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On the 3-dimensional Hopf bifurcation via averaging theory of third order

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Abstract: We apply the averaging theory of third order to polynomial quadratic vector fields in $\mathbb{R}^3$ to study the Hopf bifurcation occurring in that polynomial. Our main result shows that at most 10 limit cycles can bifurcate from a singular point having eigenvalues of the form $\pm bi$ and 0. We provide an example of a quadratic polynomial differential system for which exactly 10 limit cycles bifurcate from a such singular point.

Key words: Hopf bifurcation, limit cycle, averaging theory of third order

1. Introduction

One of the main problems in the theory of ordinary differential equations is the study of their limit cycles, their existence, their number, and their stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. Limit cycles usually arise at a Hopf bifurcation in nonlinear systems with varying parameters. The term Hopf bifurcation refers to the birth or death of a periodic solution from an equilibrium as a parameter crosses a critical value. In a differential equation a Hopf bifurcation occurs when a complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher. In general Hopf bifurcation is well studied for singular points that have an eigenvalue of the form $\alpha(\varepsilon) \pm \beta(\varepsilon)i$ with $\alpha(0) = 0$ and $\alpha'(0) \neq 0$. The Hopf bifurcation of limit cycles has been considered by several authors, e.g., [4–6,8,9,11–14,16,18–21]. We mention that in [12] the authors studied Hopf bifurcation in higher dimensions than 3 by using the first-order averaging method. They proved that l limit cycles can bifurcate from one singularity with eigenvalues $\pm bi$ and $n-2$ zeros with $l \in \{0, 1, ..., 2^{n-3}\}$. They proved for the first time that the number of bifurcated limit cycles in a Hopf bifurcation can grow exponentially with the dimension of the system. They applied their results to certain fourth-order differential equations. In [11] the authors studied the Hopf bifurcation for a class of degenerate singular point of multiplicity $2n-1$ in dimension 3 via averaging theory. In [8], the authors studied the zero-Hopf bifurcation in the generalized Michelson system. They provided sufficient conditions for the existence of two periodic solutions bifurcating from a zero-Hopf equilibrium for such system. Other studies on Hopf bifurcation using averaging theory were done by Chow and Mallet-Paret; see [14]. A related generalized Hopf bifurcation can be found in [1].

In this work we study the Hopf bifurcation occurring in vector fields in $\mathbb{R}^3$ via the averaging theory of

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third order. We investigate the quadratic polynomial differential systems in $\mathbb{R}^3$ with a singular point at the origin $(0,0,0)$ whose linear part has eigenvalues $(\varepsilon a_1+\varepsilon^2 a_2+\varepsilon^3 a_3)\pm bi$ and $\varepsilon c_1+\varepsilon^2 c_2+\varepsilon^3 c_3$, where ε is a small parameter. Thus, the quadratic polynomial differential systems that we analyze are

$$\begin{cases} \dfrac{dx}{dt} = (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3)x - by + \displaystyle\sum_{i+j+k=2} a_{ijk}x^iy^jz^k + \varepsilon \sum_{i+j+k=2} A_{ijk}x^iy^jz^k + \varepsilon^2 \sum_{i+j+k=2} A'_{ijk}x^iy^jz^k, \\ \dfrac{dy}{dt} = bx + (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3)y + \displaystyle\sum_{i+j+k=2} b_{ijk}x^iy^jz^k + \varepsilon \sum_{i+j+k=2} B_{ijk}x^iy^jz^k + \varepsilon^2 \sum_{i+j+k=2} B'_{ijk}x^iy^jz^k, \\ \dfrac{dz}{dt} = (c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3)z + \displaystyle\sum_{i+j+k=2} c_{ijk}x^iy^jz^k + \varepsilon \sum_{i+j+k=2} C_{ijk}x^iy^jz^k + \varepsilon^2 \sum_{i+j+k=2} C'_{ijk}x^iy^jz^k, \end{cases} \tag{1}$$

where $a_{ijk}, b_{ijk}, c_{ijk}, A_{ijk}, B_{ijk}, C_{ijk}, A'_{ijk}, B'_{ijk}, C'_{ijk}$ for $i+j+k=2$, a_1,a_2,a_3,c_1,c_2,c_3, and b are real parameters.
If $a_1^2+a_2^2+a_3^2\neq 0$, then the general theory of Hopf bifurcation tell us that an infinitesimal limit cycle bifurcates from the origin of system (1) when $\varepsilon=0$; see, for instance, [14].

In 2007 Buzzi et al. [7] studied the Hopf bifurcation occurring in vector fields in $\mathbb{R}^3$ via the averaging theory of first order. They obtained at most 1 limit cycle and made applications of their result to the Lorenz and Rössler systems.

In 2009 Llibre et al. [9] studied the Hopf bifurcation occurring in vector fields in $\mathbb{R}^3$ via the averaging theory of second order. They obtained at most 3 limit cycles and they provided an example for which exactly 3 limit cycles bifurcate from the origin. The aim of this paper is to improve the result of [9] using the averaging theory of third order for studying the number of limit cycles that can bifurcate from the origin of the system (1) and their stability.

Our main result is as follows:

Theorem 1 *The following statements hold.*

(a) *At most 10 limit cycles bifurcate from the origin of system (1) when $\varepsilon=0$ by applying the averaging theory of third order.*

(b) *We give an example of a quadratic polynomial differential system of the form (1) for which exactly 10 limit cycles bifurcate from the origin when $\varepsilon=0$.*

Statement (a) of Theorem 1 is the main result of this paper and it is proved in Section 3, while statement (b) is proved in section 4. In Section 2 we recall the averaging theory of first, second, and third order as was stated in [2]. This will be the main tool for proving Theorem 1.

2. The averaging theory of first, second, and third order

The aim of this section is to present the averaging method of first, second, and third order as was developed in [2, 6, 10].

Theorem 2 *Consider the differential system*

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{2}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$*. Assume that the following hypotheses* (i) *and* (ii) *hold.*

(i) $F_1(t,.) \in C^2(D), F_2(t,.) \in C^1(D)$ *for all* $t \in \mathbb{R}$, $F_1, F_2, F_3, R, D_x^2 F_1, D_x F_2$ *are locally Lipschitz with respect to* x*, and* R *is twice differentiable with respect to* ε*.*
We define $F_{k0} : D \longrightarrow \mathbb{R}$ *for* $k = 1, 2, 3$ *as*

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_z F_1(s,z).y_1(s,z) + F_2(s,z)\right] ds,$$

$$F_{30}(z) = \frac{1}{T}\int_0^T [\frac{1}{2} y_1(s,z)^T \frac{\partial^2 F_1}{\partial z^2}(s,z) y_1(s,z) + \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z) y_2(s,z) +$$

$$\frac{\partial F_2}{\partial z}(s,z)(y_1(s,z)) + F_3(s,z)]ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)dt,$$

$$y_2(s,z) = \int_0^s \left[\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)dr + F_2(t,z)\right] dt.$$

(ii) *For* $V \subset D$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$*, there exists* $a_\epsilon \in V$ *such that* $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ *and* $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$*.*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $\varphi(\cdot, \varepsilon)$ *of the system* (2) *such that* $\varphi(0, \varepsilon) = a_\varepsilon$*.*

The expression $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ *means that the Brouwer degree of the function* $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30} : V \to \mathbb{R}^n$ *at fixed point* a_ε *is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function* $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ *at* a_ε *is not zero.*

If F_{10} *is not identically zero, then the zeros of* $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ *are mainly the zeros of* F_{10} *for* ε *sufficiently small. In this case the previous result provides the averaging theory of first order.*

If F_{10} *is identically zero and* F_{20} *is not identically zero, then the zeros of* $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ *are mainly the zeros of* F_{20} *for* ε *sufficiently small. In this case the previous result provides the averaging theory of second order.*

If F_{10} *and* F_{20} *are identically zero and* F_{30} *is not identically zero, then the zeros of* $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ *are mainly the zeros of* F_{30} *for* ε *sufficiently small. In this case the previous result provides the averaging theory of third order.*

Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0,T]$; see [2]. Hence, for each $z \in D$ we denote by $x(.,z,\varepsilon)$ the solution of (2) with the initial value $x(0,z,\varepsilon) = z$.

