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Terminal value problem for causal differential equations with a Caputo fractional derivative

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Abstract: In this paper, we have given new definitions and obtained the unique solution of a fractional causal terminal value problem by combining the technique of generalized quasilinearization in the sense of upper and lower solutions.

Key words: Causal operator, terminal value problem, Caputo fractional derivative, quasilinearization method, quadratic convergence, upper and lower solutions

1. Introduction

Recently, the study of differential equations [6] with causal operators [13] has rapidly developed and some results are assembled in [8,13,19,26]. The theory of causal operators is a powerful tool unifying the fractional order differential equations [4,16,25,27], ordinary differential equations [1,8,11,28], integro-differential equations [23], differential equations with finite or infinite delay, Volterra integral equations [23], and neutral functional equations [8,13,20]. There has been rapidly growing interest in the study of fractional differential equations [2,4,5,13,16,18,21,22,25,27] because recent investigations in science and engineering have indicated that the dynamics of many systems can be described more accurately by using differential equations of a noninteger order.

It has recently been shown that causal differential equations [2,8,9,13,19,20,26] provide excellent models for real world problems [8] and its real time applications in a variety of disciplines. This is not only the main advantage of causal differential equations in comparison with the traditional models [12] and there is growing interest in this new area to study the concept of causal dynamic systems [8,13]. The theory of a terminal value problem [1,3,10,12,21,23,24] for ordinary differential equations is more complicated than that of the initial value problems of ordinary differential equations. Hence, it is an interesting theory to study.

The study of a terminal value problem for ordinary differential equations by using the method of upper and lower solutions can be found in [12], as the information is given at the end point of the interval and one has to work backwards to find the initial value at which the solution must start in order to reach the prescribed value at the end point of the interval. This problem becomes more interesting in the case of a fractional differential equation [4,16,18,21,22,25,27], where it closely resembles a boundary value problem [7,17], in the sense that the initial value is inherently involved in the definition of the differential operator and the terminal value provides the condition at the right end point of the interval.

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The method of the generalized quasilinearization technique [3,7,11,13–15,17,21,25–28] not only offers monotone sequences that converge uniformly to the solutions of nonlinear fractional causal differential equations as in the generalized monotone iterative technique, but they also show that the convergence is quadratic. It is an advantage to employ numerical methods for real world applications. In a fractional causal terminal value problem ($FCTVP$) is used to obtain upper and lower sequences in terms of the solutions of a linear $FCTVP$ and bound the solutions of a given nonlinear $FCTVP$. Moreover, we have also shown that these monotone sequences converge to the unique solution of the nonlinear equation uniformly and quadratically.

2. Preliminaries

In this section, we present some definitions and basic results that are needed in our subsequent work.

Let $E = C[J, X]$, where J is an appropriate time interval, and X represents either finite or infinite dimensional space, depending on the requirement of the context, so that E is a function space.

The operator $Q : E \to E$ is said to be a causal operator if for each couple of elements x, y in E such that $x(s) = y(s)$ for $0 \leq t_0 \leq s \leq t$ the equality $(Qx)(s) = (Qy)(s)$ holds for $0 \leq t_0 \leq s \leq t,\ t < T$; T is a given number.

If E is a space of measurable functions on $[t_0, T)$ for $t_0 \geq 0$, then the definition needs a slight modification, requiring the property to be valid almost everywhere on $[t_0, T]$. One can point out that for causal operators a notation identical to what is encountered for a general equation with a memory can be stated as follows. A representation of the form

$$(Qx)(t) = Q(t, x_t),$$

where for each $t \in [t_0, T)$, $Q(t, x_t)$ is a functional on E that takes values in X, for each t, while the whole family of functionals, $t \in [t_0, T)$, defines the operator from $E = C([t_0, T),\ X)$ to itself.

