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Convergence Analysis of Parabolic Basis Functions for Solving Systems of Linear and Nonlinear Fredholm Integral Equations

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Abstract: In this paper, a computational method based on a hybrid of parabolic and block-pulse functions is proposed to solve a system of linear and special nonlinear Fredholm integral equations of the second kind. The convergence and error bound are analyzed. Numerical examples are given to illustrate the efficiency of the method.

Key words: Hybrid, parabolic functions, block-pulse functions, Fredholm integral equations, fixed-point iteration method, error bound analysis

1. Introduction

Second kind integral equations have recently attracted attention due to their wide applications in various areas of science and engineering; for example, many problems in plasma physics [6] or electrical engineering [7] result in solving some second kind integral equations.

Usually the explicit solution of an integral equation system is difficult to derive. Hence it is necessary to seek efficient numerical solutions. There are many different basis functions such as the Adomian decomposition method [2, 21], Legendre collocation method [18], Tau method [9], method of Taylor's expansion [10, 19], homotopy perturbation method [11], method of spline collocation [5, 13, 17], Runge–Kutta [15, 25], Sinc-collocation method [8, 20], block-pulse functions [4, 12] and hat function [3] that have been used to get approximate solutions of integral equations.

Recently, the idea of hybrid functions has been exploited to improve the convergence rate of the numerical solution of integral equations. For example, the combination of block-pulse functions with Chebyshev polynomials [23, 24], the combination of block-pulse functions with Legendre polynomials [16, 22] and the combination of block-pulse functions with Bernstein polynomials [14] have been investigated.

This paper considers the combination of block-pulse function and parabolic functions (BPPFs) to solve the system of linear Fredholm integral equations (SLFIEs) and system of special nonlinear Fredholm integral equations (SSNFIEs) of the second kind

$$X(s) = Y(s) + \lambda \int_0^1 K(s,t)X(t)dt, 0 \leq s \leq 1, \tag{1.1}$$

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and

$$X(s) = Y(s) + \lambda \int_0^1 K(s,t) X^{\alpha}(t) dt, 0 \leq s \leq 1, \tag{1.2}$$

where $K(s,t)$ and $Y(s)$ are known, $X(s)$ is unknown, and α is a positive integer.

The method is based on a simple algorithm that converts a system of nonlinear Fredholm integral equations into a system of nonlinear algebraic equations, which is then solved by fixed-point iteration.

In Section 2, we introduce BPPFs and derive their properties. In section 3, we demonstrate the established numerical method to solve SLFIEs of the second kind. In section 4, the error bound and convergence of the proposed method are discussed. The SSNFIE is studied in section 5. Sections 6 and 7 are respectively devoted to numerical examples and concluding remarks.

2. BPPFs

A set of hybrid of parabolic with block-pulse functions $\varphi_i(t)$ ($i = 0, 1, 2, ..., n$) is usually defined in the unit interval $[0, 1)$ where n is even and $h = \frac{1}{n}$ as:

$$\varphi_0(t) = \begin{cases} \frac{(t-h)^2}{h^2}, & 0 \leq t < h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

$$\varphi_{2i+1}(t) = \begin{cases} \frac{(t-2ih)((2i+2)h-t)}{h^2}, & 2ih \leq t < (2i+2)h, \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

$$i = 0, ..., n/2 - 1,$$

$$\varphi_{2i+2}(t) = \begin{cases} \frac{(t-(2i+1)h)^2}{h^2}, & (2i+1)h \leq t < (2i+2)h, \\ \frac{(t-(2i+3)h)^2}{h^2}, & (2i+2)h \leq t < (2i+3)h, \\ 0 & \text{otherwise}, \end{cases} \tag{2.3}$$

$$i = 0, ..., n/2 - 2,$$

$$\varphi_n(t) = \begin{cases} \frac{(t-(1-h))^2}{h/2}, & 1-h \leq t < 1, \\ 0, & otherwise, \end{cases} \tag{2.4}$$

