i = 1,2,3,4, \quad J,k \in \mathbb{Z},$$

and

$$\text{ß}_{i,j,k} = \text{supp}\left[\phi_i^{j,k}(x)\right] = \text{clos}\{x : \phi_i^{j,k}(x) \neq 0\}, \quad i = 1,2,3,4,$$

It is easy to see that

$$\text{ß}_{i,j,k} = \left[2^{-j}(k-1), 2^{-j}(k+1)\right], \quad j,k \in \mathbb{Z}.$$

Define the set of indices

$$S_{i,j} = \{k : \text{ß}_{i,j,k} \cap (0,\mathtt{L}) \neq \emptyset\}, \quad j \in \mathbb{Z},$$

and it is easy to see that $S_{i,j} = \left\{0, \ldots, \mathtt{L} \times 2^j\right\}, j \in \mathbb{Z}$.

We need the biorthogonal Hermite functions intrinsically defined on $[0,1]$, and so we put

$$\phi_i^{J,k}(x) = \phi_i^{J,k}(x)\chi_{[0,\mathtt{L}]}(x), \qquad j \in \mathbb{Z}, \qquad k \in S_{i,j}, \quad i = 1,2,3,4.$$

2.1. Function approximation

For a fixed $j = J$, a function $f(x)$ defined on $L^2[0,\mathtt{L}]$ may be approximated by biorthogonal multiscaling functions as [16,23–26]

$$f(x) \approx \sum_{k=0}^{\mathtt{L}\times 2^J} \sum_{m=1}^{4} c_m^{J,k} \phi_m^{J,k}(x) = C^T \Phi_J(x), \tag{7}$$

where $\Phi_J(x)$ and C are N-vectors given by

$$\Phi_J(x) = \left[\phi_1^{J,0}(x), \phi_2^{J,0}(x), \phi_3^{J,0}(x), \phi_4^{J,0}(x)|...|\phi_1^{J,\mathtt{L}\times 2^J}(x), \phi_2^{J,\mathtt{L}\times 2^J}(x), \phi_3^{J,\mathtt{L}\times 2^J}(x), \phi_4^{J,\mathtt{L}\times 2^J}(x)\right]^T, \tag{8}$$

$$C = \left[c_1^{J,0}, c_2^{J,0}, c_3^{J,0}, c_4^{J,0}|...|c_1^{J,\mathtt{L}\times 2^J}, c_2^{J,\mathtt{L}\times 2^J}, c_3^{J,\mathtt{L}\times 2^J}, c_4^{J,\mathtt{L}\times 2^J}\right]^T, \tag{9}$$

in which $N = 4(\mathtt{L} \times 2^J + 1)$.

Because of the interpolatory nature of multiscaling function defined in (6), the coefficients $c_i^{J,k}$ are computed by

$$c_1^{J,k} = f(k/2^J), \quad c_2^{J,k} = 2^{-J} f'(k/2^J),$$
$$c_3^{J,k} = 2^{-2J} f''(k/2^J), \quad c_4^{J,k} = 2^{-3J} f'''(k/2^J), \quad k = 0,1...,\mathtt{L} \times 2^J.$$

The approximation of the two-dimensional function $u(x,t)$ for $0 \le x \le \mathtt{L}$ and $0 \le t \le T_1$, $L^2([0,\mathtt{L}] \times [0,T_1])$, is [16, 23, 25]

$$u(x,t) \approx \Phi_J^T(t) U \Phi_J(x), \tag{10}$$

where U is a block matrix with dimension $M \times N$, with $M = 4(T_1 \times 2^J + 1)$ and $N = 4(\mathtt{L} \times 2^J + 1)$. The matrix U is given by

$$U = \begin{bmatrix} U_0^0 & \cdots & U_{T_1.2^J}^0 \\ \vdots & & \vdots \\ U_0^{\mathtt{L}.2^J} & \cdots & U_{T_1.2^J}^{\mathtt{L}.2^J} \end{bmatrix}. \tag{11}$$

In Eq. (11), U_s^l, $s = 0, \dots, T_1 \times 2^J, l = 0, \dots, \mathtt{L} \times 2^J$ are 4×4 matrices. Using Eq. (6) we have

$$U_s^l(i,j) = 2^{(i+j-2)} \frac{\partial^{j-1} x}{\partial x^{j-1}} \left(\frac{\partial^{i-1} u(x,t)}{\partial t^{i-1}} \Big|_{t=s/T_1.2^J} \right) \Big|_{x=l/\mathtt{L}.2^J}, \qquad i,j = 1, \dots, 4.$$

2.2. The operational matrix of derivative for the cardinal Hermite interpolant multiscaling functions

Using Eq. (7) we can approximate the derivative of $\phi_i(2^J x - l)$ by the multiscaling functions as [16,23–26,31]

$$\phi_i'(2^J x - l) = \sum_{k=l-1}^{k=l+1} \left\{ (\phi_i'(k-l)\phi_1^{J,k}(x) + \phi_i''(k-l)\phi_2^{J,k}(x) + \phi_i'''(k-l)\phi_3^{J,k}(x) + \phi_i''''(k-l)\phi_4^{J,k}(x) \right\}, \tag{12}$$

in which $i = 1,2,3,4$ and $l = 0, \dots, \mathtt{L} \times 2^J$.