Let $0 \leq t_0$ and $t_0 < T$ be arbitrary and let $E = C[[t_0, T], \mathbb{R}^n]$ be a function space. The map $Q : E \to E$ is said to be a causal or a nonanticipative map if $x,\ y \in E$ has the property that if $x(s) = y(s)$ for $t_0 \leq s < t$ then $Q(s, x_s) = Q(s, y_s),\ t_0 \leq s \leq t,\ t < T$.

Next, we give the definition of and relationship between the Riemann–Liouville and Caputo fractional differential equations. The Riemann–Liouville fractional causal terminal value problem is given by

$$D^q u(t) = Q(t, u_t),\ u(T) = u^T = u(t)(T-t)^{1-q}\,|_{t=T}, \tag{2.1}$$

where $0 < q < 1$. The corresponding Volterra fractional integral equation is given by

$$u(t) = u^T(t) + \frac{1}{\Gamma(q)} \int_t^T (t-\tau)^{q-1} Q(\tau, u_\tau)\, d\tau, \tag{2.2}$$

where $u^T(t) = \frac{u^T (T-t)^{q-1}}{\Gamma(q)}$ and $\Gamma(q)$ is the standard Gamma function.

The fractional causal terminal value problem of the Caputo type is given by

$${}^c D^q u(t) = Q(t, u_t),\ u(T) = u_T \tag{2.3}$$

where $0 < q < 1$ and the terminal value T and the solution $u(T, t_0, u_0) = u_T$. If $u \in C^q([[t_0, T], \mathbb{R}^n])$ satisfies (2.3), it also satisfies the Volterra fractional integral

$$u(t) = u_T(t) + \frac{1}{\Gamma(q)} \int_t^{\infty} (t-\tau)^{q-1} Q(\tau, u_\tau)\, d\tau, \tag{2.4}$$

and vice versa. The relationship between the two types of fractional derivatives is given by

$$^cD^q u(t) = D^q(u(t) - u(T)). \tag{2.5}$$

Let $p = 1-q$ and $C_p([t_0, T], \mathbb{R}) = \{u : u \in C([t_0, T], \mathbb{R}) \text{ and } (T-t)^p u(t) \in C([t_0, T], \mathbb{R})\}$ and consider the fractional terminal value problem

$$D^q(u(t) - u^T) = f\ (t, u(t)),\ u(T) = u^T = u(t)(T-t)^{1-q}\ |_{t=T}, \tag{2.6}$$

where $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and $u^T(t) = \frac{u^T (T-t)^{q-1}}{\Gamma(q)}$. Terminal conditions where $u(T) = u^T$ and $u(t)$ function is a solution of the fractional terminal value problem.

Definition 2.1 *([22]) A function $f : (t_0, T] \to \mathbb{R}$ is Hölder continuous if there are nonnegative real constants $C,\ \alpha$ such that $|f(x) - f(y)| \leq C|x-y|^{\alpha}$ for all $x,\ y \in (t_0, T]$.*

Lemma 2.1 *([16, 25]) Let $m \in C_p[[t_0, T], \mathbb{R}]$ be locally Hölder continuous with exponent $\lambda > q$, and for any $t_1 \in (t_0, T]$, we have that on $(t_1, T]$*

$$m(t_1) = 0,\ m(t) \leq 0 \text{ for } t_0 \leq t \leq t_1.$$

Then

$$^cD^q m(t_1) \leq 0. \tag{2.7}$$

Lemma 2.2 *([18]) Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for $\epsilon > 0$, such that*

$$D^q u_\epsilon(t) = f(t, u_\epsilon(t))$$

$$u_\epsilon^T = u_\epsilon(t)(T-t)^{1-q}\ |_{t=T},\ \text{and } |f(t, u_\epsilon(t))| \leq M \text{ for } t_0 \leq t \leq T.$$

Then the family of the sequence of the functions $\{u_\epsilon(t)\}$ is equicontinuous on $[t_0, T]$.