According to the definition of BPPFs, we have

$$\varphi_i(t)\varphi_j(t) = 0, |i-j| \geq 2, \tag{2.5}$$

and

$$\varphi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.6}$$

We can consider $\Phi(t)$ as an $(n+1)$ -vector:

$$\Phi(t) = [\varphi_0(t), ..., \varphi_n(t)]^T. \tag{2.7}$$

By representation above, it follows that

$$\Phi(t)\Phi^T(t) = \begin{bmatrix} \varphi_0^2 & \varphi_0\varphi_1 & & & \\ \varphi_0\varphi_1 & \varphi_1^2 & \varphi_1\varphi_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \varphi_{n-1}\varphi_n \\ & & & \varphi_{n-1}\varphi_n & \varphi_n^2 \end{bmatrix}, \tag{2.8}$$

$$\int_0^1 \Phi(t)\Phi^T(t)dt = P, \tag{2.9}$$

where

$$P = \frac{h}{5}\begin{bmatrix} 1 & 2/3 & & & & & \\ 2/3 & 16/3 & 2/3 & & & & \\ & 2/3 & 2 & 2/3 & & & \\ & & 2/3 & 16/3 & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & & & 2/3 \\ & & & & & 2/3 & 1 \end{bmatrix}. \tag{2.10}$$

The expansion of a function $x(t)$ over $[0,1)$ in a series of $\varphi_i(t)$, $i = 0, 1, ..., n$, may be written as

$$x(t) \approx M_n x(t) := \sum_{i=0}^{n} x_i \varphi_i(t) = X^T \Phi(t),$$

where

$$X = [x_0, x_1, ..., x_n]^T, \Phi(t) = [\varphi_0(t), ..., \varphi_n(t)]^T, x_i = x(ih).$$

Lemma 2.1 *Let $D = C^3([0,1))$, $x \in D$, and M_n be defined by $M_n x(t) = \sum_{i=0}^{n} x_i \varphi_i(t)$, where n is even and $\varphi_i(t)$ is i th BPPF. Then*

$$\|x - M_n x\|_\infty \leq \frac{h^3}{15}\left\|x^{(3)}\right\|_\infty.$$

Proof $M_n x(t)$ is written within $t \in [t_i, t_{i+2}]$, $t_i = ih$,

$$M_n x(t) = x_i \varphi_i(t) + x_{i+1}\varphi_{i+1}(t) + x_{i+2}\varphi_{i+2}(t),$$

which yields a polynomial of degree two for $M_n x(t)$; by using interpolation function error at the points t_i, t_{i+1}, and t_{i+2}, we have

$$x(t) - M_n x(t) = \frac{1}{6}(t - t_i)(t - t_{i+1})(t - t_{i+2})x^{(3)}(\eta_i), t_i \leq \eta_i \leq t_{i+2}.$$

Now, $|(t - t_i)(t - t_{i+1})(t - t_{i+2})| \leq .385h^3$; hence

$$|x - M_n x| \leq \frac{h^3}{15}\left|x^{(3)}(\eta)\right|, \quad 0 \leq \eta \leq 1.$$

Therefore $\|x - M_n x\|_\infty \leq \frac{h^3}{15} \left\|x^{(3)}\right\|_\infty$.

Now the expansion of a function $k(s,t)$ in a series of $\varphi_i(t)$, $i = 0, 1, ..., n$, may be written as

$$k(s,t) \cong \Phi^T(x)\Psi\Phi(t),$$

where $\Phi(x)$ is $n+1$ component BPPFs vectors and Ψ is an $(n+1)\times(n+1)$ coefficient matrix with entries $a_{ij} = k(ih, jh)$. □