Consider the function $\xi : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ defined by

$$\xi(z,\varepsilon) = \int_0^T (\varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon))dt.$$

For every $z \in D$ the following relation holds:

$$x(T,z,\varepsilon) - x(0,z,\varepsilon) = \xi(z,\varepsilon).$$

Moreover, the function ξ can be written in the form

$$\xi(z,\varepsilon) = \varepsilon F_{10}(z) + \varepsilon^2 F_{20}(z) + \varepsilon^3 F_{30}(z) + O(\varepsilon^4),$$

where F_{10}, F_{20}, and F_{30} are defined in the statement of Theorem 2, and the symbol $O(\varepsilon^4)$ denotes a bounded function on every compact subset of $D \times (-\varepsilon_f, \varepsilon_f)$ multiplied by ε^4. It follows that the stability of the limit cycles associated to the simple zero a_ε is controlled by the eigenvalues of the Jacobian of $\xi(z,\varepsilon)$ evaluated at a_ε and from Theorem 3.5.1 of [15] we know that the limit cycle associated to the zero a_ε of $F_{30}(z)$ when $F_{10}(z) = 0$ and $F_{20}(z) = 0$ is the following:

$$x(t,a_\varepsilon,\varepsilon) = a_\varepsilon + \varepsilon y_1(t,a_\varepsilon) + \varepsilon^2 y_2(t,a_\varepsilon) + O(\varepsilon^3). \tag{3}$$

For more information about the averaging theory, see [15, 17].

3. Proof of statement (a) of Theorem 1

Changing to cylindrical coordinates $x = Rcos(\theta), y = Rsin(\theta)$, and $z = z$, system (1) becomes

$$\begin{cases} \dfrac{dR}{dt} = \varepsilon(a_1 + a_2\varepsilon + a_3\varepsilon^2)R + h_{11}(\theta)R^2 + h_{12}(\theta)Rz + h_{13}(\theta)z^2, \\ \dfrac{d\theta}{dt} = \dfrac{1}{R}\left[bR + h_{21}(\theta)R^2 + h_{22}(\theta)Rz + h_{23}(\theta)z^2\right], \\ \dfrac{dz}{dt} = \varepsilon(c_1 + c_2\varepsilon + c_3\varepsilon^2)z + h_{31}(\theta)R^2 + h_{32}(\theta)Rz + h_{33}(\theta)z^2, \end{cases} \tag{4}$$

where

$h_{11}(\theta) = (a_{200} + \varepsilon A_{200} + \varepsilon^2 A'_{200})cos^3(\theta) + [(a_{020} + b_{110}) + \varepsilon(A_{020} + B_{110}) + \varepsilon^2(A'_{020} + B'_{110})]sin^2(\theta)cos(\theta) + [(a_{110} + b_{200}) + \varepsilon(A_{110} + B_{200}) + \varepsilon^2(A'_{110} + B'_{200})]cos^2(\theta)sin(\theta) + (b_{020} + \varepsilon B_{020} + \varepsilon^2 B'_{020})sin^3(\theta),$

$h_{12}(\theta) = (a_{101} + \varepsilon A_{101} + \varepsilon^2 A'_{101})cos^2(\theta) + [(a_{011} + b_{101}) + \varepsilon(A_{011} + B_{101}) + \varepsilon^2(A'_{011} + B'_{101})]sin(\theta)cos(\theta) + (b_{011} + \varepsilon B_{011} + \varepsilon^2 B'_{011})sin^2(\theta)],$

$h_{13}(\theta) = (a_{002} + \varepsilon A_{002} + \varepsilon^2 A'_{002})cos(\theta) + (b_{002} + \varepsilon B_{002} + \varepsilon^2 B'_{002})sin(\theta),$

$h_{21}(\theta) = (b_{200} + \varepsilon B_{200} + \varepsilon^2 B'_{200})cos^3(\theta) + [(b_{020} - a_{110}) + \varepsilon(B_{020} - A_{110}) + \varepsilon^2(B'_{020} - A'_{110})]sin^2(\theta)cos(\theta) + [(b_{110} - a_{200}) + \varepsilon(B_{110} - A_{200}) + \varepsilon^2(B'_{110} - A'_{200})]cos^2(\theta)sin(\theta) - (a_{020} + \varepsilon A_{020} + \varepsilon^2 A'_{020})sin^3(\theta)$,

$h_{22}(\theta) = (b_{101} + \varepsilon B_{101} + \varepsilon^2 B'_{101})cos^2(\theta) + [(b_{011} - a_{101}) + \varepsilon(B_{011} - A_{101}) + \varepsilon^2(B'_{011} - A'_{101})]cos(\theta)sin(\theta) - (a_{011} + \varepsilon A_{011} + \varepsilon^2 A'_{011})sin^2(\theta)$,

$h_{23}(\theta) = (b_{002} + \varepsilon B_{002} + \varepsilon^2 B'_{002})cos(\theta) - (a_{002} + \varepsilon A_{002} + \varepsilon^2 A'_{002})sin(\theta)$,

$h_{31}(\theta) = (c_{200} + \varepsilon C_{200} + \varepsilon^2 C'_{200})cos^2(\theta) + (c_{020} + \varepsilon C_{020} + \varepsilon^2 C'_{020})sin^2(\theta) + (c_{110} + \varepsilon C_{110} + \varepsilon^2 C'_{110})sin(\theta)cos(\theta)$,

$h_{32}(\theta) = (c_{101} + \varepsilon C_{101} + \varepsilon^2 C'_{101})cos(\theta) + (c_{011} + \varepsilon C_{011} + \varepsilon^2 C'_{011})sin(\theta)$,

$h_{33}(\theta) = c_{002} + \varepsilon C_{002} + \varepsilon^2 C'_{002}$.

Therefore, the solutions of system (4) in the region $\dot{\theta} \neq 0$ can be studied analyzing the solutions of the system

$$\begin{cases} \dfrac{dR}{d\theta} = \dfrac{[\varepsilon(a_1 + a_2\varepsilon + a_3\varepsilon^2)R + h_{11}(\theta)R^2 + h_{12}(\theta)Rz + h_{13}(\theta)z^2]R}{bR + h_{21}(\theta)R^2 + h_{22}(\theta)Rz + h_{23}(\theta)z^2}, \\ \dfrac{dz}{d\theta} = \dfrac{[\varepsilon(c_1 + c_2\varepsilon + c_3\varepsilon^2)z + h_{31}(\theta)R^2 + h_{32}(\theta)Rz + h_{33}(\theta)z^2]R}{bR + h_{21}(\theta)R^2 + h_{22}(\theta)Rz + h_{23}(\theta)z^2}. \end{cases} \tag{5}$$

By performing the rescaling

$$(R, z) = (\rho\varepsilon, \xi\varepsilon),$$

system (1) comes into the normal form for applying the averaging theory. That is, in the variables (ρ, ξ), system (1) is written as follows:

$$\begin{cases} \dfrac{d\rho}{d\theta} = \varepsilon F_{11}(\theta, \rho, \xi) + \varepsilon^2 F_{21}(\theta, \rho, \xi) + \varepsilon^3 F_{31}(\theta, \rho, \xi) + O(\varepsilon^4), \\ \dfrac{d\xi}{d\theta} = \varepsilon F_{12}(\theta, \rho, \xi) + \varepsilon^2 F_{22}(\theta, \rho, \xi) + \varepsilon^3 F_{32}(\theta, \rho, \xi) + O(\varepsilon^4), \end{cases} \tag{6}$$

where $F_{11}, F_{21}, F_{31}, F_{12}, F_{22}$, and F_{32} are given in the Appendix.
Taking $x = (\rho, \xi), t = \theta, F_1(t, x) = (F_{11}(\theta, \rho, \xi), F_{12}(\theta, \rho, \xi)), F_2(t, x) = (F_{21}(\theta, \rho, \xi), F_{22}(\theta, \rho, \xi)), F_3(t, x) = (F_{31}(\theta, \rho, \xi), F_{32}(\theta, \rho, \xi))$, and $T = 2\pi$, system (6) is equivalent to system (2).
For $i = 1, 2$, we have

$$f_{1i}(\rho, \xi) = \frac{1}{2\pi} \int_0^{2\pi} F_{1i}(\theta, \rho, \xi) d\theta.$$

The unique limit cycle that bifurcates from the origin of system (1) is provided from the unique zero of the system

$$\begin{cases} f_{11}(\rho, \xi) = \dfrac{\rho(2a_1 + (a_{101} + b_{011})\xi)}{2b} = 0, \\ f_{12}(\rho, \xi) = \dfrac{(c_{020} + c_{200})\rho^2 + 2\xi(c_1 + c_{002}\xi)}{2b} = 0. \end{cases} \tag{7}$$