Definition 2.2 *The functions $v, w \in C_p[[t_0, T], \mathbb{R}]$ are said to be lower and upper solutions (2.3) of $FCTVP$ if v and w satisfy the differential inequalities respectively*

$$\begin{aligned} ^cD^q v(t) &\geq Q(t, v_t),\ v(T) \leq u_T \\ ^cD^q w(t) &\leq Q(t, w_t),\ w(T) \geq u_T, \end{aligned}$$

where the causal operator $Q \in E = C(\mathbb{R}_+, \mathbb{R})$, $Q : E \to E$ is continuous.

Definition 2.3 *([14]) $Q : E \to E$ is said to be seminondecreasing in t for each x if*

$$Q(t_1, x_{t_1}) = Q(t_1, y_{t_1}) \text{ and } Q(t, x_t) \leq Q(t, y_t),\ 0 \leq t < t_1 < T,\ T \in \mathbb{R}_+$$

provided that

$$x_{t_1} = y_{t_1},\ x(t) < y(t),\ 0 \leq t < t_1 < T,\ T \in \mathbb{R}_+.$$

Definition 2.4 *([26]) Let $Q \in C(\mathbb{R}_+, \mathbb{R})$. At $x \in E$*

$$(Q(x+h))(t) = Q(t, x_t) + L(x,h)(t) + \|h\| \eta(x,h)(t),$$

where $\lim_{\|h\|\to 0} \|\eta(x,h)(t)\| = 0$ and $L(x,\cdot)(t)$ is a linear operator. $L(x,h)(t)$ is said to be the Fréchet derivative of Q at x with the increment h for the remainder $\eta(x,h)(t)$.

Theorem 2.1 *Assume that $Q(t,u_t) \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ is continuous causal operator $Q \in E = C(\mathbb{R}_+, \mathbb{R})$, $Q : E \to E$. In addition to $v, w \in C_p[[t_0, T], \mathbb{R}]$ with continuous exponent $\lambda > q$, assume that*

(i) ${}^cD^q v(t) \geq Q(t, v_t)$;

(ii) ${}^cD^q w(t) \leq Q(t, w_t)$;

(iii) $Q(t,u_t)$ is nondecreasing in u for each t, $t_0 \leq t \leq T$ with one of the inequalities (i) or (ii) being strict.

Then $v(T) \leq w(T)$ implies $v(t) \leq w(t)$ for $t \in [t_0, T]$.

Proof Assume that one of the inequalities is strict; let $m(t) = v(t) - w(t)$. If the conclusion of the theorem is not true, there exists $t_1 \in (t_0, T]$ such that $m(t_1) = 0$, $m(t) \leq 0$ for $t_0 \leq t \leq t_1$.

Consider the case when $t_1 \in (t_0, T]$, then $m(t_1) = 0$, $m(t) \leq 0$ on (t_0, t_1). By using Lemma 2.1, we get that to be ${}^cD^q m(t_1) \leq 0$. Thus

$$Q(t_1, v_{t_1}) < {}^cD^q v(t_1) \leq {}^cD^q w(t_1) \leq Q(t_1, w_{t_1})$$

$$Q(t_1, v_{t_1}) < Q(t_1, w_{t_1})$$

is a contradiction. Therefore, $v(t) < w(t)$.

Now let us define for ϵ, $L > 0$ arbitrary,

$$\widetilde{v}(t) = v(t) - \epsilon E_q(-2L(t-t_0)^q).$$

where E_q is the Mittag–Leffler function defined as $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, q > 0$. Then $\widetilde{v}(t) < v(t)$ for $t \in [t_0, T]$ and $\widetilde{v}(T) < v(T)$.