Since the function vector $\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x))^T$ has accuracy order 8, Eqs. (12) are exact. Then the differentiation of vector $\Phi_J(x)$ in Eq. (8) can be expressed as

$$\Phi_J(x) = D_\phi \Phi_J(x), \tag{13}$$

where D_ϕ is an $N \times N$ matrix as defined as

$$D_\phi = 2^J \begin{bmatrix} A_0 & B_0 & & & & \\ C_0 & A_1 & B_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & C_{\mathtt{L}.2^J-2} & A_{\mathtt{L}.2^J-1} & B_{\mathtt{L}.2^J-1} \\ & & & & C_{\mathtt{L}.2^J-1} & A_{\mathtt{L}.2^J} \end{bmatrix}. \tag{14}$$

in which

$$A_k = \begin{bmatrix} 0 & 0 & 0 & -840 \\ 1 & 0 & 0 & -480\gamma_k \\ 0 & 1 & 0 & -120 \\ 0 & 0 & 1 & -16\omega_k \end{bmatrix}, \quad B_k = \begin{bmatrix} 0 & 0 & 0 & 840\lambda_k \\ 0 & 0 & 0 & 360\lambda_k \\ 0 & 0 & 0 & 60\lambda_k \\ 0 & 0 & 0 & 4\lambda_k \end{bmatrix}, \quad C_k = \begin{bmatrix} 0 & 0 & 0 & 840\tau_k \\ 0 & 0 & 0 & -360\tau_k \\ 0 & 0 & 0 & 60\tau_k \\ 0 & 0 & 0 & -4\tau_k \end{bmatrix},$$

with

$$\gamma_k := \mathrm{sgn}(\phi_2(2^J x - k)), \;\; \omega_k := \mathrm{sgn}(\phi_4(2^J x - k)), \;\; \lambda_k := \kappa_1(2^J x - k - 1), \;\; \tau_k := \kappa_2(2^J x - k),$$

where

$$\kappa_1(x) = \begin{cases} 1 & x \le 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\kappa_2(x) = \begin{cases} 1 & x \ge 0 \\ 0 & \text{otherwise} \end{cases},$$

and

$$\mathrm{sgn}(x) = \begin{cases} 1 & x \ge 0 \\ -1 & \text{otherwise} \end{cases}.$$

2.3. The operational matrix of integration for the cardinal Hermite interpolant multiscaling functions

The integration of vector $\Phi_J(x)$ in Eq. (8) can be expressed as [16, 23]

$$\int_0^x \Phi_J(x')dx' \approx I_\Phi \Phi_J(x), \tag{15}$$

where I_Φ is $N \times N$ operational matrix of integration for multiscaling functions and can be obtained by the following process. The function $\int_0^x \phi_i(2^J x' - l)dx'$ using Eq. (7) can be approximated as

$$\int_0^x \phi_i(2^J x' - l)dx' = \sum_{k=0}^{\mathtt{L}\times 2^J} (\{\int_0^{2^J} \phi_i(2^J x - l)dt\}\phi_1^{J,k}(x) + \frac{1}{2^J}\phi_i(k-l)\phi_2^{J,k}(x)$$
$$+\frac{1}{2^{2J}}\phi_i'(k-l)\phi_3^{J,k}(x) + \frac{1}{2^{3J}}\phi_i''(k-l)\phi_4^{J,k}(x)\}),$$

for $i = 1, 2, 3, 4$ and $l = 0, \dots, \mathtt{L} \times 2^J$. Then it can be shown that

$$I_\Phi = \frac{1}{2^J}\begin{bmatrix} R_1 & R_2 & R_2 & \cdots & \cdots & R_2 \\ & R_3 & R_4 & \cdots & \cdots & R_4 \\ & & R_3 & \ddots & \cdots & R_4 \\ & & & \ddots & \ddots & \vdots \\ & & & & R_3 & R_4 \\ & & & & & R_3 \end{bmatrix},$$

where

$$R_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; R_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{3}{28} & 0 & 0 & 0 \\ \frac{1}{84} & 0 & 0 & 0 \\ \frac{1}{1680} & 0 & 0 & 0 \end{bmatrix}, \; R_3 = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 \\ -\frac{3}{28} & 0 & 1 & 0 \\ \frac{1}{84} & 0 & 0 & 1 \\ -\frac{1}{1680} & 0 & 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2.4. The operational matrix of the product for the cardinal Hermite interpolant multiscaling functions

The property of the product of two multiscaling functions vectors will be as follows [27, 28]

$$Z\Phi_J(x)\Phi_J^T(x) \approx \Phi_J^T(x)\tilde{Z}, \tag{16}$$

where $Z = \left[z_1^{J,0}, z_2^{J,0}, z_3^{J,0}, z_4^{J,0}|...|z_1^{J,\mathtt{L}\times 2^J}, z_2^{J,\mathtt{L}\times 2^J}, z_3^{J,\mathtt{L}\times 2^J}, z_4^{J,\mathtt{L}\times 2^J}\right]^T$ is a known constant vector, and $\tilde{Z}$ is an $N \times N$ matrix. This matrix is called the operational matrix of product and can be obtained as

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_0 & & & \\ & \tilde{Z}_1 & & \\ & & \ddots & \\ & & & \tilde{Z}_{\mathtt{L}.2^J} \end{bmatrix}, \tag{17}$$

where $\tilde{Z}_k$ is a 4×4 matrix given by

$$\tilde{Z}_k = \begin{bmatrix} z_1^{J,k} & & & \\ z_2^{J,k} & z_1^{J,k} & & \\ z_3^{J,k} & 2z_2^{J,k} & z_1^{J,k} & \\ z_4^{J,k} & 3z_3^{J,k} & 3z_2^{J,k} & z_1^{J,k} \end{bmatrix}, \qquad k = 0, \dots, \mathtt{L} \times 2^J. \tag{18}$$

2.5. Convergence of multiscaling bases

Theorem 1. Suppose that the function $f : [0, \mathtt{L}] \to \mathbb{R}$ is eight times continuously differentiable, $f \in C^8[0, \mathtt{L}]$, and the interpolation operator P_J mapping function f into space V_J is as Eq. (7); then the error bound is given by

$$||f - P_J||_\infty := \max_{x\in[0,\mathtt{L}]} |f(x) - P_J(x)| \le \frac{2^{-8J-8}}{8!}||f^{(8)}||_\infty.$$

Moreover, the first derivative of f we have

$$||f' - P_J'||_\infty := \max_{x\in[0,\mathtt{L}]} |f'(x) - P_J'(x)| \le \frac{2^{-7J-6}}{6!}||f^{(8)}||_\infty, \tag{19}$$

where $P_J(x) = C^T\Phi_J(x)$.