3. Applying BPPFs to solve SLFIEs

Definition 3.1 *Let $D = C^3([0,1))$ and subspace $D_n \subset C^3([0,1))$; the set of all functions that are piecewise linear on $[0,1)$ then D^m is a Banach space with norm that is defined as*

$$\|X\| = \max_{1\leq i\leq m} \sup_{s\in[0,1]} |x_i(s)|,$$

Definition 3.2 *A projection operator $M_n : D^m \to D_n^m$ is defined as*

$$M_n X(s) = [M_{n1}x_1(s), ..., M_{nm}x_m(s)]^T, \tag{3.1}$$

where

$$M_{ni}x_i(s) = \sum_{j=0}^{n} x_i(t_j)\varphi_j(s), \quad i-1, \dots, m.$$

It [25] can be shown that

$$\forall \quad X \in D_n^m, \quad M_n X(s) = X(s).$$

Definition 3.3 *An operator $\kappa : D^m \to D^m$ is defined as $\kappa X = [\kappa_1 X, ..., \kappa_m \mathrm{X}]^T$, where $\kappa_i X = \int_0^1 (k_{i1}(s,t)x_1(t) + \cdots + k_{im}(s,t)x_m(t))dt$, and the norm of κ is defined as*

$$\|\kappa\| = \max_{1\leq i\leq m} \max_{1\leq j\leq m} \sup_{s\in[0,1]} \int_0^1 |k_{ij}(s,t)|\, dt.$$

We consider BPPFs to solve the SLFIEs of the second kind:

$$X(s) = Y(s) + \lambda \int_0^1 K(s,t)X(t)dt, \tag{3.2}$$

where

$$X(s) = [x_1(s), ..., x_m(s)]^T, Y(s) = [y_1(s), ..., y_m(s)]^T, K(s,t) = [k_{i,j}(s,t)]_{i,j=1,\dots,m}.$$

We can write the system of integral equations (3.2) in the operator form

$$(I - \lambda\kappa)X = Y, \tag{3.3}$$

where $\kappa = [\kappa_{i,j}]_{i,j=1,\dots,m}$ and $\kappa_{ij}x_j(s) = \int_0^1 k_{ij}(s,t)x_j(t)dt$.

We consider the ith equation of (3.2) as

$$x_i(s) = y_i(s) + \lambda \int_0^1 (k_{i1}(s,t)x_1(t) + \cdots + k_{im}(s,t)x_m(t))dt; \tag{3.4}$$

according to the expansion of a function respect to BPPFs, we can write

$$k_{ij}(s,t) \cong \Phi^T(s)\Psi_{ij}\Phi(t). \tag{3.5}$$

$$x_i(t) \cong \Phi^T(t)X_i, \tag{3.6}$$

Substituting (3.6) into (3.4) we have

$$\Phi^T(s)X_i \cong \Phi^T(s)Y_i + \lambda\Phi^T(s)(\Psi_{i1}(\int_0^1 \Phi(t)\Phi^T(t)dt)X_1 + \cdots$$
$$+ \Psi_{im}(\int_0^1 \Phi(t)\Phi^T(t)dt)X_m))$$

Replacing the approximation sign with $=$, we have

$$X_i = Y_i + \lambda(\Psi_{i1}PX_1 + \cdots + \Psi_{im}PX_m) \tag{3.7}$$

The matrix form of the above system can be expressed as

$$(I - \lambda\hat{\Psi})\hat{X} = \hat{Y}, \tag{3.8}$$

where I is $(m(n+1) \times m(n+1))$ identity matrix.

By solving the system (3.8) we obtain $m(n+1)$ unknowns and then it also allows getting approximate numerical values at other points by another method such as interpolation or applying BPPFs as

$$x_i(s) \approx \sum_{j=0}^{n} x_i(s_j)\varphi_j(s) = X_i^T\Phi(s), i = 1, ..., m. \tag{3.9}$$

Now, we approximate the solution of (3.2) by solving X_n from the equation

$$M_n(I - \lambda\kappa)X_n = Y, X_n \in D_n^m,$$

or

$$(I - \lambda M_n\kappa)X_n = Y \tag{3.10}$$

4. Error bound analysis

In this section, error bound analysis is discussed.