The averaged function of the order $(f_{11}(\rho,\xi), f_{12}(\rho,\xi))$ is identically zero if and only if

$$a_1 = 0, b_{011} = -a_{101}, c_{200} = -c_{020}, c_{002} = 0, c_1 = 0.$$

Considering this condition to apply the averaging theory of second order, we compute the following expression:

$$\begin{pmatrix} \frac{\partial F_{11}}{\partial \rho} & \frac{\partial F_{11}}{\partial \xi} \\ \frac{\partial F_{12}}{\partial \rho} & \frac{\partial F_{12}}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix} + \begin{pmatrix} F_{21}(s,\rho,\xi) \\ F_{22}(s,\rho,\xi) \end{pmatrix}.$$

Now integrating between 0 and 2π and dividing by 2π we obtain the system

$$\begin{cases} f_{21}(\rho,\xi) = \frac{1}{b^2}\rho[U_0 + U_1\xi + U_2\rho^2 + U_3\xi^2], \\ f_{22}(\rho,\xi) = \frac{1}{b^2}[V_0\xi + V_1\rho^2 + V_2\xi^2 + V_3\rho^2\xi + V_4\xi^3] \end{cases},$$

where:
$U_0 = a_2 b$,
$U_1 = \frac{b(A_{101}+B_{011})}{2}$,
$U_2 = [a_{110}a_{200} + a_{020}(a_{110} + 2b_{020}) - 2a_{200}b_{200} - b_{110}(b_{020} + b_{200}) - c_{020}(a_{011} + b_{101}) - c_{110}a_{101}]/8$,
$U_3 = b_{002}(c_{101} - a_{200} - \frac{1}{2}b_{110}) + a_{002}(\frac{1}{2}a_{110} + b_{020} - c_{011})$,
$V_0 = bc_2$,
$V_1 = b(C_{020} + C_{200})/2$,
$V_2 = bC_{002}$,
$V_3 = [c_{011}(a_{020} + a_{200}) + c_{020}(a_{011} + b_{101}) - c_{101}(b_{020} + b_{200}) + a_{101}c_{110}]/2$,
$V_4 = a_{002}c_{011} - b_{002}c_{101}$.
To look for the limit cycles we solve the system

$$\begin{cases} f_{21}(\rho,\xi) = 0, \\ f_{22}(\rho,\xi) = 0. \end{cases} \tag{8}$$

The first equation (avoiding the solutions with $\rho = 0$) has the following two solutions:

$$\rho_1 = \frac{\sqrt{-U_0 - U_1\xi - U_3\xi^2}}{\sqrt{U_2}}, \quad \rho_2 = -\frac{\sqrt{-U_0 - U_1\xi - U_3\xi^2}}{\sqrt{U_2}}.$$

Since ρ must be positive, we keep ρ_1. Then the second equation becomes

$$\frac{U_2V_4 - U_3V_3}{U_2}\xi^3 + \frac{U_2V_2 - U_3V_1 - U_1V_3}{U_2}\xi^2 + \frac{U_2V_0 - U_1V_1 - U_0V_3}{U_2}\xi - \frac{U_0V_1}{U_2} = 0.$$

The coefficients of this cubic equation can take arbitrary values when we play with the coefficients of system (1), so they can be chosen in such a way that the cubic equation has three positive real zeros. Let $\bar{\xi}$ be one of these zeros. Let $(\bar{\rho}, \bar{\xi})$ be a solution of system (8). In order to have a limit cycle according to the theory in Section 2, we must have

$$D(\bar{\rho},\bar{\xi}) = det \begin{pmatrix} \frac{\partial f_{21}}{\partial \rho} & \frac{\partial f_{21}}{\partial \xi} \\ \frac{\partial f_{22}}{\partial \rho} & \frac{\partial f_{22}}{\partial \xi} \end{pmatrix} |_{(\rho,\xi)=(\bar{\rho},\bar{\xi})} \neq 0. \tag{9}$$

In short, the solutions $(\bar{\rho}, \bar{\xi})$ of system (8), which verify condition (9), satisfy the assumptions (i) and (ii) of Section 2. Applying the averaging theory of second order, we conclude that system (6) has at most 3 limit cycles. Therefore, due to the rescaling, system (1) has at most 3 limit cycles bifurcating from the origin.

Example 1 *Consider the quadratic polynomial differential system*

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{2}\varepsilon^2 x - y + x^2 - 2xy - xz + 5\varepsilon z^2 - \varepsilon^2 yz, \\ \dfrac{dy}{dt} = x + \dfrac{1}{2}\varepsilon^2 y + y^2 - z^2 + 4xy - xz + yz + \varepsilon^2 xy, \\ \dfrac{dz}{dt} = \dfrac{-1}{4}\varepsilon^2 z - 2x^2 + 2y^2 + xz + 4yz - 4\varepsilon z^2 + \varepsilon^2 x^2. \end{cases} \tag{10}$$

The eigenvalues of the singular point $(0,0,0)$ *of system (10) are* $\frac{\varepsilon^2}{2} \pm i$ *and* $\frac{-\varepsilon^2}{4}$.
The corresponding system (6) *associated to system (10) satisfies*

$F_{11}(\theta,\rho,\xi) = \rho^2(\cos^3(\theta) + \sin^3(\theta)) + 2\rho^2\cos(\theta)\sin(\theta)(2\sin(\theta) - \cos(\theta)) - \rho\xi cos(2\theta) - \rho\xi\cos(\theta)\sin(\theta) - \xi^2\sin(\theta)$,

$F_{21}(\theta,\rho,\xi) = \frac{1}{2}\rho - 3\rho^3 cos^5(\theta)sin(\theta) + 3\rho^3 cos^4(\theta)sin^2(\theta) - 6\rho^3 cos^3(\theta)sin^3(\theta) - 15\rho^3 cos^2(\theta)sin^4(\theta) - 3\rho^3 cos(\theta)sin^5(\theta) + \rho^2\xi cos^5(\theta) - \rho^2\xi cos^4(\theta)sin(\theta) + 17\rho^2\xi cos^3(\theta)sin^2(\theta) - 7\rho^2\xi cos^2(\theta)sin^3(\theta) - 5\rho^2\xi cos(\theta)sin^4(\theta) - \rho\xi^2 cos^3(\theta)sin(\theta) + 10\rho\xi^2 cos^2(\theta)sin^2(\theta) + 2\rho\xi^2 cos(\theta)sin^3(\theta) - \xi^3 cos^3(\theta) - 2\xi^3 cos^2(\theta)sin(\theta) + 3\xi^3 cos(\theta)sin^2(\theta) - \frac{\xi^4}{\rho}cos(\theta)sin(\theta) + 5\xi^2 cos(\theta)$,

$F_{12}(\theta,\rho,\xi) = -2\rho^2 cos^2(\theta) + 2\rho^2 sin^2(\theta) + \rho\xi cos(\theta) + 4\rho\xi sin(\theta)$,

$F_{22}(\theta,\rho,\xi) = \frac{-1}{4}\xi - 4\xi^2 + 6\rho^3 cos^4(\theta)sin(\theta) + 6\rho^3 cos^3(\theta)sin^2(\theta) - 6\rho^3 cos^2(\theta)sin^3(\theta) - 6\rho^3 cos(\theta)sin^4(\theta) - 2\rho^2\xi cos^4(\theta) + \rho^2\xi cos^3(\theta)sin(\theta) - 13\rho^2\xi cos^2(\theta)sin^2(\theta) - 16\rho^2\xi cos(\theta)sin^3(\theta) - \rho\xi^2 cos^3(\theta) + 2\rho\xi^2 cos^2(\theta)sin(\theta) - 6\rho\xi^2 cos(\theta)sin^2(\theta) + \xi^3 cos^2(\theta) + 4\xi^3 cos(\theta)sin(\theta)$.
To look for the limit cycles we must solve the following system:

$$\begin{cases} f_{21}(\rho,\xi) = \dfrac{\rho}{2}[1 - \rho^2 + 4\xi^2] = 0, \\ f_{22}(\rho,\xi) = \xi[\dfrac{-1}{4} - 4\xi + \xi^2 + \dfrac{1}{2}\rho^2] = 0. \end{cases} \tag{11}$$