Thus, it follows from (i) and the fact that $Q(t, u_t)$ is nondecreasing that

$$\begin{aligned} {}^cD^q \widetilde{v}(t) &= {}^cD^q v(t) + 2L\epsilon E_q(-2L(t-t_0)^q) \geq Q(t, v_t) + 2L\epsilon E_q(-2L(t-t_0)^q) \\ &\geq Q(t, \widetilde{v}_t) + 2L\epsilon E_q(-2L(t-t_0)^q) > Q(t, \widetilde{v}_t). \end{aligned}$$

It then follows by the foregoing argument that $\widetilde{v}(t) < w(t)$. Finally, and letting $\varepsilon \to 0$, we have $v(t) \leq w(t)$.

The proof is complete. □

Theorem 2.2 *Let* $v, w \in C_p[[t_0, T], \mathbb{R}]$ *such that* $v(t) \leq w(t),\ t \in [t_0, T]$ *and* $Q : \Omega \to \mathbb{R}$ *is the continuous causal operator, where* $\Omega = [(t, u) : v(t) \leq u \leq w(t),\ t \in [t_0, T]]$.

Suppose that

(i) ${}^cD^q v(t) \geq Q(t, v_t)$;

(ii) ${}^cD^q w(t) \leq Q(t, w_t)$;

(iii) $Q(t, u_t) \leq \lambda(t)$ *on* Ω *such that* $\lambda \in L^1[\mathbb{R}_+, \mathbb{R}]$.

Then the $FCTVP$ *has a solution that satisfies* $v(t) \leq u(t) \leq w(t)$ *on* $[t_0, T]$ *provided that* $v(T) \leq u(T) \leq w(T)$ *for some* $t_0 \geq 0$.

Proof Consider $P : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ defined by

$$P(t, u_t) = \max\{v(t),\ \min\{u,\ w(t)\}\}. \tag{2.8}$$

Then Q is a continuous causal operator and by the assumption (iii), we get $Q(t, u_t) \leq \lambda(t)$, so that $Q(t, P(t, u_t))$ defines a continuous extension of Q to $[t_0, T] \times \mathbb{R}$, which is also bounded. Therefore, the $FCTVP$ of

$${}^cD^q u = Q(t, P(t, u_t)),\ u(T) = u_T \tag{2.9}$$

has a solution $u(t)$ on $[t_0, T]$. We show $v(t) \leq u(t) \leq w(t)$ for $t \in [t_0, T]$, and therefore $u(t)$ is a solution of (2.3).

For $\epsilon,\ L > 0$, consider

$$\begin{aligned} \widetilde{v}(t) &= v(t) - \epsilon E_q(-2L(t - t_0)^q), \\ \widetilde{w}(t) &= w(t) + \epsilon E_q(-2L(t - t_0)^q). \end{aligned} \tag{2.10}$$

Then $\widetilde{w}(t) > w(t)$, $\widetilde{v}(t) < v(t)$ and $\widetilde{v}(T) < u(T) < \widetilde{w}(T)$. We claim that $\widetilde{v}(t) < u(t) < \widetilde{w}(t)$ on $[t_0, T]$. Suppose that it is not true and thus there exists $t_1 \in [t_0, T]$ such that $u(t_1) = \widetilde{w}(t_1)$ and $\widetilde{v}(t) < u(t) < \widetilde{w}(t),\ t_0 \leq t \leq t_1$.

Then $u(t_1) > w(t_1)$ and hence $P(t_1, u_{t_1}) = w(t_1)$. Also $v(t_1) \leq P(t_1, u_{t_1}) \leq w(t_1)$. Setting $m(t) = u(t) - \widetilde{w}(t)$, we have $m(t_1) = 0$ and $m(t) \leq 0,\ t_0 \leq t \leq t_1$. Hence by Lemma 2.1 we get ${}^cD^q m(t_1) \leq 0$, which yields

$$\begin{aligned} Q(t_1, P(t_1, u_{t_1})) &= {}^cD^q u(t_1) \leq {}^cD^q \widetilde{w}(t_1) = {}^cD^q w(t_1) - 2L\epsilon E_q(-2L(t_1 - t_0)^q) \\ &\leq Q(t_1, w_{t_1}) - 2L\epsilon E_q(-2L(t_1 - t_0)^q) = Q(t_1, P(t_1, u_{t_1})) - 2L\epsilon E_q(-2L(t_1 - t_0)^q) \\ &< Q(t_1, P(t_1, u_{t_1})), \end{aligned}$$

which is a contradiction. The other case can be proved in a similar manner.