Proof See [7]. □

Remark. By using the above theorem, we can conclude if $f \in L^2[0, \mathtt{L}]$ and $C^T\Phi_J(x)$ is the approximation of f out of $\Phi_J(x)$ in Eq. (8) then

$$|| \int_0^x f(s)ds - \int_0^x C^T\Phi_J(s)ds||_\infty := \max_{x\in[0,\mathtt{L}]} | \int_0^x (f(s) - C^T\Phi_J(s))ds| \le \gamma_f, \tag{20}$$

where

$$\gamma_f := \frac{2^{-8J-8}}{8!}||f^{(8)}||_\infty.$$

Lemma 1 *Suppose that the function* $f:[0,L]\to\mathbb{R}$ *and* $f\in L^2[0,L]$. *If* $C^T\Phi_J(x)$ *is the approximation of* f *out of* Φ_J *in Eq. (8) and we use Eq. (15) for approximation of integration of* f *then the error bounds are given by*

$$||\int_0^x f(s)ds - C^T I_\phi \Phi_J(x)||_\infty \leq \Lambda_f + \gamma_f,$$

where $\Lambda := M_1 2^{-J-7}$. *Constant* M_1 *is a bound for*

$$f_{M_1}(x) := \frac{5}{2}f(x) + \frac{5}{4}\times 2^{-J}f'(x) + \frac{1}{4}\times 2^{-2J}f''(x) + 2^{-3J}f'''(x) - \frac{5}{2}f(x+\frac{1}{2^J}) + \frac{5}{4}\times 2^{-J}f'(x+\frac{1}{2^J}) - \frac{1}{4}\times 2^{-2J}f''(x+\frac{1}{2^J}) + 2^{-3J}f'''(x+\frac{1}{2^J}), \tag{21}$$

such that $|f_{M_1}(x)|\leq M_1$ *for all* $x\in[0,\mathbf{L}]$.

Proof Theorem 1 provides a interpolation error for $x\in[x_i,x_{i+1}]$ that arises if the polynomial P_i replaces f

$$f(x) - P_i(x) = \frac{(x-x_i)^4(x-x_{i+1})^4}{8!}f^{(8)}(\xi), \quad x,\xi\in[x_i,x_{i+1}], \tag{22}$$

so that Hermite interpolation conditions

$$P_i(x_i) = f(x_i), \qquad P_i(x_{i+1}) = f(x_{i+1}), \qquad x_i = \frac{i}{2^J}, \quad i = 0,\dots,\mathtt{L}\times 2^J - 1$$

are satisfied.

By taking the seventh derivative of function vector $\phi = (\phi_1(x),\phi_2(x),\phi_3(x),\phi_4(x))^T$ and using Eq. (22) we get

$$(\int_0^x \phi_1(2^Js-k)ds - I_\phi\Phi_J(x))[4k+1,1] = -2^{7J}100800\frac{(x-\frac{k-1}{2^J})^4(x-\frac{k}{2^J})^4}{8!}\chi_{(\frac{k-1}{2^J},\frac{k}{2^J})}(x) + 2^{7J}100800\frac{(x-\frac{k}{2^J})^4(x-\frac{k+1}{2^J})^4}{8!}\chi_{(\frac{k}{2^J},\frac{k+1}{2^J})}(x),$$

$$(\int_0^x \phi_1(2^Js-k)ds - I_\phi\Phi_J(x))[4k+2,1] = 2^{7J}50400\frac{(x-\frac{k-1}{2^J})^4(x-\frac{k}{2^J})^4}{8!}\chi_{(\frac{k-1}{2^J},\frac{k}{2^J})}(x) + 2^{7J}50400\frac{(x-\frac{k}{2^J})^4(x-\frac{k+1}{2^J})^4}{8!}\chi_{(\frac{k}{2^J},\frac{k+1}{2^J})}(x),$$

$$(\int_0^x \phi_1(2^Js-k)ds - I_\phi\Phi_J(x))[4k+3,1] = -2^{7J}10080\frac{(x-\frac{k-1}{2^J})^4(x-\frac{k}{2^J})^4}{8!}\chi_{(\frac{k-1}{2^J},\frac{k}{2^J})}(x) + 2^{7J}10080\frac{(x-\frac{k}{2^J})^4(x-\frac{k+1}{2^J})^4}{8!}\chi_{(\frac{k}{2^J},\frac{k+1}{2^J})}(x),$$

$$(\int_0^x \phi_1(2^Js-k)ds - I_\phi\Phi_J(x))[4k+4,1] = 2^{7J}840\frac{(x-\frac{k-1}{2^J})^4(x-\frac{k}{2^J})^4}{8!}\chi_{(\frac{k-1}{2^J},\frac{k}{2^J})}(x) + 2^{7J}840\frac{(x-\frac{k}{2^J})^4(x-\frac{k+1}{2^J})^4}{8!}\chi_{(\frac{k}{2^J},\frac{k+1}{2^J})}(x),$$

where $k = 0,\dots,\mathtt{L}\times 2^J$.

Using the above equations and after simplification we have

$$C^T(\int_0^x \Phi_J(s)ds - I_\phi \Phi_J(x)) =$$

$$2^{7J} \sum_{k=0}^{\mathtt{L}\times 2^J-1} \{\frac{5}{2}f(\frac{k}{2^J}) + \frac{5}{4}\times 2^{-J}f'(\frac{k}{2^J}) + \frac{1}{4}\times 2^{-2J}f''(\frac{k}{2^J}) + 2^{-3J}f'''(\frac{k}{2^J}) - \frac{5}{2}f(\frac{k+1}{2^J})$$

$$+\frac{5}{4}\times 2^{-J}f'(\frac{k}{2^J}) - \frac{1}{4}\times 2^{-2J}f''(\frac{k}{2^J}) + 2^{-3J}f'''(\frac{k}{2^J})\}(x-\frac{k}{2^J})^4(x-\frac{k+1}{2^J})^4|_{(\frac{k}{2^J},\frac{k+1}{2^J})}.$$