This system possesses the roots $(0,0), (0, \frac{(4+\sqrt{17})}{2}), (0, \frac{(4-\sqrt{17})}{2}), (1,0), (-1,0)$,
$(\frac{1}{3}\sqrt{38+8\sqrt{13}}, \frac{2}{3} + \frac{1}{6}\sqrt{13}), (\frac{1}{3}\sqrt{38-8\sqrt{13}}, \frac{2}{3} - \frac{1}{6}\sqrt{13}), (-\frac{1}{3}\sqrt{38+8\sqrt{13}}, \frac{2}{3} + \frac{1}{6}\sqrt{13})$,
$(-\frac{1}{3}\sqrt{38-8\sqrt{13}}, \frac{2}{3} - \frac{1}{6}\sqrt{13})$.
Since ρ *must be positive and* ξ *real, the unique admissible roots are*

$$(1,0), (\frac{1}{3}\sqrt{38+8\sqrt{13}}, \frac{2}{3} + \frac{1}{6}\sqrt{13}), (\frac{1}{3}\sqrt{38-8\sqrt{13}}, \frac{2}{3} - \frac{1}{6}\sqrt{13}).$$

Now we verify that the determinant is different from zero at these roots, where

$$D(\bar{\rho},\bar{\xi}) = det(M) = det\begin{pmatrix} \frac{1}{2}-\frac{3\bar{\rho}^2}{2}+2\bar{\xi}^2 & 4\bar{\rho}\bar{\xi} \\ \bar{\rho}\bar{\xi} & \frac{-1}{4}-8\bar{\xi}+3\bar{\xi}^2+\frac{1}{2}\bar{\rho}^2 \end{pmatrix}.$$

We get

$$D(1,0)=\frac{-1}{4}, D(\frac{1}{3}\sqrt{38+8\sqrt{13}}, \frac{2}{3}+\frac{1}{6}\sqrt{13}) = -\frac{455}{27}-\frac{128}{27}\sqrt{13},$$

$$D(\frac{1}{3}\sqrt{38-8\sqrt{13}}, \frac{2}{3}-\frac{1}{6}\sqrt{13}) = -\frac{455}{27}+\frac{128}{27}\sqrt{13}.$$

Hence, this system has exactly 3 limit cycles bifurcating from the origin.

Now we study the stability of these 3 limit cycles. The eigenvalues of the matrix M at the points

$$(1,0), (\frac{1}{3}\sqrt{38+8\sqrt{13}}, \frac{2}{3}+\frac{1}{6}\sqrt{13}), (\frac{1}{3}\sqrt{38-8\sqrt{13}}, \frac{2}{3}-\frac{1}{6}\sqrt{13})$$

are $\left(\frac{1}{4},-1\right)$, $\left(-\frac{5}{9}\sqrt{13}+\frac{\sqrt{36065+9944\sqrt{13}}-95}{36}, -\frac{5}{9}\sqrt{13}-\frac{\sqrt{36065+9944\sqrt{13}}+95}{36}\right)$, and $\left(\frac{5}{9}\sqrt{13}+\frac{\sqrt{36065-9944\sqrt{13}}-95}{36}, \frac{5}{9}\sqrt{13}-\frac{\sqrt{36065-9944\sqrt{13}}+95}{36}\right)$, respectively. Therefore, the corresponding limit cycles are semistable, semistable, and stable, respectively.

The limit cycles Γ_i for $i=1,2,3$ of system (5) associated to system (10) and corresponding to the zeros $(\overline{\rho},\overline{\xi})$ given by (11) can be written as $\left\{(R_i(\theta), z_i(\theta)), \theta\in\mathbb{S}^1\right\}$, where from (3) we have

$$\begin{pmatrix} R_i(\theta) \\ z_i(\theta) \end{pmatrix} = \varepsilon\left[\begin{pmatrix} \overline{\rho} \\ \overline{\xi} \end{pmatrix} + \varepsilon\begin{pmatrix} \int_0^\theta F_{11}(s,\overline{\rho},\overline{\xi})ds \\ \int_0^\theta F_{12}(s,\overline{\rho},\overline{\xi})ds \end{pmatrix} + \begin{pmatrix} O(\varepsilon^2) \\ O(\varepsilon^2) \end{pmatrix}\right],$$

where

$$\begin{pmatrix} \int_0^\theta F_{11}(s,\overline{\rho},\overline{\xi})ds \\ \int_0^\theta F_{12}(s,\overline{\rho},\overline{\xi})ds \end{pmatrix} = \begin{pmatrix} -\overline{\xi}^2-\frac{1}{2}\overline{\rho}\overline{\xi}+(\overline{\xi}^2-\overline{\rho}^2)\cos(\theta)-\overline{\rho}^2\cos^2(\theta)\sin(\theta)+ \\ 2\overline{\rho}^2\sin(\theta)+\frac{1}{2}\overline{\rho}\overline{\xi}\cos^2(\theta)+\overline{\rho}^2\cos^3(\theta)-\overline{\rho}\overline{\xi}\cos(\theta)\sin(\theta) \\ 4\overline{\rho}\overline{\xi}(1-\cos(\theta))+\overline{\rho}\overline{\xi}\sin(\theta)-2\overline{\rho}^2\cos(\theta)\sin(\theta) \end{pmatrix}.$$

Therefore, the limit cycle Γ_1 can be written as

$$\begin{aligned} R_1(\theta) &= \varepsilon+\varepsilon^2(\cos^3(\theta)-\cos(\theta)-\cos^2(\theta)\sin(\theta)+2\sin(\theta))+O(\varepsilon^3), \\ Z_1(\theta) &= -2\varepsilon^2\cos(\theta)\sin(\theta)+O(\varepsilon^3). \end{aligned}$$

The limit cycle Γ_2 can be written as

$$\begin{aligned} R_2(\theta) &= \frac{\sqrt{38+8\sqrt{13}}}{3}\varepsilon+\varepsilon^2((\frac{29}{36}+\frac{2\sqrt{13}}{9})(\cos(\theta)-1)+\frac{38+8\sqrt{13}}{9}(\cos^3(\theta)-\cos^2(\theta)\sin(\theta) \\ &+ 2\sin(\theta)-\cos(\theta))+\sqrt{38+8\sqrt{13}}(\frac{4+\sqrt{13}}{18})(\frac{1}{2}(\cos^2(\theta)-1)-\cos(\theta)\sin(\theta)))+O\left(\varepsilon^3\right), \\ Z_2(\theta) &= \frac{4+\sqrt{13}}{6}\varepsilon+\varepsilon^2(\sqrt{38+8\sqrt{13}}(\frac{4+\sqrt{13}}{18})(4(1-\cos(\theta))+\sin(\theta)) \\ &- \frac{2(38+8\sqrt{13})}{9}\cos(\theta)\sin(\theta))+O\left(\varepsilon^3\right). \end{aligned}$$

The limit cycle Γ_3 can be written as

$$\begin{aligned} R_3(\theta) &= \frac{\sqrt{38-8\sqrt{13}}}{3}\varepsilon+\varepsilon^2((\frac{29}{36}-\frac{2\sqrt{13}}{9})(\cos(\theta)-1)+\frac{38-8\sqrt{13}}{9}(\cos^3(\theta)-\cos^2(\theta)\sin(\theta) \\ &+ 2\sin(\theta)-\cos(\theta))+\sqrt{38-8\sqrt{13}}(\frac{4-\sqrt{13}}{18})(\frac{1}{2}(\cos^2(\theta)-1)-\cos(\theta)\sin(\theta)))+O\left(\varepsilon^3\right), \\ Z_3(\theta) &= \frac{4-\sqrt{13}}{6}\varepsilon+\varepsilon^2(\sqrt{38-8\sqrt{13}}(\frac{4-\sqrt{13}}{18})(4(1-\cos(\theta))+\sin(\theta)) \\ &- \frac{2(38-8\sqrt{13})}{9}\cos(\theta)\sin(\theta))+O\left(\varepsilon^3\right). \end{aligned}$$

Now to prove the main result of this paper we shall need the third order averaging theory. According to the theorem of Section 2, we must verify that the averaged function of the second order $(f_{21}(\rho,\xi), f_{22}(\rho,\xi))$ is identically zero.
For this, we take

$$a_2=0, c_2=0, a_{200}=0, a_{020}=0, a_{002}=0, a_{011}=-b_{101},$$
$$b_{002}=0, b_{200}=-b_{020}, c_{110}=0, A_{101}=-B_{011}, C_{020}=-C_{200}, C_{002}=0.$$

Now applying the averaging theory of third order, we must compute the two expressions.