Consequently, we have $\widetilde{v}(t) < u(t) < \widetilde{w}(t)$ on $t \in [t_0, T]$ and letting $\epsilon \to 0$ we get $v(t) \leq u(t) \leq w(t)$, on $[0, T]$. The proof is completed. □

3. Quasilinearization method

In this section, we will extend the generalized quasilinearization method for nonlinear terminal value problems in [3] and prove the main theorem that gives several different conditions to apply the method of quasilinearization to the nonlinear $FCTVP$.

Theorem 3.1 *Let* $Q,\ \Phi : C[\mathbb{R}_+,\mathbb{R}] \to C[\mathbb{R}_+,\mathbb{R}]$ *be a continuous causal operator,* $Q(t,u_t),\ \Phi(t,u_t) \in C[\mathbb{R}_+\times\mathbb{R},\mathbb{R}]$,

(M_1) $|Q(t,u_t)| \le \lambda(t)|u(t)|$ *on* $\Omega = [(t,u)\in[t_0,T]\times C^q[[t_0,T],\mathbb{R}] : v(t)\le u\le w(t)]$, *where* $\lambda \in L^1[0,\infty)$;

(M_2) $v,w \in C^q[[t_0,T],\mathbb{R}]$ *are the lower and upper solutions (2.3) of* $FCTVP$ *such that* $v(t)\le w(t)$, $t\in[t_0,T]$;

(M_3) $v_0,w_0\in C^q[[t_0,T],\mathbb{R}]$ *with* $v_0(t)\le w_0(t)$ *on* $[t_0,T]$, $v_0(T)$, $w_0(T)$ *exist and*

(*a*) ${}^cD^q v_0(t) \ge Q(t,v_{0t})$, $v_0(T)\le u_T$ *for* $t\in[t_0,T]$;

(*b*) ${}^cD^q w_0(t) \le Q(t,w_{0t})$, $w_0(T)\ge u_T$ *for* $t\in[t_0,T]$;

(M_4) $Q,\ \Phi\in C^q[\mathbb{R}_+,\mathbb{R}]$ *and for* $(t,u)\in\Omega$, *the Fréchet derivatives* $Q_u(t,u_t)$, $\Phi_u(t,u_t)$, $Q_{uu}(t,u_t)$ *and* $\Phi_{uu}(t,u_t)$ *exist and are continuous on* $[0,\infty)$ *such that* $Q_u(t,u_t)\le B$, $Q_{uu}(t,u_t)+\ \Phi_{uu}(t,u_t)\le 0$ *for some function* Φ *with* $|\Phi(t,u_t)|\le\lambda_1(t)|u(t)|$, $|\Phi_u(t,u_t)|\le F$ *and* $Q_{uu}(t,u_t)\ge 0$, $\Phi_{uu}(t,u_t)\le 0$ *on* $\mathbb{R}_+\times\mathbb{R}$, *where* $B,\ F,\ \lambda_1\in L^1[0,\infty)$.

Then there exist monotone sequences $\{v_n\}$, $\{w_n\}$ *converging uniformly to the unique solution* $u(T,t_0,u_0)=u_T$ *of (2.3) on* $[t_0,T]$ *and the convergence is quadratic.*