We can write the above expression as

$$C^T(\int_0^x \Phi_J(s)ds - I_\phi \Phi_J(x)) = \sum_{k=0}^{\mathtt{L}\times 2^J-1} H[k]h_k(x),$$

where

$$H[k] := \frac{5}{2}f(\frac{k}{2^J}) + \frac{5}{4}\times 2^{-J}f'(\frac{k}{2^J}) + \frac{1}{4}\times 2^{-2J}f''(\frac{k}{2^J}) + 2^{-3J}f'''(\frac{k}{2^J}) - \frac{5}{2}f(\frac{k+1}{2^J})$$
$$+\frac{5}{4}\times 2^{-J}f'(\frac{k}{2^J}) - \frac{1}{4}\times 2^{-2J}f''(\frac{k}{2^J}) + 2^{-3J}f'''(\frac{k}{2^J}),$$
$$h_k(x) := 2^{7J}(x-\frac{k}{2^J})^4(x-\frac{k+1}{2^J})^4|_{(\frac{k}{2^J},\frac{k+1}{2^J})}.$$

Then by computing the maximum value of the function $h_k(x)$, we get

$$||\int_0^x C^T\Phi_J(x')dx' - C^T I_\phi \Phi_J(x)||_\infty \leq M_1 2^{-J-8}, \tag{23}$$

where M_1 is a bound for $f_{M_1}(x)$ given in Eq. (21) On the other hand,

$$||\int_0^x f(x')dx' - C^T I_\phi \Phi(x)||_\infty \leq ||\int_0^x f(x')dx' - \int_0^x C^T\Phi(x)||_\infty + ||\int_0^x C^T\Phi(x) - C^T I_\phi\Phi(x)||_\infty.$$

By using Eqs. (20) and (23), the result can be obtained. □

Lemma 2 *Suppose that the function* $f:[0,\mathtt{L}]\to\mathbb{R}$ *and* $f\in L^2[0,\mathtt{L}]$. *If* $C^T\Phi_J(x)$ *is the approximation of* f *out of* Φ_J *in Eq.* (8) *and we use Eq.* (14) *for approximation of derivative* f *then the error bounds are given by*

$$||f'(x) - C^T D\Phi_J(x)||_\infty \leq \frac{2^{-7J-6}}{6!}||f^{(8)}||_\infty \tag{24}$$

Proof By using Eqs (13) and (19) the result can be obtained. □

3. Solving the inverse problem

In this section, we solve the inverse problem in Eq. (1) with initial and boundary conditions in Eqs. (2)–(5). Employing a pair of transformations [11, 23]

$$r(t) = e^{-\int_0^t p(s)ds}, \tag{25}$$

$$u(x,t) = r(t)v(x,t), \tag{26}$$

Eqs. (1)–(5) can be replaced by

$$u_t = \alpha u_{xx} + \beta u_x + r(t)q(x,t), \qquad 0 \le x \le \mathtt{L}, \quad 0 < t \le T_1, \tag{27}$$

subject to

$$u(x,0) = f(x), \qquad 0 \le x \le \mathtt{L}, \tag{28}$$

$$u(0,t) = r(t)g_0(t), \qquad 0 < t \le T_1, \tag{29}$$

$$u(\mathtt{L},t) = r(t)g_1(t), \qquad 0 < t \le T_1, \tag{30}$$

and

$$u(x^*,t) = r(t)E(t), \qquad 0 < t \le T_1. \tag{31}$$

With this transformation, $p(t)$ disappeared, and its role is represented implicitly by $r(t)$. It is seen that if we have $u(x,t)$ and $r(t)$, then by using Eqs. (25) and (26), $v(x,t)$ and $p(t)$ can be obtained as

$$v(x,t) = \frac{u(x,t)}{r(t)}, \qquad 0 \le x \le \mathtt{L}, \qquad 0 < t \le T_1. \tag{32}$$

$$p(t) = -\frac{r'(t)}{r(t)}, \qquad 0 < t \le T_1. \tag{33}$$

By integrating Eq. (27) from 0 to t and using Eq. (28) we have

$$u(x,t) - f(x) = \alpha \int_0^t u_{xx}(x,t')dt' + \beta \int_0^t u_x(x,t')dt' + \int_0^t r(t')q(x,t')dt' \tag{34}$$

Similarly to Eq. (10), we expand $q(x,t)$ as

$$q(x,t) \approx \Phi_J^T(t) Q \Phi_J(x), \tag{35}$$

where Q is a $M \times N$ known matrix, and the entries of matrix Q can be found similarly to Eq. (12). The function $r(t)$ may be expanded in terms of multiscaling functions as

$$r(t) \approx \sum_{k=0}^{T_1 \times 2^J} \sum_{m=1}^{4} h_m^{J,k} \phi_m^{J,k}(t) = H^T \Phi_J(t), \tag{36}$$

where $H = \left[h_1^{J,0}, h_2^{J,0}, h_3^{J,0}, h_4^{J,0} | ... | h_1^{J,T_1\times 2^J}, h_2^{J,T_1\times 2^J}, h_3^{J,T_1\times 2^J}, h_4^{J,T_1\times 2^J}\right]^T$ is an unknown vector. Using Eqs. (10), (14), and (15), we get

$$\int_0^t u_{xx}(x,t')dt' = \left(\int_0^t \Phi_J^T(t')dt'\right) U \left(\frac{d^2\Phi_J(x)}{dx^2}\right) \approx \Phi_J^T(t) I_\Phi^T U D_\Phi^2 \Phi_J(x), \tag{37}$$

$$\int_0^t u_{x}(x,t')dt' = \left(\int_0^t \Phi_J^T(t')dt'\right) U \left(\frac{d\Phi_J(x)}{dx}\right) \approx \Phi_J^T(t) I_\Phi^T U D_\Phi \Phi_J(x), \tag{38}$$

and by using Eqs. (10), (35), and (36) we have

$$\int_0^t r(t')q(x,t')dt' \approx \left(\int_0^t H^T \Phi_J(t')\Phi_J^T(t')dt'\right) Q\Phi_J(x). \tag{39}$$

Thus we get

$$H^T\Phi_J(t)\Phi_J^T(t) \approx \Phi_J^T(t)\tilde{H}, \tag{40}$$

where $\tilde{H}$ can be calculated similarly to matrix $\tilde{Z}$ in Eq. (17). Employing Eqs. (15) and (40) in Eq. (39) we have

$$\int_0^t r(t')q(x,t')dt' \approx \Phi_J^T(t) I_\phi^T \tilde{H} Q \Phi_J(x). \tag{41}$$

Expanding $f(x)$ in terms of the multiscaling functions yields

$$f(x) \approx \sum_{k=0}^{\mathtt{L}\times 2^J} \sum_{m=1}^{4} f_m^{J,k}\phi_m^{J,k}(x) = \Phi_J^T(t) F \Phi_J(x), \tag{42}$$

in which F is a known $M \times N$ matrix and can be obtained similarly to Eq. (11).