The first is $\frac{1}{2}\left[\begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta & \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}.\begin{pmatrix} \frac{\partial^2 F_{11}}{\partial\rho^2} & \frac{\partial^2 F_{11}}{\partial\rho\partial\xi} \\ \frac{\partial^2 F_{11}}{\partial\xi\partial\rho} & \frac{\partial^2 F_{11}}{\partial\xi^2} \end{pmatrix}.\begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}\right]+$
$\frac{1}{2}\left[\begin{pmatrix} \frac{\partial F_{11}}{\partial\rho} & \frac{\partial F_{11}}{\partial\xi} \end{pmatrix}.\int_0^s\left[\begin{pmatrix} \frac{\partial F_{11}}{\partial\rho} & \frac{\partial F_{11}}{\partial\xi} \\ \frac{\partial F_{12}}{\partial\rho} & \frac{\partial F_{12}}{\partial\xi} \end{pmatrix}.\begin{pmatrix} \int_0^t F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^t F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}+\begin{pmatrix} F_{21}(t,\rho,\xi) \\ F_{22}(t,\rho,\xi) \end{pmatrix}\right]dt\right]+\begin{pmatrix} \frac{\partial F_{21}}{\partial\rho} & \frac{\partial F_{21}}{\partial\xi} \end{pmatrix}.$
$\begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}+F_{31}(s,\rho,\xi),$ and the second is

$\frac{1}{2}\left[\begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta & \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}.\begin{pmatrix} \frac{\partial^2 F_{12}}{\partial\rho^2} & \frac{\partial^2 F_{12}}{\partial\rho\partial\xi} \\ \frac{\partial^2 F_{12}}{\partial\xi\partial\rho} & \frac{\partial^2 F_{12}}{\partial\xi^2} \end{pmatrix}.\begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}\right]+$
$\frac{1}{2}\left[\begin{pmatrix} \frac{\partial F_{12}}{\partial\rho} & \frac{\partial F_{12}}{\partial\xi} \end{pmatrix}.\int_0^s\left[\begin{pmatrix} \frac{\partial F_{11}}{\partial\rho} & \frac{\partial F_{11}}{\partial\xi} \\ \frac{\partial F_{12}}{\partial\rho} & \frac{\partial F_{12}}{\partial\xi} \end{pmatrix}.\begin{pmatrix} \int_0^t F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^t F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix}+\begin{pmatrix} F_{21}(t,\rho,\xi) \\ F_{22}(t,\rho,\xi) \end{pmatrix}\right]dt\right]+$

$$\begin{pmatrix} \frac{\partial F_{22}}{\partial \rho} & \frac{\partial F_{22}}{\partial \xi} \end{pmatrix} . \begin{pmatrix} \int_0^s F_{11}(\theta,\rho,\xi)d\theta \\ \int_0^s F_{12}(\theta,\rho,\xi)d\theta \end{pmatrix} + \ F_{32}(s,\rho,\xi).$$

Now we compute the integral of these expressions between 0 and 2π, and dividing by 2π, we get the two equations

$$f_{31}(\rho,\xi) = \frac{1}{2b^3}\rho[I_0 + I_1\rho^2\xi + I_2\rho^2 + I_3\xi^2 + I_4\xi],$$

$$f_{32}(\rho,\xi) = \frac{1}{2b^3}[J_0\xi + J_1\rho^4 + J_2\rho^3\xi + J_3\rho^2\xi^2 + J_4\rho^2\xi + J_5\rho^2 + J_6\xi^3 + J_7\xi^2],$$

where:

$I_0 = 2b^2a_3$

$I_1 = \frac{71}{384}a_{101}\left(a_{110}^2 - b_{110}^2\right) - \frac{29}{96}a_{101}b_{020}^2 - \frac{1}{24}a_{101}c_{011}(7a_{110} + 10b_{020}) + a_{101}(\frac{7}{24}c_{101}b_{110} + \frac{7}{32}a_{110}b_{020}) + \frac{1}{4}b_{101}(b_{110}c_{011} - c_{101}a_{110} - 2c_{101}b_{020}) + \frac{1}{8}a_{101}(c_{011}^2 - c_{101}^2)$,

$I_2 = b[\frac{1}{24}a_{110}(A_{200} - B_{110} + 5A_{020}) + \frac{1}{6}b_{020}(A_{200} - B_{110} + 2A_{020}) + \frac{1}{24}b_{110}(A_{110} - B_{020} - 5B_{200}) - \frac{1}{4}c_{020}(A_{011} + B_{101}) - \frac{1}{8}a_{101}C_{110}]$,

$I_3 = b[\frac{1}{8}a_{101}(A_{011} + B_{101}) + \frac{1}{4}A_{002}(3a_{110} + 6b_{020}) - \frac{3}{4}b_{110}B_{002} + 2(c_{101}B_{002} - c_{011}A_{002})]$,

$I_4 = b^2(A'_{101} + B'_{011})$,

$J_0 = 2b^2c_3$

$J_1 = \frac{1}{192}c_{020}(a_{110}^2 - b_{110}^2) + \frac{5}{48}c_{020}b_{020}^2 + \frac{1}{12}c_{020}c_{011}(a_{110} + b_{020}) + \frac{1}{16}a_{110}c_{020}b_{020} - \frac{1}{12}c_{020}c_{101}b_{110} + \frac{1}{8}c_{020}(c_{101}^2 - c_{011}^2)$,

$J_2 = \frac{1}{12}c_{101}(a_{110}^2 + 2b_{020}^2) + \frac{1}{4}b_{020}c_{101}a_{110} - \frac{1}{12}b_{110}c_{011}(a_{110} + b_{020})$,

$J_3 = \frac{9}{8}a_{101}(c_{101}^2 - c_{011}^2) + \frac{1}{6}a_{101}c_{011}(5a_{110} - b_{020}) - \frac{1}{8}b_{101}c_{101}(a_{110} + 2b_{020}) + \frac{1}{8}b_{110}b_{101}c_{011} - \frac{5}{6}b_{110}a_{101}c_{101}$,

$J_4 = b[a_{101}C_{110} + \frac{1}{2}(C_{011}c_{101} - C_{101}c_{011}) + \frac{3}{4}c_{020}(A_{011} + B_{101}) + \frac{1}{8}c_{101}(A_{110} - 7B_{200} - 5B_{020}) + \frac{1}{8}c_{011}(7A_{020} - B_{110} + 5A_{200})$,

$J_5 = b^2(C'_{200} + C'_{020})$,

$J_6 = \frac{3}{2}b(c_{011}A_{002} - c_{101}B_{002})$,

$J_7 = 2b^2C'_{002}$.

To look for the limit cycles, we must solve the system

$$\begin{cases} f_{31}(\rho,\xi) = 0, \\ f_{32}(\rho,\xi) = 0. \end{cases} \tag{12}$$

Solving the first equation with respect to ρ and avoiding the solutions with $\rho = 0$, we obtain the two solutions

$$\rho_1 = \frac{\sqrt{-I_3\xi^2 - I_4\xi - I_0}}{\sqrt{I_1\xi + I_2}}, \quad \rho_2 = -\frac{\sqrt{-I_3\xi^2 - I_4\xi - I_0}}{\sqrt{I_1\xi + I_2}}.$$

Since ρ must be positive, we keep ρ_1. Then the second equation becomes

$$\tfrac{1}{\Lambda}\left[K_0^2\xi^{10} + \Lambda_0\xi^9 + \Lambda_1\xi^8 + \Lambda_2\xi^7 + \Lambda_3\xi^6 + \Lambda_4\xi^5 + \Lambda_5\xi^4 + \Lambda_6\xi^3 + \Lambda_7\xi^2 + 2K_4K_5\xi + K_5^2\right],$$

where:

$\Lambda = 2b^3(I_1\xi + I_2)^2((K_0\xi^5 + K_1\xi^4 + K_2\xi^3 + K_3\xi^2 + K_4\xi + K_5) - \left(-J_2I_3\xi^3 - J_2I_4\xi^2 - J_2I_0\xi\right) \times$

$\sqrt{-I_3\xi^2 - I_4\xi - I_0}\sqrt{I_1\xi + I_2}),$

$\Lambda_0 = 2K_0K_1 + J_2^2I_3^3I_1,$

$\Lambda_1 = 3J_2^2I_3^2I_4I_1 + J_2^2I_3^3I_2 + 2K_0K_2 + K_1^2,$

$\Lambda_2 = 2K_0K_3 + 2K_1K_2 + 3J_2^2I_3I_4^2I_1 + 3J_2^2I_3^2I_4I_2 + 3J_2^2I_3^2I_0I_1,$