Proof Let us initially define a continuous causal operator $\Psi : C[\mathbb{R}_+,\mathbb{R}]\to C[\mathbb{R}_+,\mathbb{R}]$ and $(\Psi u)(t)=\Psi(t,u_t)\in C[\mathbb{R}_+\times\mathbb{R},\mathbb{R}]$, such that

$$\Psi(t,u_t)=Q(t,u_t)+\Phi(t,u_t). \tag{3.1}$$

In view of (M_4), we have $\Psi_{uu}(t,u_t)\le 0$, and $|\Psi(t,u_t)|\le(\lambda(t)+\lambda_1(t))|u(t)| = P|u(t)|$, where $P=(\lambda(t)+\lambda_1(t))\in L^1[0,\infty)$. Moreover, $|\Psi_u(t,u_t)|\le B+F=P_1\in L^1[0,\infty)$. Using the generalized mean value theorem and (3.1), we have

$$Q(t,u_t)\le\Psi(t,\alpha_t)+\Psi_u(t,\alpha_t)(u-\alpha)-\Phi(t,u_t),$$

where $u,\ \alpha\in C^q[[t_0,T],\mathbb{R}]$ such that $\alpha(t)\le u(t)$, $t\in[t_0,T]$. We get

$$G(t,u_t,\alpha_t)=\Psi(t,\alpha_t)+\Psi_u(t,\alpha_t)(u-\alpha)-\Phi(t,u_t), \tag{3.2}$$

and see that

$$\begin{aligned} G(t,u_t,\alpha_t) &\ge Q(t,u_t) \\ G(t,u_t,u_t) &= Q(t,u_t). \end{aligned} \tag{3.3}$$

Moreover, using the nonincreasing property of $\Phi_u(t,u_t)$,

$$G_u(t,u_t,\alpha_t)=\Psi_u(t,\alpha_t)-\Phi_u(t,u_t)\ge\Psi_u(t,\alpha_t)-\Phi_u(t,\alpha_t)\ge Q_u(t,\alpha_t)\ge 0.$$

Thus, $G(t,u_t,\alpha_t)$ is nondecreasing in u for each fixed $(t,\alpha)\in[t_0,T]\times C^q[[t_0,T],\mathbb{R}]$. Further,

$$G(t,u_t,\alpha_t)=\Psi(t,\alpha_t)+\Psi_u(t,u_t)(u-\alpha)-\Phi(t,u_t),$$

which, together with (M_1), (M_4), and (3.2), implies that

$$G(t,u_t,\alpha_t)=P|\alpha|+B(|u|+|\alpha|)+\lambda_1|u|=P_2(t)|\alpha|+P_3(t)|u|=H(t,|u_t|), \tag{3.4}$$

where $P_2 = P + B,\ P_3 = \lambda_1 + B \in L^1[0,\infty)$. Now, using the mean value theorem and exploiting the nonincreasing nature of $\Psi_u(t,u_t)$, we obtain

$$\begin{aligned} G(t,u_t,\alpha_{t1}) - G(t,u_t,\alpha_{t2}) &\leq \Psi_u(t,\mu_{t1})(\alpha_1-\alpha_2) + \Psi_u(t,\alpha_{t2})(\alpha_2-\alpha_1) \\ &= \Psi_{uu}(t,\mu_{t2})(\mu_1-\alpha_2)(\alpha_1-\alpha_2) \\ &\leq 0 \end{aligned} \tag{3.5}$$

where $\alpha_2 \leq \mu_2 \leq \mu_1 \leq \alpha_1$. Expression (3.5) implies that $G(t,u_t,\alpha_t)$ is nonincreasing in α for each fixed $(t,u) \in [t_0,T] \times C^q[[t_0,T],\mathbb{R}]$.