Lemma 4.1 *Let $D = C^3([0,1))$, $X \in D^m$, and $M_n : D^m \to D_n^m$ be defined by (3.1).*

Then

$$\|X - M_nX\|_\infty \leq \frac{h^3}{15}\left\|X^{(3)}\right\|_\infty .$$

Proof For $t_i \le s < t_{i+2}$, we can write

$$X - M_n X \cong [\frac{1}{6}(s-t_i)(s-t_{i+1})(s-t_{i+2})x_1^{(3)}(\eta_1), \ldots,$$
$$\frac{1}{6}(s-t_i)(s-t_{i+1})(s-t_{i+2})x_m^{(3)}(\eta_m)]^T.$$

Therefore

$$\|X - M_n X\|_\infty \le \frac{h^3}{15}\left\|X^{(3)}\right\|_\infty.$$

□

Lemma 4.2 *Let D be a Banach space, and let $\{M_n\}$ be a family of bounded projections on D with*

$$\lim_{n\to\infty} M_{\mathrm{n}}X = X, \qquad X \in D.$$

Let $\kappa : D \to D$ be compact. Then

$$\lim_{n\to\infty} \|\kappa - M_{\mathrm{n}}\kappa\| = 0.$$

Proof See [1] . □

Now with the assumptions of lemma 4.1, we conclude that $\lim_{n\to\infty} \|\kappa - M_{\mathrm{n}}\kappa\| = 0$.

Theorem 4.3 *Assume $\kappa : D \to D$ is bounded, with D a Banach space, and assume $I - \kappa : D \to D$ is one-to-one and onto. Further assume $\lim_{n\to\infty} \|\kappa - M_{\mathrm{n}}\kappa\| = 0$; then*

1. *for all sufficiently large n, say $n \ge N$, the operator $(I - M_n\kappa)^{-1}$ exists as a bounded operator from D to D and is uniformly bounded, i.e. $\sup_{n\ge N} \left\|(\mathrm{I} - M_{\mathrm{n}}\kappa)^{-1}\right\| < \infty$,*
2. $\lim_{n\to\infty} \left\|(I - M_n\kappa)^{-1}\right\| = \left\|(I-\kappa)^{-1}\right\|$,
3. $X - X_n = (I - X_n\kappa)^{-1}(X - M_n X)$,
4. $\frac{1}{\|(I - M_n\kappa)\|}\|X - M_n X\| \le \|X - X_n\| \le \left\|(I - M_n\kappa)^{-1}\right\| \ \|X - M_n X\|$,
5. $\|X - X_n\| \le \left\|(I-\kappa)^{-1}\right\|(\|Y - M_n Y\| + \|\kappa - M_n\kappa\| \ \|X_n\|)$.

Proof See [3, 1]. □

Now if $\lim_{n\to\infty} \|\kappa - M_{\mathrm{n}}\kappa\| = 0$, then we conclude that X_{n} is convergent to X.

5. Nonlinear SFIEs

Consider a (2×2) special nonlinear SFIE of the second kind

$$\begin{cases} x_1(s) = y_1(s) + \int_0^1 k_{11}(s,t)x_1^{\alpha_1}(t)dt + \int_0^1 k_{12}(s,t)x_2^{\alpha_2}(t)dt, \\ x_2(x) = y_2(s) + \int_0^1 k_{21}(s,t)x_1^{\alpha_1}(t)dt + \int_0^1 k_{22}(s,t)x_2^{\alpha_2}(t)dt. \end{cases} \tag{5.1}$$

Now, we approximate functions x_i, y_i, k_{ij}, and $x_i^{\alpha_i}$ $(i,j=1,2)$ with respect to BPPFs. Substituting into (5.1) and eliminating $\Phi^T(x)$, we obtain

$$\begin{cases} X_1(s) \cong Y_1(s) + K_{11}PX_{\alpha_1 1} + K_{12}PX_{\alpha_2 2}, \\ X_2(s) \cong Y_2(s) + K_{21}PX_{\alpha_1 2} + K_{22}PX_{\alpha_2 2}, \end{cases} \tag{5.2}$$

where

$$X_{\alpha_i i} = [x_i^{\alpha_i}(0), x_i^{\alpha_i}(h)..., x_i^{\alpha_i}(nh)]^T.$$

The system of Equations (5.2) is a $(2(n+1)\times 2(n+1))$ nonlinear system of algebraic equations that we solve by fixed-point iteration method.