$\Lambda_3 = 3J_2^2I_3I_4^2I_2 + K_2^2 + 2K_1K_3 + J_2^2I_4^3I_1 + 2K_0K_4 + 6J_2^2I_3I_4I_0I_1 + 3J_2^2I_3^2I_0I_2,$

$\Lambda_4 = J_2^2I_4^3I_2 + 3J_2^2I_3I_0^2I_1 + 2K_2K_3 + 6J_2^2I_3I_4I_0I_2 + 2K_1K_4 + 2K_0K_5 + 3J_2^2I_4^2I_0I_1,$

$\Lambda_5 = 3J_2^2I_3I_0^2I_2 + 3J_2^2I_4I_0^2I_1 + K_3^2 + 2K_2K_4 + 2K_1K_5 + 3J_2^2I_4^2I_0I_2,$

$\Lambda_6 = 3J_2^2I_4I_0^2I_2 + 2K_3K_4 + 2K_2K_5 + J_2^2I_0^3I_1,$

$\Lambda_7 = J_2^2I_0^3I_2 + K_4^2 + 2K_3K_5,$

and

$K_0 = J_6I_1^2 - J_3I_3I_1,$

$K_1 = J_1I_3^2 + 2J_6I_1I_2 - J_4I_3I_1 - J_3I_3I_2 + J_7I_1^2 - J_3I_4I_1,$

$K_2 = J_6I_2^2 - J_3I_0I_1 - J_4I_4I_1 + 2J_7I_1I_2 - J_5I_3I_1 + J_0I_1^2 + 2J_1I_3I_4 - J_3I_4I_2 - J_4I_3I_2,$

$K_3 = 2J_0I_1I_2 - J_4I_0I_1 - J_5I_3I_2 - J_5I_4I_1 + 2J_1I_3I_0 - J_3I_0I_2 + J_1I_4^2 - J_4I_4I_2 + J_7I_2^2,$

$K_4 = J_0I_2^2 - J_4I_0I_2 + 2J_1I_4I_0 - J_5I_0I_1 - J_5I_4I_2,$

$K_5 = J_1I_0^2 - J_5I_0I_2.$

It is easy to verify that the coefficients of this equation can take arbitrary values when we play with the coefficients of system (1), so they can be chosen in such a way that this equation has ten real zeros different from zero. Let $\bar{\xi}$ be one these zeros. Let $(\bar{\rho}, \bar{\xi})$ be a solution of system (12). In order to have a limit cycle according to the theory in section 2, we must have

$$D(\bar{\rho}, \bar{\xi}) = det\begin{pmatrix} \frac{\partial f_{31}}{\partial \rho} & \frac{\partial f_{31}}{\partial \xi} \\ \frac{\partial f_{32}}{\partial \rho} & \frac{\partial f_{32}}{\partial \xi} \end{pmatrix}|_{(\rho,\xi)=(\bar{\rho},\bar{\xi})} \neq 0. \tag{13}$$

In short, the solutions $(\bar{\rho}, \bar{\xi})$ of system (12), which verify condition (13), satisfy the assumptions (i) and (ii) of Section 2. We conclude that applying the averaging theory of third order, system (5) has at most 10 limit cycles. Therefore, due to the rescaling, system (1) has at most 10 limit cycles bifurcating from the origin. This completes the proof of statement (a) of Theorem 1.

4. Proof of statement (b) of Theorem 1

We consider the quadratic polynomial differential system

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{1}{4}\varepsilon^3 x - y + 2xy - 4xz - \dfrac{199}{36}yz - 2\varepsilon x^2 + \varepsilon y^2 + \varepsilon^2 y^2 + 3\varepsilon xy + 3\varepsilon yz - \varepsilon^2 xz, \\ \dfrac{dy}{dt} = x - \dfrac{1}{4}\varepsilon^3 y - 2x^2 + 2y^2 + \dfrac{199}{36}xz + 4yz + \varepsilon x^2 + 2\varepsilon^2 x^2 - 2\varepsilon y^2 + 5\varepsilon xz + 3\varepsilon xy - \varepsilon^2 yz, \\ \dfrac{dz}{dt} = \dfrac{1}{8}\varepsilon^3 z + 2x^2 - 2y^2 - yz - xz + 24\varepsilon xz - 2\varepsilon yz + \varepsilon^2 x^2 + \varepsilon^2 y^2 - 9\varepsilon^2 z^2. \end{cases} \tag{14}$$

The eigenvalues of the singular point $(0,0,0)$ of system (14) are $-\frac{1}{4}\varepsilon^3 \pm i$ and $\frac{1}{8}\varepsilon^3$.
The corresponding system (5) associated to system (14) satisfies

$$F_{11}(\theta,\rho,\xi) = -4\rho\xi\cos^2(\theta) + 2\rho^2\sin^3(\theta) + 4\rho\xi\sin^2(\theta),$$

$$F_{21}(\theta,\rho,\xi) = -2\rho^2\cos^3(\theta) + 4\rho^2\cos^2(\theta)\sin(\theta) + 4\rho^2\cos(\theta)\sin^2(\theta) + 8\rho\xi\cos(\theta)\sin(\theta)$$
$$- 2\rho^2\sin^3(\theta) - 8\rho^2\xi\cos^5(\theta) + 32\rho\xi^2\cos^3(\theta)\sin(\theta) + \tfrac{199}{9}\rho\xi^2\cos^4(\theta) + 8\rho^2\xi\cos^3(\theta)\sin^2(\theta)$$
$$- 32\rho\xi^2\cos(\theta)\sin^3(\theta) - \tfrac{199}{9}\rho\xi^2\sin^4(\theta) + 4\rho^3\cos^3(\theta)\sin^3(\theta) - 16\rho^2\xi\cos(\theta)\sin^4(\theta)$$
$$- \tfrac{199}{18}\rho^2\xi\cos^2(\theta)\sin^3(\theta) - \tfrac{199}{18}\rho^2\xi\sin^5(\theta),$$

$$F_{31}(\theta,\rho,\xi) = -\tfrac{1}{4}\rho - \rho\xi + \rho^2\cos(\theta)\sin^2(\theta) + 2\rho^2\cos^2(\theta)\sin(\theta) + \tfrac{271}{18}\rho^2\xi\cos^5(\theta)$$
$$- \tfrac{1387}{18}\rho^2\xi\cos^2(\theta)\sin^3(\theta) - \tfrac{1207}{18}\rho^2\xi\cos^3(\theta)\sin^2(\theta) + \tfrac{269}{9}\rho^2\xi\cos^4(\theta)\sin(\theta) + 20\rho\xi^2\cos^4(\theta)$$
$$- 96\rho\xi^2\cos^2(\theta)\sin^2(\theta) - 6\rho^3\cos^3(\theta)\sin^3(\theta) + (2\rho^3 + \tfrac{39601}{324}\rho\xi^3)\sin^6(\theta) + (10\rho^3 + \tfrac{3184}{9}\rho\xi^3)\cos(\theta)\sin^5(\theta)$$
$$+ (-10\rho^3 + \tfrac{122545}{324}\rho\xi^3)\cos^2(\theta)\sin^4(\theta) + \tfrac{379}{18}\rho^2\xi\sin^5(\theta) + \tfrac{125}{9}\rho^2\xi\cos(\theta)\sin^4(\theta) + 12\rho\xi^2\sin^4(\theta)$$
$$+ (-4\rho^3 - \tfrac{39601}{324}\rho\xi^3)\cos^6(\theta) + (8\rho^3 - \tfrac{3184}{9}\rho\xi^3)\cos^5(\theta)\sin(\theta) - \tfrac{398}{9}\rho\xi^2\cos^3(\theta)\sin(\theta) - \tfrac{398}{9}\rho\xi^2\cos(\theta)\sin^3(\theta) +$$
$$(8\rho^3 - \tfrac{122545}{324}\rho\xi^3)\cos^4(\theta)\sin^2(\theta) - 16\rho^3\xi\cos^8(\theta) + 128\rho^2\xi^2\cos^6(\theta)\sin(\theta) + \tfrac{796}{9}\rho^2\xi^2\cos^7(\theta) + 8\rho^4\cos^6(\theta)\sin^3(\theta) -$$
$$64\rho^3\xi\cos^4(\theta)\sin^4(\theta) - \tfrac{398}{9}\rho^3\xi\cos^5(\theta)\sin^3(\theta) - \tfrac{398}{9}\rho^3\xi\cos^3(\theta)\sin^5(\theta) +$$
$$\tfrac{81073}{324}\rho^2\xi^2\cos^2(\theta)\sin^5(\theta) + \tfrac{796}{9}\rho^2\xi^2\cos^3(\theta)\sin^4(\theta) + \tfrac{1592}{9}\rho^2\xi^2\cos(\theta)\sin^6(\theta) -$$
$$\tfrac{43343}{648}\rho^2\xi^2\cos^4(\theta)\sin^3(\theta) + 16\rho^3\xi\cos^6(\theta)\sin^2(\theta) + \tfrac{39601}{648}\rho^2\xi^2\sin^7(\theta),$$