Set $v = \beta_0$ and consider the following $FCTVP$

$$^cD^q u(t) = G(t,u_t,\beta_{t0}),\ u(T) = \gamma_T \tag{3.6}$$

Because of expression (3.4), the problem (3.6) has a unique solution $\beta_1(t)$ on $[a,\infty)$, $a > 0$ satisfying $u_1(T) = u_T$. Furthermore, in view of (M_2) and (3.3), we have

$$^cD^q\beta_0 \geq Q(t,\beta_{0t}) = G(t,\beta_{t0},\beta_{t0}),\ \beta_0(T) \leq \gamma_T$$

and

$$^cD^q w(t) \leq Q(t,w_t) \leq G(t,w_t,\beta_{t0}),\ w(T) \geq \gamma_T$$

which imply that

$$v(t) \leq u_1(t) \leq w(t) \text{ for some } a \geq 0.$$

Next, consider the $FCTVP$

$$^cD^q u(t) = G(t,u_t,\beta_{t1}),\ u(T) = \gamma_T \tag{3.7}$$

In a similar manner, we can also show that $FCTVP$ (3.7) has a unique solution $\beta_2(t)$ satisfying $\beta_2(T) = \gamma_T$. Using (3.3) and the nonincreasing property of $G(t,u_t,\alpha_t)$ in α, we have

$$^cD^q\beta_1(t) = G(t,\beta_{t1},\beta_{t0}) \geq G(t,\beta_{t1},\beta_{t1}),\ \beta_1(T) = \gamma_T,$$

which implies that $\beta_1(t)$ is a lower solution of (3.7) and

$$^cD^q w(t) \leq Q(t,w_t) \leq G(t,w_t,\beta_{t1}),\ \beta(T) \geq \gamma_T$$

implies that $w(t)$ is an upper solution of (3.7). Further, $\beta_1(T) \leq \beta_2(T) \leq w(T)$. Again, by virtue of Theorem 2.2, we get

$$\beta_1(t) \leq \beta_2(t) \leq w(t),\ t \in [a,T) \text{ for some } a \geq 0.$$

Using mathematical induction, we obtain a monotone sequence $\{\beta_n\}$ satisfying

$$v \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq ... \leq \beta_{n-1} \leq \beta_n \leq w \text{ on } [t_0,T],$$

where β_n is the solution of the following $FCTVP$

$$^cD^q u(t) = G\left(t,u_t,\beta_{t(n-1)}\right),\ u(T) = \gamma_T.$$

The monotone sequence $\{\beta_n\}$ converges $\beta(t)$ as a pointwise limit as n approaches infinity. To show that $\beta(t)$ is in fact a solution of (2.3), we observe that β_n is a solution of the following linear $FCTVP$

$$^{c}D^{q}u(t) = G\left(t, \beta_{tn}, \beta_{t(n-1)}\right) = F_n(t), \ \beta_n(T) = \gamma_T, \tag{3.8}$$

where G is continuous on $\mathbb{R}_+$ and

$$F_n(t) = \Psi\left(t, \beta_{t(n-1)}\right) + \Psi_u\left(t, \beta_{t(n-1)}\right)(\beta_n - \beta_{n-1}) - \Phi\left(t, \beta_{t(n)}\right).$$

Therefore, in view of (3.4), it follows that for each $n \in \mathbb{N}$, the sequence $\{F_n(t)\}$ is a sequence of continuous functions and is bounded by $H(t, \beta_{tn}) \in L^1[0, \infty)$. Consequently, $\int_t^\infty F_n(s)\,ds < \infty$. Now, taking the limits both side as $n \to \infty$, we have

$$\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty} G\left(t, \beta_{tn}, \beta_{t(n-1)}\right) = Q(t, \beta_t).$$

Now using the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_t^\infty F_n(s)\,ds = \int_t^\infty Q(s, u_s)\,ds < \infty.$$

Hence, the solution of (3.8) is given by

$$\beta_n(t) = \gamma_T - \int_t^\infty F_n(s)\,ds,$$

which, by taking the limit as $n \to \infty$, yields

$$\beta(t) = \gamma_T - \int_t^\infty Q(s, u_s)\,ds$$

which is a solution of (2.3).