Applying Eqs. (10), (37), (38), (41), and (42) in Eq. (34), we get

$$\Phi_J^T(t)[U - F - I_\phi^T \tilde{H} Q - \alpha I_\phi^T U D_\Phi^2 - \beta I_\phi^T U D_\Phi]\Phi_J(x) = 0. \tag{43}$$

By collocating Eq. (43) in $M(N-2)$ points (x_i, t_j), $i = 1, \cdots, N-2, j = 1, \ldots, M$, where $x_i = \mathtt{L}\frac{i}{(N-1)}, i = 1, \ldots, N-2$ and $t_j = T_1\frac{j}{M}, j = 1, \cdots, M$, we obtain

$$R(x_i, t_j) = \Phi_J^T(t_j)[U - F - I_\phi^T \tilde{H} Q - \alpha I_\phi^T U D_\Phi^2 - \beta I_\phi^T U D_\Phi]\Phi_J(x_i) = 0, \tag{44}$$

Moreover, using Eqs. (29)–(31) and Eq. (10) we obtain

$$\Phi_J^T(t) U \Phi_J(0) = g_0(t)\Phi_J^T(t) H, \tag{45}$$

$$\Phi_J^T(t) U \Phi_J(\mathtt{L}) = g_1(t)\Phi_J^T(t) H, \tag{46}$$

$$\Phi_J^T(t) U \Phi_J(x^*) = E(t)\Phi_J^T(t) H. \tag{47}$$

Eqs. (45)–(47) are collocated at points

$$t_j = T_1 \frac{j-1}{M-1}, \quad j = 1, \cdots, M. \tag{48}$$

The number of unknown coefficients $u_{i,j}$ and z_i is equal to $M(N+1)$ and can be obtained from Eqs. (44)–(47). Consequently, $u(x,t)$, and $r(t)$ given in Eqs. (10) and (36) can be calculated. Finally, using Eqs. (32) and (33), the unknowns $v(x,t)$ and $p(t)$ can be found.

4. Illustrative examples

In this section, three examples are given to demonstrate the applicability and accuracy of our method. These examples are chosen such that their analytical solutions are known. Example 1 was considered in [33] by sinc-collocation method, and the absolute value of the errors (AVE) for $p(t)$ and $u(x, 0.5)$ for $N_1 = 5, 10, 15$ and 20 were given. In [33], the step size in the definition of the sinc-collocation method are appropriately chosen depending on N_1. For this example we report the AVE of our method with $J = 1$ and $J = 2$ with [33], for $N_1 = 15$ and $N_1 = 20$. Example 2 was first considered in [15] by using the finite-difference method and also solved in [33]. For this example we report the AVE of our method with $J = 2$ for $p(t)$ with [15] by using the second-order three-point forward time centered space (FTCS) method, the second-order three-point backward time centered space (BTCS) procedure, and the Crank–Nicolson (3,3) technique, and with [33] with $N_1 = 10$. Example 3 was first considered in [1] by using the finite difference method and also solved in [36] by the reproducing kernel space. For this example the root-mean-square (RMS) errors for $v(x,t)$ and $p(t)$ for several $N_2 \times M_2$, where N_2 and M_2 are step sizes, were considered in [1] and in [36]. For example 3, we report the RMS errors of our method for $J = 2$ and $J = 3$ with the RMS errors in [1] and [36] for $N_2 \times M_2 = 56 \times 56$ and $N_2 \times M_2 = 8 \times 8$, respectively.

Example 1 *Consider the equation [33]*

$$v_t = v_{xx} + 2v_x + p(t)v,$$

with

$$v(x,0) = e^{-x}(1+\cos x),$$
$$v(0,t) = e^{t^2-\sin t}(1+e^{-t}),$$
$$v(1,t) = e^{t^2-1-\sin t}(1+e^{-t}\cos 1),$$
$$v(0.26,t) = e^{(t^2-0.26-\sin t)}(1+0.96639e^{-t}).$$

This problem has the exact solution

$$v(x,t) = e^{t^2-x-\sin t}(1+e^{-t}\cos x),$$

and $p(t) = 2t - 1 + \cos t$. We solved this example by using the present method. In Tables 1 and 2, we compare the absolute error of our method with $J = 1$ and $J = 2$ together with the sinc-collocation method presented in [33] for $p(t)$ and $v(x, 0.5)$, respectively.

Example 2 *In this example to show the stability of our method, consider the following perturbed equation [15, 33]:*

Table 1. The computational results for $p(t)$, for Example 1.