$$F_{12}(\theta,\rho,\xi) = 2\rho^2\cos^2(\theta) - 2\rho^2\sin^2(\theta) - \rho\xi\cos(\theta) - \rho\xi\sin(\theta),$$

$$F_{22}(\theta,\rho,\xi) = 24\rho\xi\cos(\theta) - 2\rho\xi\sin(\theta) + 4\rho^3\cos^5(\theta) - 18\rho^2\xi\cos^3(\theta)\sin(\theta) - \tfrac{235}{18}\rho^2\xi\cos^4(\theta)$$
$$- 4\rho^3\cos^3(\theta)\sin^2(\theta) + 16\rho^2\xi\cos(\theta)\sin^3(\theta) + \tfrac{199}{18}\rho^2\xi\sin^4(\theta) + \tfrac{487}{36}\rho\xi^2\cos(\theta)\sin^2(\theta) + \tfrac{487}{36}\rho\xi^2\cos^2(\theta)\sin(\theta)$$
$$+ \tfrac{199}{36}\rho\xi^2\cos^3(\theta) + \tfrac{199}{36}\rho\xi^2\sin^3(\theta),$$

$$F_{32}(\theta,\rho,\xi) = \tfrac{1}{8}\xi + \rho^2 - 9\xi^2 + 39\rho^2\xi\cos^4(\theta) - \tfrac{3167}{18}\rho\xi^2\cos^2(\theta)\sin(\theta) - \tfrac{383}{3}\rho\xi^2\cos^3(\theta) - \tfrac{359}{3}\rho\xi^2\cos(\theta)\sin^2(\theta) +$$
$$2\rho^2\xi\cos^3(\theta)\sin(\theta) + \tfrac{145}{18}\rho\xi^2\sin^3(\theta) + (-2\rho^3 - \tfrac{39601}{1296}\rho\xi^3)\cos^5(\theta) +$$
$$(12\rho^3 - \tfrac{138385}{648}\rho\xi^3)\cos^2(\theta)\sin^3(\theta) + (12\rho^3 - \tfrac{138385}{648})\cos^3(\theta)\sin^2(\theta) - (10\rho^3 + \tfrac{154225}{1296}\rho\xi^3)\cos^4(\theta)\sin(\theta)$$
$$+16\rho^2\xi\cos^2(\theta)\sin^2(\theta) - (2\rho^3 + \tfrac{39601}{1296}\rho\xi^3)\sin^5(\theta) - (10\rho^3 + \tfrac{154225}{1296}\rho\xi^3)\cos(\theta)\sin^4(\theta) - 7\rho^2\xi\sin^4(\theta) - 6\rho^2\xi\cos(\theta)\sin^3(\theta) +$$
$$8\rho^4\cos^8(\theta) - 68\rho^3\xi\cos^6(\theta)\sin(\theta) - \tfrac{434}{9}\rho^3\xi\cos^7(\theta) + \tfrac{157609}{648}\rho^2\xi^2\cos^4(\theta)\sin^2(\theta)$$

$+231\rho^2\xi^2\cos^5(\theta)\sin(\theta)+\frac{199}{9}\rho^2\xi^2\cos^3(\theta)\sin^3(\theta)+\frac{53929}{648}\rho^2\xi^2\cos^6(\theta)-\frac{122545}{648}\rho^2\xi^2\cos^2(\theta)\sin^4(\theta)-8\rho^4\cos^6(\theta)\sin^2(\theta)+64\rho^3\xi\cos^4(\theta)\sin^3(\theta)+\frac{398}{9}\rho^3\xi\cos^3(\theta)\sin^4(\theta)-\frac{1592}{9}\rho^2\xi^2\cos(\theta)\sin^5(\theta)-\frac{39601}{648}\rho^2\xi^2\sin^6(\theta).$

To look for the limit cycles we must solve the system

$$\begin{aligned} f_{31}(\rho,\xi) &= \frac{1}{2}\rho[-\frac{1}{2}+\rho^2\xi-4\xi^2-2\xi+3\rho^2]=0,\\ f_{32}(\rho,\xi) &= \frac{1}{2}[\frac{-17}{24}\rho^4-2\rho^3\xi+\frac{455}{48}\rho^2\xi^2+\rho^2\xi+2\rho^2-18\xi^2+\frac{1}{4}\xi]=0. \end{aligned} \tag{15}$$

Solving the first equation with respect to ρ and avoiding the solutions with $\rho=0$, we obtain the two solutions

$$\rho_1=\frac{\sqrt{2}\sqrt{(3+\xi)(1+8\xi^2+4\xi)}}{2(3+\xi)},\rho_2=-\frac{\sqrt{2}\sqrt{(3+\xi)(1+8\xi^2+4\xi)}}{2(3+\xi)}.$$

Since ρ must be positive, we keep ρ_1. Then the second equation becomes

$\frac{1}{L(\xi)}(13249600\xi^{10}+72682944\xi^9+70849632\xi^8-165550520\xi^7-219277487\xi^6+105231662\xi^5+121637617\xi^4-35209606\xi^3-3958090\xi^2+797824\xi+73441),$
where

$L(\xi)=-192(3+\xi)^2((-3640\xi^5-10308\xi^4+3405\xi^3+11275\xi^2-1472\xi-271)-((384\xi^3+192\xi^2+48\xi)\sqrt{2}\sqrt{(3+\xi)(1+8\xi^2+4\xi)}).$

Solving this equation we get the roots

$$\xi_1=0.2075448556,\xi_2=0.2648446151,\xi_3=0.7360098291,\xi_4=1.245269006,\xi_5=-0.1024648022,$$

$$\xi_6=-0.1060657437,\xi_7=-1.080408535,\xi_8=-1.337103376,\xi_9=-2.433134674,\xi_{10}=-2.880161987.$$

Since ρ must be positive and ξ real, the unique admissible roots are

$$(0.5822454911, 0.2075448556), (0.6335012386, 0.2648446151), (1.052534750, 0.7360098291),$$

$$(1.471578806, 1.245269006), (0.3410699520, -0.1024648022), (0.3391500426, -0.1060657437),$$

$$(1.251863940, -1.080408535), (1.730050696, -1.337103376), (5.837133551, -2.433134674),$$

$$(15.26399619, -2.880161987).$$

According to statement (a) of Theorem 1, system (14) has at most 10 limit cycles.
Now we must verify that the determinant is different from zero at these roots where

$$D(\bar\rho,\bar\xi)=det(M)=det\begin{pmatrix} -\frac{1}{4}+\frac{3}{2}\bar\rho^2\bar\xi-2\bar\xi^2-\bar\xi+\frac{9}{2}\bar\rho^2 & \frac{1}{2}\bar\rho^3-4\bar\rho\bar\xi-\bar\rho \\ \frac{-17}{12}\bar\rho^3-3\bar\rho^2\bar\xi+\frac{455}{48}\bar\rho\bar\xi^2+\bar\rho\bar\xi+2\bar\rho & -\bar\rho^3+\frac{455}{48}\bar\rho^2\bar\xi+\frac{1}{2}\bar\rho^2-18\bar\xi+\frac{1}{8} \end{pmatrix}.$$

Easy computations show that

$$D(0.5822454911, 0.2075448556) = -2.233224452, D(0.6335012386, 0.2648446151) = -3.446707469,$$

$$D(1.052534750, 0.7360098291) = -9.92066588, D(1.471578806, 1.245269006) = 110.2318284,$$

$$D(0.3410699520, -0.1024648022) = 0.7518306652, D(0.3391500426, -0.1060657437) = 0.7607482745,$$

$$D(1.251863940, -1.080408535) = -81.87173135, D(1.730050696, -1.337103376) = -442.0333741,$$

$$D(5.837133551, -2.433134674) = -61762.27228, D(15.26399619, -2.880161987) = 3.291647160 \cdot 10^6.$$

In short, this proves that system (14) has exactly 10 limit cycles bifurcating from the origin. Hence, statement (b) of Theorem 1 is proved. Now we shall study the stability of these 10 limit cycles.