Finally, in order to show that the convergence is quadratic, we set

$$\sigma_n(t) = \beta(t) - \beta_n(t), \ n = 1, 2, 3, ...$$

Observe that $\sigma_n(t) \geq 0$ and $\sigma_n(\infty) = 0$. Now, by using mean value theorem and the assumption (M_4) we obtain

$$\begin{aligned}
{}^cD^q\sigma_{n+1}(t) &= {}^cD^q\beta(t) - {}^cD^q\beta_{n+1}(t) \\
&= Q(t,\beta_t) - \left[\Psi\left(t,\beta_{t(n)}\right) + \Psi_u\left(t,\beta_{t(n)}\right)(\beta_{n+1}-\beta_n) - \Phi\left(t,\beta_{t(n+1)}\right)\right] \\
&= \Psi_u\left(t,\beta_{t(n)}\right)(\beta-\beta_n) + \Psi_{uu}(t,\xi_t)\frac{(\beta-\beta_n)^2}{2!} - \\
&\quad \Psi_u\left(t,\beta_{t(n)}\right)(\beta_{n+1}-\beta_n) - \left(\Phi(t,\beta_t) - \Phi\left(t,\beta_{t(n+1)}\right)\right) \qquad (3.9) \\
&= \Psi_u\left(t,\beta_{t(n)}\right)(\beta-\beta_{n+1}) + \Psi_{uu}(t,\xi_t)\frac{(\beta-\beta_n)^2}{2!} - \Psi_u\left(t,\xi_{t(1)}\right)(\beta-\beta_{n+1}) \\
&\geq Q_u\left(t,\beta_{t(n)}\right)\sigma_{n+1}(t) + \Psi_{uu}\left(t,\zeta_{t(1)}\right)\frac{(\sigma_n(t))^2}{2!} \\
&\geq -B(t)\sigma_{n+1}(t) - \frac{{}^cD^qP(t)}{2}(\sigma_n(t))^2, \ \sigma_{n+1}(\infty) = 0, \text{ where } \beta_n \leq \zeta \leq \beta.
\end{aligned}$$

By the application of Theorem 2.2, we obtain

$$\sigma_{n+1}(t) \leq r(t) \text{ for some } t \geq a > 0,$$

where

$$r(t) = \exp\left(\int_t^\infty B(s)\,ds\right)\left[\int_t^\infty \frac{{}^cD^qP(s)}{2}(\sigma_n(s))^2 \exp\left(-\int_t^\infty B(l)\,dl\right)ds\right]$$

and solution of the following linear $FCTVP$

$${}^cD^qr(t) = -B(t)r(t) - \frac{{}^cD^qP(t)}{2}(\sigma_n(t))^2, \ \beta(\infty) = 0.$$

Thus,

$$\sigma_{n+1}(t) \leq \exp\left(\int_t^\infty B(s)\,ds\right)\left[\int_t^\infty \frac{{}^cD^qP(s)}{2}(\sigma_n(s))^2 \exp\left(-\int_t^\infty B(l)\,dl\right)ds\right].$$

Hence, it follows that

$$\begin{aligned}
|\sigma_{n+1}(t)| &\leq \left|\exp\left(\int_t^\infty B(s)\,ds\right)\right|\left|\int_t^\infty \frac{{}^cD^qP(s)}{2}(\sigma_n(s))^2 \exp\left(-\int_t^\infty B(l)\,dl\right)ds\right| \\
&\leq K|\sigma_n(s)|^2 T \\
&= A|\sigma_n(s)|^2,
\end{aligned}$$

where $\left|\exp\left(\int_t^\infty B(s)\,ds\right)\right| \leq K$, $\left|\int_t^\infty \frac{{}^cD^qP(s)}{2}(\sigma_n(s))^2\exp\left(-\int_t^\infty B(l)\,dl\right)ds\right| \leq 2T$, and $A = KT$.

This shows that the convergence is quadratic. These complete the proof of the theorem. □

4. Conclusion

We have investigated a new concept for the fractional causal terminal value problem and obtained the unique solution of the fractional causal terminal value problem by combining the technique of the generalized quasilinearization method.

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