t_i	$p(t)$ exact	$N_1 = 15$ error in[33]	$N_1 = 20$ error in [33]	$J = 1$ present method	$J = 2$ present method
0.05	0.10125	$3.5e-03$	$1.1e-03$	$8.3e-06$	$1.8e-07$
0.1	0.204996	$2.9e-04$	$4.0e-05$	$3.0e-05$	$1.8e-07$
0.15	0.311229	$1.4e-04$	$2.8e-05$	$1.3e-05$	$2.3e-07$
0.2	0.419933	$9.7e-05$	$2.8e-05$	$3.2e-05$	$1.9e-07$
0.25	0.531088	$1.4e-04$	$2.3e-05$	$6.4e-05$	$5.0e-09$
0.3	0.644664	$7.6e-05$	$1.7e-05$	$5.2e-05$	$2.4e-07$
0.35	0.760627	$7.8e-05$	$1.2e-05$	$2.6e-06$	$2.4e-07$
0.4	0.878939	$1.0e-04$	$1.9e-05$	$4.3e-05$	$3.0e-07$
0.45	0.999553	$8.4e-06$	$1.7e-06$	$3.8e-05$	$2.6e-07$
0.5	1.12242	$1.1e-04$	$1.7e-05$	$1.2e-06$	$3.3e-09$
0.55	1.24748	$8.6e-05$	$1.4e-05$	$3.2e-05$	$3.2e-07$
0.6	1.37466	$6.9e-05$	$7.0e-06$	$5.4e-05$	$3.3e-07$
0.65	1.50392	$1.8e-04$	$2.1e-05$	$1.7e-05$	$3.9e-07$
0.7	1.63516	$4.6e-05$	$1.4e-06$	$5.1e-05$	$3.4e-07$
0.75	1.76831	$2.8e-04$	$3.9e-05$	$9.6e-05$	$9.2e-09$
0.8	1.90329	$1.7e-04$	$4.1e-05$	$7.8e-05$	$4.0e-07$
0.85	2.04002	$6.3e-04$	$1.1e-04$	$3.0e-06$	$4.0e-07$
0.9	2.17839	$8.1e-04$	$3.2e-04$	$6.2e-05$	$5.5e-07$
0.95	2.31832	$7.2e-03$	$3.4e-04$	$4.5e-05$	$5.8e-07$

Table 2. The computational results for $v(x, 0.5)$, for Example 1.

x	$v(x, 0.5)$ exact	$N_1 = 15$ error in [33]	$N_1 = 20$ error in [33]	$J = 1$ present method	$J = 2$ present method
0.05	1.21431	$8.4e-04$	$7.2e-05$	$2.6e-08$	$2.6e-11$
0.1	1.15346	$1.3e-04$	$9.7e-06$	$2.4e-08$	$5.3e-11$
0.15	1.09462	$2.0e-04$	$2.2e-05$	$2.1e-08$	$3.4e-11$
0.2	1.03779	$6.0e-05$	$1.3e-05$	$2.0e-08$	$2.2e-11$
0.25	0.982992	$9.1e-05$	$9.7e-06$	$2.1e-08$	$4.4e-11$
0.3	0.930201	$6.4e-05$	$7.2e-06$	$2.4e-08$	$1.9e-11$
0.35	0.87941	$8.7e-05$	$2.0e-05$	$2.9e-08$	$3.9e-11$
0.4	0.830602	$6.4e-05$	$4.7e-06$	$3.4e-08$	$4.0e-11$
0.45	0.783755	$2.5e-05$	$6.8e-06$	$3.8e-08$	$2.6e-11$
0.5	0.738844	$4.9e-05$	$1.2e-05$	$4.2e-08$	$1.3e-11$
0.55	0.695839	$3.2e-05$	$6.8e-06$	$4.6e-08$	$1.8e-12$
0.6	0.654708	$2.1e-04$	$7.2e-05$	$5.0e-08$	$1.4e-11$
0.65	0.615414	$8.2e-05$	$6.7e-06$	$5.2e-08$	$3.2e-11$
0.7	0.577919	$4.3e-05$	$2.1e-05$	$5.3e-08$	$5.2e-11$
0.75	0.542182	$2.8e-04$	$3.6e-05$	$5.3e-08$	$5.2e-11$
0.8	0.508161	$1.6e-04$	$2.7e-05$	$5.1e-08$	$4.0e-11$
0.85	0.475809	$2.2e-04$	$2.0e-05$	$4.7e-08$	$3.6e-12$
0.9	0.445081	$5.0e-04$	$6.3e-05$	$3.9e-08$	$3.9e-11$
0.95	0.415928	$6.7e-04$	$8.0e-05$	$2.5e-08$	$8.4e-11$

$$v = v_{xx} + p(t)v + (1+\varepsilon_1)q(x,t),$$

with

$$v(x,0) = (1+\varepsilon_2)f(x),\ v(0,t) = e^{-t^2},\ v(1,t) = -e^{-t^2},\ v(0.25,t) = \sqrt{2}e^{-t^2},$$

where

$$q(x,t) = (\pi^2 - (t+1)^2)e^{-t^2}(\cos(\pi x) + \sin(\pi x)),$$
$$f(x) = \cos(\pi x) + \sin(\pi x),$$

and $0 < \varepsilon_1, \varepsilon_2 \ll 1$ *are small perturbation parameters.*

The exact solution of the original problem (when $\varepsilon_1 = 0, \varepsilon_2 = 0$) is [15, 33]

$$v(x,t) = e^{-t^2}(\cos(\pi x) + \sin(\pi x)),$$

and

$$p(t) = 1 + t^2.$$

In this Example we have reported the MaxErrors in Table 3 for different values of ε_1 and ε_2. This table shows that small errors in the initial data cannot result in larger errors in the answers. In Figures 1(a) and 1(b), errors of approximate solutions for $p(t)$ and $v(x,1)$ with perturbation parameters $\varepsilon_1 = 10^{-3}$ and $\varepsilon_2 = 10^{-3}$ are presented, respectively.

Table 3. Max errors for $v(x,1)$ and $p(t)$ for different values of ε_1 and ε_2.