6. Numerical examples

In this section, we present some examples and their numerical results.

Example 6.1 *Consider the following SLFIEs:*

$$\begin{cases} x_1(s) = \sin(s) - .301s^2 - .382s + \int_0^1 s^2 t x_1(t)dt + \int_0^1 st x_2(t)dt, \\ x_2(s) = \cos(s) - .301s^2 - .382s + \int_0^1 s^2 t x_1(t)dt + \int_0^1 st x_2(t)dt, \end{cases}$$

where $y_1(s) = \sin(s) - .301s^2 - .382s$, $y_2(s) = \cos(s) - .301s^2 - .382s$, $k_{11}(s,t) = s^2 t$, $k_{12}(s,t) = st$,

$$k_{21}(s,t) = s^2 t, k_{22}(s,t) = st,$$

with the exact solutions $x_1(s) = \sin(s)$ *and* $x_2(s) = \cos(s)$. *The system* $(I - \lambda\hat{\Psi})\hat{X} = \hat{Y}$ *for* $n = 2$ *is calculated as*

$$\hat{Y}^T = [0, 0.2132, 0.1585, 1, 0.611, -0.1426]$$

and

$$I - \hat{\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -.0104 & .9375 & -.0521 & -.0208 & -.125 & -.1041 \\ -.0417 & -.25 & .7917 & -.0417 & -.25 & -.2083 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -.0104 & -.0625 & -.052 & -.0208 & .875 & -.1042 \\ -.0417 & -.25 & -.2083 & -.0417 & -.25 & .7917 \end{bmatrix}.$$

The numerical results are shown in Table 1 and Figures 1 and 2.

Example 6.2 *Consider the following SLFIEs:*

Table 1.

Table 1	True solution	BPPF method	True solution	BPPF method
$N = 32$	$x-1$	$x-1$	$x-2$	$x-2$
$s = 0.0$	0.	0.	1.	1.
h	0.312449	0.0312463	0.999512	0.999513
$4h$	0.124657	0.124679	0.992198	0.992202
$8h$	0.247404	0.247408	0.968912	0.968916
$12h$	0.366273	0.366272	0.930508	0.930507
$16h$	0.479426	0.479417	0.877583	0.877574
$20h$	0.585097	0.585077	0.810963	0.810943
$24h$	0.681639	0.681603	0.731689	0.731653
$28h$	0.767544	0.767488	0.640997	0.640941
1	0.841471	0.841391	0.540302	0.540223