ε_1	ε_2	MaxError of $p(t)$	MaxError of $v(x,1)$
$\frac{1}{2}\times 10^{-2}$	$\frac{1}{2}\times 10^{-2}$	$4.43e-02$	$1.8253e-07$
10^{-3}	10^{-3}	$8.92e-03$	$1.8245e-07$
10^{-4}	10^{-3}	$1.37e-03$	$1.8241e-07$
10^{-5}	10^{-4}	$9.53e-04$	$1.8239e-07$
0	0	$8.10e-04$	$1.8239e-07$

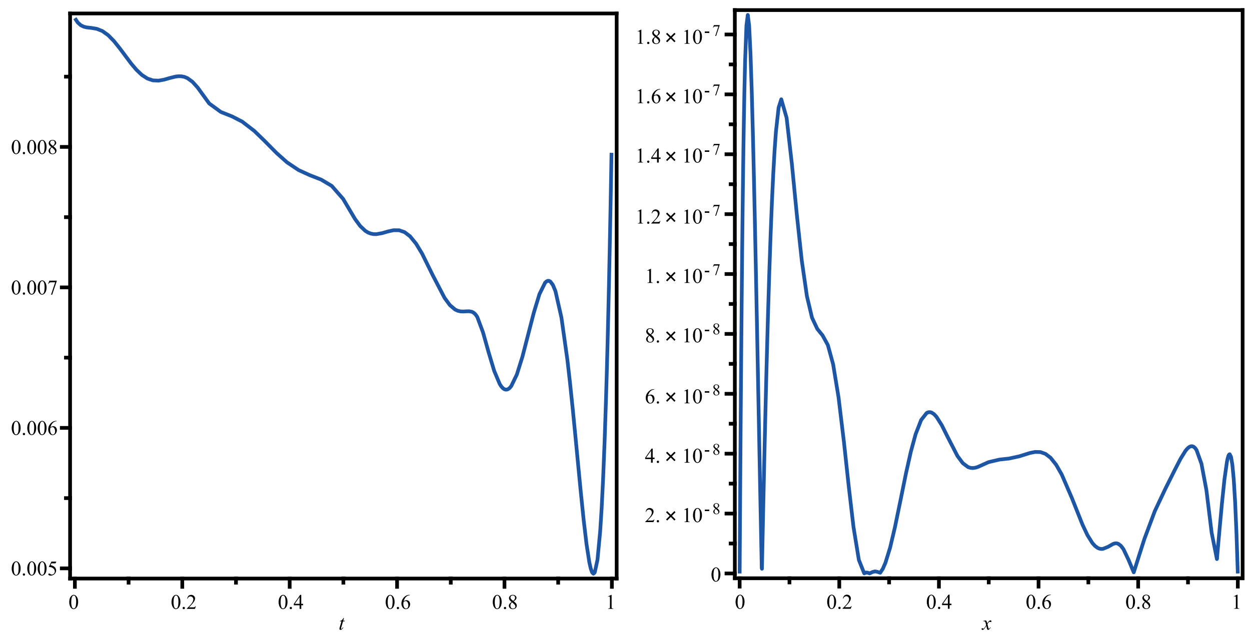
Furthermore, for the purpose of comparison, in Table 4, we compare the absolute error of our method for the original equation with $J = 2$ for $p(t)$ with [15] by using the FTCS method, the BTCS procedure, the Crank–Nicolson (3,3) technique, and with [33] $N = 10$. The graphs of error of approximate solutions for $v(x,1)$ at levels of $J = 1$ and $J = 2$ are reported in Figures 2(a) and 2(b), respectively.

Example 3 *Consider the equation [36]*

$$v_t = v_{xx} + 2v_x + p(t)v - (2 + xt^2)e^t, \tag{49}$$

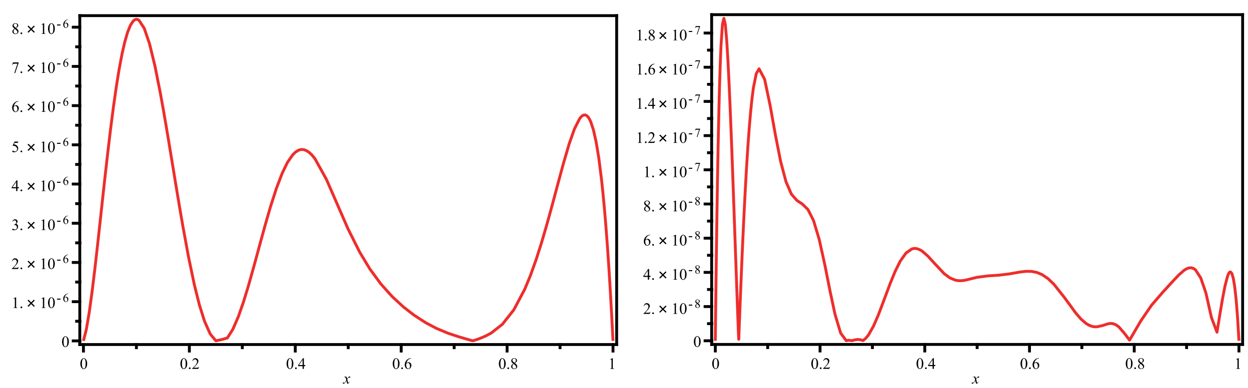
with

$$v(x,0) = x, \quad v(0,t) = 0, \quad v(1,t) = e^t, \quad v(0.5,t) = \frac{1}{2}e^t.$$



(a) (b)

Figure 1. Plot of absolute error for the solutions $p(t)$ (a) and $v(x,1)$ (b) for Example 2 with $J = 2$ and perturbation parameters $\varepsilon_1 = 10^{-3}$ and $\varepsilon_2 = 10^{-3}$..



(a) (b)

Figure 2. Plot of absolute errors for $v(x,1)$ for Example 2 with $\varepsilon_1 = 0$, $\varepsilon_2 = 0$; (a): $J = 1$, (b): $J = 2$.

The exact solution of this problem is

$$v(x,t) = xe^t,$$

and

$$p(t) = 1 + t^2.$$

We solved Eq. (49) *by using the present method with* $J = 2$ *and* $J = 3$. *In Table 5, we report the RMS errors*

Table 4. The computational results for $p(t)$ with $\varepsilon_1 = 0, \varepsilon_2 = 0$, for Example 2.

t	$p(t)$ exact	FTCS error in [15]	BTCS error in [15]	Crank-Nicolson error in [15]	$N=10$ error in [33]	$J=2$ present method
0.05	1.0025	$6.9e-03$	$6.5e-03$	$5.3e-03$	$5.4e-03$	$6.0e-05$
0.1	1.0100	$6.8e-03$	$6.3e-03$	$5.2e-03$	$2.5e-03$	$4.2e-05$
0.15	1.0225	$6.7e-03$	$6.1e-03$	$5.1e-03$	$2.2e-03$	$7.6e-05$
0.2	1.0400	$6.7e-03$	$6.2e-03$	$5.0e-03$	$8.3e-04$	$7.2e-05$
0.25	1.0625	$6.7e-03$	$6.6e-03$	$5.5e-03$	$6.9e-04$	$2.7e-06$
0.3	1.0900	$6.6e-03$	$6.7e-03$	$5.4e-03$	$4.2e-04$	$2.5e-05$
0.35	1.1225	$6.5e-03$	$6.8e-03$	$5.7e-03$	$4.3e-04$	$8.2e-06$
0.4	1.1600	$6.4e-03$	$6.5e-03$	$5.6e-03$	$3.1e-04$	$3.4e-05$
0.45	1.2025	$6.2e-03$	$6.9e-03$	$5.5e-03$	$3.0e-04$	$1.3e-05$
0.5	1.2500	$6.3e-03$	$6.7e-03$	$5.3e-03$	$3.2e-04$	$7.4e-06$
0.55	1.3025	$6.3e-03$	$6.7e-03$	$5.2e-03$	$2.7e-04$	$8.3e-05$
0.6	1.3600	$6.3e-03$	$6.6e-03$	$5.2e-03$	$2.9e-04$	$9.9e-05$
0.65	1.4225	$6.2e-03$	$6.6e-03$	$5.1e-03$	$4.7e-04$	$6.9e-05$
0.7	1.4900	$6.2e-03$	$6.5e-03$	$5.1e-03$	$4.9e-04$	$1.1e-05$
0.75	1.5625	$6.1e-03$	$6.5e-03$	$5.1e-03$	$6.5e-04$	$2.2e-05$
0.8	1.6400	$6.0e-03$	$6.4e-03$	$5.0e-03$	$8.1e-04$	$3.6e-05$
0.85	1.7225	$6.0e-03$	$6.3e-03$	$5.0e-03$	$8.0e-04$	$3.1e-04$
0.9	1.8100	$6.1e-03$	$6.3e-03$	$5.3e-03$	$4.5e-03$	$6.5e-04$
0.95	1.9025	$6.0e-03$	$6.2e-03$	$5.3e-03$	$5.4e-03$	$8.1e-04$

of our method for $J=2$ and $J=3$ for $v(x,t)$ and $p(t)$. Figures 3(a) and 3(b) show the plot of error for $p(t)$ and $v(x,1)$, respectively.

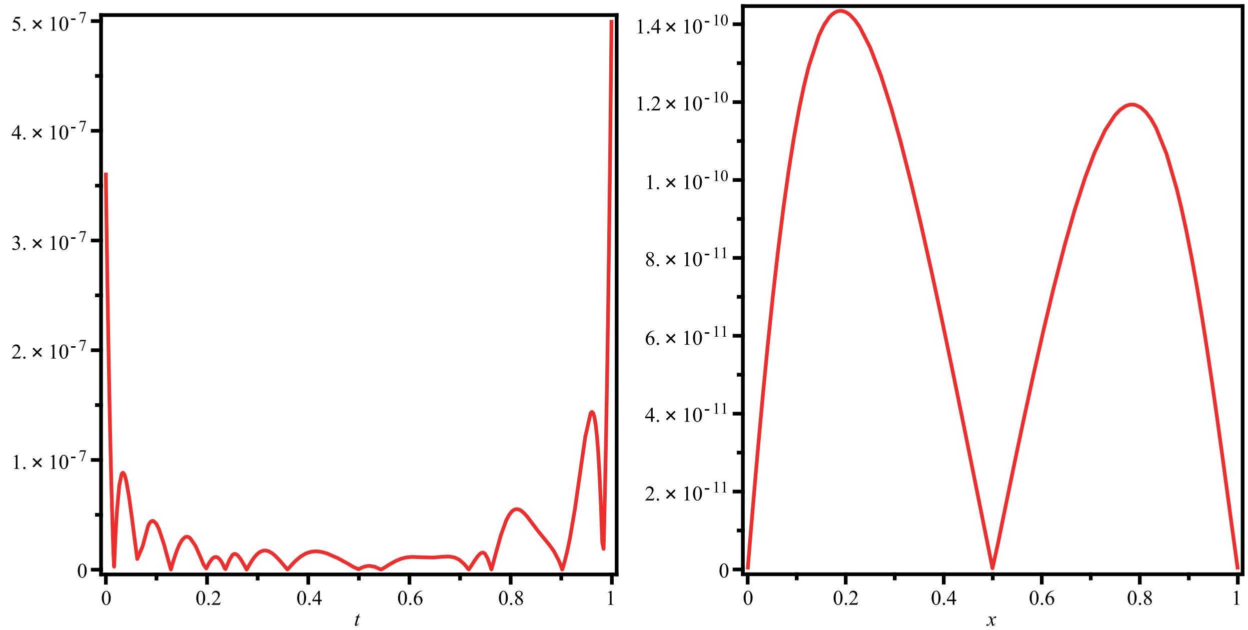
Table 5. RMS errors for $v(x,t)$ and $p(t)$.

The methods	RMS errors of $v(x,t)$	RMS errors of $p(t)$
Method in [1] for $N_2 \times M_2 = 56 \times 56$	$4.3e-04$	$6.0e-02$
Method in [36] for $N_2 \times M_2 = 8 \times 8$	$2.0e-05$	$9.1e-05$
Present method for $J=2$	$5.3e-10$	$6.6e-08$
Present method for $J=3$	$6.5e-11$	$2.3e-08$

From the obtained results presented in Figures 1, 2, and 3 and Tables 1–5 one can observe that the propose method is working well and provide good results.

5. Conclusion

This paper focused on solving the one-dimensional parabolic inverse problem subject to temperature overspecification. The cardinal Hermite interpolant multiscaling functions on $[0,1]$ were employed and the operational matrices of derivative, integration, and product for them were calculated. As it is seen from numerical examples, the method provides accurate solutions.



(a) (b)

Figure 3. Plot of absolute error for the solutions $p(t)$ (a) and $v(x,1)$ (b) for Example 3 with $J=2$.

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