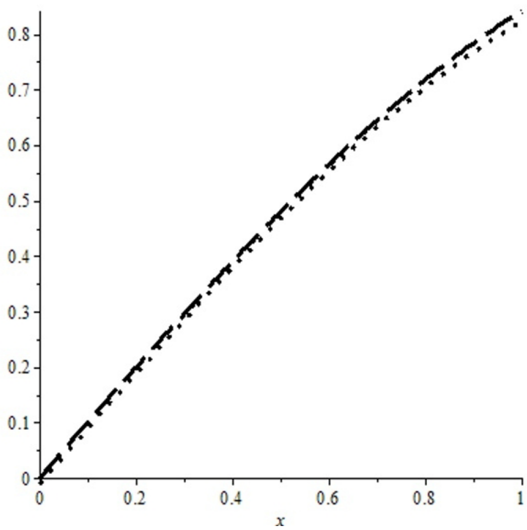

Figure 1. Dot Approximate solution Dash: Exact solution $X1$

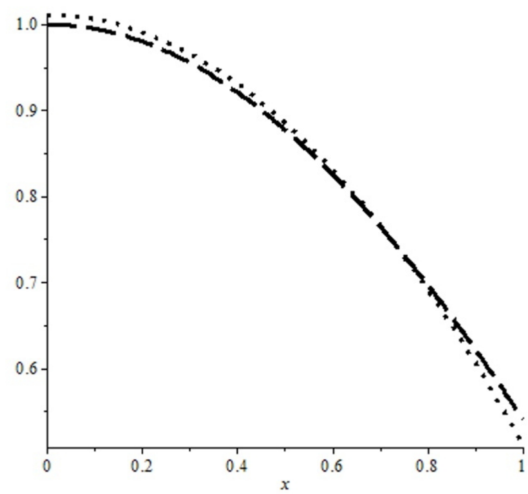

Figure 2. Dot Approximate solution Dash: Exact solution $X2$

$$\begin{cases} x_1(s) = -\frac{4s^3}{3} + \frac{7s^2}{4} + \frac{s}{15} + \frac{5}{6} + \int_0^1 (s-t)^3 x_1(t)dt + \int_0^1 (s-t)^2 x_2(t)dt, \\ x_2(s) = -\frac{3s^3}{2} + \frac{7s^2}{6} + \frac{3s}{4} + \frac{11}{12} + \int_0^1 (s-t)^2 x_1(t)dt + \int_0^1 (s-t)^3 x_2(t)dt, \end{cases}$$

with the exact solutions $x_1(s) = s^2 + 1$ *and* $x_2(s) = s + 1$. *The numerical results are shown in Table 2*

Example 6.3 *Consider the following nonlinear SFIEs:*

$$\begin{cases} x_1(s) = -3/4e^s + 0.5e^{s-2} + e^{-s} + \int_0^1 e^{s+t} x_1^3(t)dt - \int_0^1 te^s x_2^2(t)dt, \\ x_2(s) = -.01s + 2\sin(s) + 0.1se^{-3s}\cos(1) + 0.3s\sin(1)e^{-3} - \sin(s-1) - 2\cos(s-1) + s \\ + \int_0^1 s\sin(t) x_1^3(t)dt - \int_0^1 \cos(s-t) x_2^2(t)dt, \end{cases}$$

with the exact solutions $x_1(s) = e^{-s}$ *and* $x_2(s) = s$.

The numerical results are shown in Table 3.

Table 2.

Table 2	True solution	BPPF method	True solution	BPPF method
$N = 32$	$x-1$	$x-1$	$x-2$	$x-2$
$s = 0.0$	1.	1.00007	1.	0.999962
h	1.00098	1.00104	1.03125	1.03121
$4h$	1.01562	1.01567	1.12500	1.12495
$8h$	1.06250	1.06253	1.25000	1.24995
$12h$	1.14062	1.14065	1.37500	1.37497
$16h$	1.25000	1.25004	1.50000	1.49999
$20h$	1.39062	1.39068	1.62500	1.62503
$24h$	1.56250	1.56259	1.75000	1.75008
$28h$	1.76562	1.76576	1.87500	1.87515
1	2.	2.00020	2.	2.00022

Table 3.

Table 3	True solution	BPPF method	True solution	BPPF method
$N = 32$	$x-1$	$x-1$	$x-2$	$x-2$
$s = 0.0$	1.	1.0	0.	-0.000231
h	0.969233	0.969236	0.0312500	0.0310190
$4h$	0.882497	0.882508	0.125000	0.124659
$8h$	0.778801	0.778821	0.250000	0.249769
$12h$	0.687289	0.687316	0.375000	0.374859
$16h$	0.606531	0.606561	0.500000	0.499769
$20h$	0.535261	0.535296	0.625000	0.624659
$24h$	0.472367	0.472403	0.750000	0.749859
$28h$	0.416862	0.416900	0.875000	0.874659
1	0.367879	0.367918	1.	0.999769

7. Conclusion

As a main feature of our approach, we show that the suggested hybrid block-pulse and parabolic functions can improve the convergence rate to be of $O(h^3)$ order. The order of convergence of the hat function [3] method is $O(h^2)$ and the order of convergence of the block-pulse functions [12] method is $O(h)$.

References

[1] Atkinson KE. The Numerical Solution of Integral Equations of the Second Kind. New York, NY, USA: Cambridge University Press, 1997.

[2] Babolian E, Biazar J, Vahidi AR. The decomposition method applied to systems of Fredholm integral equations of the second kind. Appl Math Comput 2004; 148: 443-452.

[3] Babolian E, Mordad M. A numerical method for solving systems of linear and nonlinear integral equations of the second kind by hat basis functions. Comput Math Appl 2011; 62: 187-198.

[4] Balakumar V, Murugesan K. Numerical solution of Volterra integral-algebraic equations using block pulse functions. Appl Comput Math 2015; 263: 165-170.

[5] Chuong NM, Tuan NV. Spline collocation methods for a system of nonlinear FredholmVolterra integral equations. Acta Math Vietnamica 1996; 21: 155-169.

[6] Farengo R, Lee YC, Guzdar PN. An electromagnetic integral equation: application to micro tearing modes. Phys Fluids 1983; 26: 3515-3523.

[7] Graham IG. Collocation method for two dimensional weakly singular integral equations. J Austral Math Soc 1981; 22: 456-473.

[8] Hashemizadeh E, Rostami M. Numerical solution of Hammerstein integral equations of mixed type using the Sinc-collocation method. J Comput Appl Math 2015; 279: 31-39.

[9] Ivaz K, Mostahkam BS. Newton-Tau numerical solution of a system of nonlinear Fredholm integral equations of second kind. Appl Comput Math 2006; 5: 201208.

[10] Jafarzadeh Y, Keramati B. Numerical method for a system of integro-differential equations by Lagrange interpolation. Asian-European Journal of Mathematics. Doi: 10.1142/S1793557116500777, 2016; 9: 1-6.

[11] Javidi M, Golbabai A. A numerical solution for solving system of Fredholm integral equations by using homotopy perturbation method. Appl Math Comput 2007; 189: 1921-1928.

[12] Jiang ZH, Schaufelberger W. Block Pulse Functions and Their Applications in Control Systems. Berlin: Springer-Verlag 1992.

[13] Mahmoudi Z, Rashidinia J, Babolian E. Spline collocation for nonlinear Fredholm integral equations. International Journal of Mathematical Modeling and Computations 2011; 1: 69-75.

[14] Maleknejad K, Mohsenyzadeh M, Hashemizadeh E. Hybrid Orthonormal Bernstein and Block-Pulse Functions for Solving Fredholm Integral Equations. Proceedings of the World Congress on Engineering; July 3–5, London, UK. 2013, pp. 143-151.

[15] Maleknejad K. Shahrezaee, M. Using RungeKutta method for numerical solution of the system of Volterra integral equations. Appl Math Comput 2004; 149: 399-410.

[16] Marzban HR, Razzaghi M. Optimal control of linear delay system via hybrid of block-pulse and Legendre polynomials. J Franklin Inst B 2004; 341: 279-293.

[17] Micula S. A spline collocation method for FredholmHammerstein integral equations of the second kind in two variables. Appl Math Comput 2015; 265: 352-357.

[18] Nemati S, Numerical solution of VolterraFredholm integral equations using Legendre collocation method. J Comput Appl Math 2015; 278: 29-36.

[19] Rashidinia J, Tahmasebi A. System of nonlinear Volterra integro-differential equations. Numer Algorithms 2012; 59: 197-212.

[20] Rashidinia J, Zarebnia M. Convergence of approximate solution of system of Fredholm integral equations. J Math Anal Appl 2007; 333: 1216-1227.

[21] Sadeghi Goghary H, Javadi SH, Babolian E. Restarted Adomian method for system of nonlinear Volterra integral equations. Appl Math Comput 2005; 161: 745-751.

[22] Singh VK, Pandcy RK, Singh S. A stable algorithm for Hankel transforms using hybrid of block-pulse and Legendre polynomials. Comput Phys Commun 2010; 181: 1-10.

[23] Tavassoli Kajani M, Vencheh AH. Solving second kind integral equations with hybrid Chebyshev and block-pulse functions. Appl Math Comput 2005; 163: 71-77.

[24] Wang XT, Li YM. Numerical solutions of integro differential system by hybrid of general block-pulse functions the second Chebyshev polynomials. Appl Math Comput 2009; 209: 266-272.

[25] Zhang L, Fuming M. PouzetRungeKuttaChebyshev method for Volterra integral equations of the second kind. J Comput Appl Math 2015; 288: 323-331.