The eigenvalues of the matrix M at the points

$(0.5822454911, 0.2075448556), (0.6335012386, 0.2648446151), (1.052534750, 0.7360098291), (1.471578806, 1.245269006), (0.3410699520, -0.1024648022), (0.3391500426, -0.1060657437), (1.251863940, -1.080408535), (1.730050696, -1.337103376), (5.837133551, -2.433134674)$ and $(15.26399619, -2.880161987)$
are

$(0.8244374073, -2.7087859343), (1.01563922728, -3.39363366), (2.3514989, -4.2188689), (5.1809028 \pm 9.1318165i), (0.419430749, 1.79250249535), (0.408750104, 1.86115738), (12.112016, -6.75954614), (15.7026074, -28.1503168), (63.788027488, -968.24239)$ and $(-351.3102078, -9369.63141676)$, respectively.

Therefore, the corresponding limit cycles Γ_i for $i = 1, ..., 10$ are as follows:

Γ_1 semistable, Γ_2 semistable, Γ_3 semistable, Γ_4 unstable, Γ_5 unstable, Γ_6 unstable, Γ_7 semistable, Γ_8 semistable, Γ_9 semistable, and Γ_{10} stable.

The 10 limit cycles Γ_i for $i = 1, ..., 10$ of system (5) associated to system (14) and corresponding to the zeros $(\bar{\rho}, \bar{\xi})$ given by (15) can be written as $\{(R_i(\theta), z_i(\theta)), \theta \in \mathbb{S}^1\}$, where from (3) we have

$$\begin{pmatrix} R_i(\theta) \\ z_i(\theta) \end{pmatrix} = \varepsilon \left[\begin{pmatrix} \bar{\rho} \\ \bar{\xi} \end{pmatrix} + \varepsilon \begin{pmatrix} \int_0^\theta F_{11}(s, \bar{\rho}, \bar{\xi}) ds \\ \int_0^\theta F_{12}(s, \bar{\rho}, \bar{\xi}) ds \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \int_0^\theta F_{21}(s, \bar{\rho}, \bar{\xi}) ds \\ \int_0^\theta F_{22}(s, \bar{\rho}, \bar{\xi}) ds \end{pmatrix} + \begin{pmatrix} O(\varepsilon^3) \\ O(\varepsilon^3) \end{pmatrix} \right]$$

where

$$\begin{pmatrix} \int_0^\theta F_{11}(s, \bar{\rho}, \bar{\xi}) ds \\ \int_0^\theta F_{12}(s, \bar{\rho}, \bar{\xi}) ds \end{pmatrix} = \begin{pmatrix} \frac{4}{3}\bar{\rho}^2(1 - \cos(\theta)) - 4\bar{\rho}\bar{\xi}\cos(\theta)\sin(\theta) - \frac{2}{3}\bar{\rho}^2\cos(\theta)\sin^2(\theta) \\ \bar{\rho}\bar{\xi}(\cos(\theta) - \sin(\theta) - 1) + 2\bar{\rho}^2\cos(\theta)\sin(\theta) \end{pmatrix},$$

and

$$\begin{pmatrix} \int_0^\theta F_{21}(s,\overline{\rho},\overline{\xi})ds \\ \int_0^\theta F_{22}(s,\overline{\rho},\overline{\xi})ds \end{pmatrix} = \begin{pmatrix} -\frac{32}{5}\overline{\rho}^2\overline{\xi}\cos^4(\theta)\sin(\theta) - \frac{8}{15}\overline{\rho}^2\overline{\xi}\cos^2(\theta)\sin(\theta) + \frac{199}{9}\overline{\rho}\overline{\xi}^2\cos(\theta)\sin(\theta) \\ -4\overline{\rho}\overline{\xi}\cos^2(\theta) - \frac{199}{54}\overline{\xi}\overline{\rho}^2\cos^3(\theta) + 16\overline{\rho}\overline{\xi}^2\cos^2(\theta) - 16\overline{\rho}\overline{\xi}^2\cos^4(\theta) \\ -\frac{16}{3}\overline{\rho}^2\overline{\xi}\sin^3(\theta) + \frac{199}{18}\overline{\xi}\overline{\rho}^2\cos(\theta) - 2\overline{\rho}^2\sin(\theta)\cos^2(\theta) - \frac{16}{15}\overline{\rho}^2\overline{\xi}\sin(\theta) \\ -2\overline{\rho}^2\cos^3(\theta) + 2\overline{\rho}^2\cos(\theta) + \frac{2}{3}\overline{\rho}^3\cos^6(\theta) - \overline{\rho}^3\cos^4(\theta) + 4\overline{\rho}\overline{\xi} \\ +\frac{1}{3}\overline{\rho}^3 - \frac{199}{27}\overline{\rho}^2\overline{\xi} \\ \\ 2\overline{\rho}\overline{\xi}(\cos(\theta) + 12\sin(\theta) - 1) + \frac{295}{36}\overline{\rho}\overline{\xi}^2 - \frac{1}{2}\overline{\rho}^2\overline{\xi} - \frac{8}{3}\overline{\rho}\overline{\xi}^2\cos^3(\theta) - \\ \frac{1}{2}\overline{\rho}^2\overline{\xi}\cos^3(\theta)\sin(\theta) - \frac{425}{36}\overline{\rho}^2\overline{\xi}\cos(\theta)\sin(\theta) - \frac{3}{4}\overline{\rho}^2\overline{\xi}\theta + \frac{199}{36}\overline{\rho}\overline{\xi}^2(\sin(\theta) - \cos(\theta)) \\ +\frac{17}{2}\overline{\rho}^2\overline{\xi}\cos^4(\theta) + \frac{8}{5}\overline{\rho}^3\cos^4(\theta)\sin(\theta) + \frac{4}{5}\overline{\rho}^3\cos^2(\theta)\sin(\theta) \\ +\frac{8}{5}\overline{\rho}^3\sin(\theta) + \frac{8}{3}\overline{\rho}\overline{\xi}^2\sin^3(\theta) - 8\overline{\rho}^2\overline{\xi}\cos^2(\theta) \end{pmatrix}.$$

Therefore, the limit cycle Γ_1 can be written as

$$R_1(\theta) = 0.5822454911\varepsilon + \varepsilon^2(0.4520130825(1-\cos(\theta)) - 0.4833682256\cos(\theta)\sin(\theta) -$$
$$0.2260065413\cos(\theta)\sin^2(\theta)) + \varepsilon^3(0.0305865093 - 0.0820858714\cos^2(\theta) - 0.9373083043\cos^3(\theta) -$$
$$0.5986692886\cos^4(\theta) - 0.3752519598\sin^3(\theta) + 1.45588566\cos(\theta) - 0.7155448198\cos^2(\theta)\sin(\theta) -$$
$$0.07505039196\sin(\theta) - 0.4503023517\sin(\theta)\cos^4(\theta) + 0.554549920\cos(\theta)\sin(\theta) +$$
$$0.1315912896\cos^6(\theta)) + O(\varepsilon^4)$$
$$z_1(\theta) = 0.2075448556\varepsilon + \varepsilon^2(0.1208420564(\cos(\theta) - \sin(\theta) - 1) + 0.6780196238\cos(\theta)\sin(\theta)) +$$
$$\varepsilon^3(-0.07134611163 + 0.103046632\cos(\theta) + 3.354665929\sin(\theta) - 0.06688039237\cos^3(\theta) -$$
$$0.03517987123\cos^3(\theta)\sin(\theta) - 0.8306358485\cos(\theta)\sin(\theta) - 0.052769806 84\theta + 0.598057810\cos^4(\theta)$$
$$+0.315819095\cos^4(\theta)\sin(\theta) + 0.1579095475\cos^2(\theta)\sin(\theta) + 0.0668803923\sin^3(\theta) -$$
$$0.5628779397\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_2 can be written as

$$R_2(\theta) = 0.6335012386\varepsilon + \varepsilon^2(0.5350984257(1-\cos(\theta)) - 0.6711175668\cos(\theta)\sin(\theta) -$$
$$0.2675492129\cos(\theta)\sin^2(\theta)) + \varepsilon^3(-0.0275213153 + 0.0398499279\cos^2(\theta) - 1.194340269\cos^3(\theta) -$$
$$0.9652066313\cos^4(\theta) - 0.5668717467\sin^3(\theta) + 1.97772553\cos(\theta) - 0.8593348133\cos^2(\theta)\sin(\theta) -$$
$$0.1133743493\sin(\theta) - 0.6802460960\sin(\theta)\cos^4(\theta) + 0.9825175795\cos(\theta)\sin(\theta) +$$
$$0.1694927577\cos^6(\theta)) + O(\varepsilon^4)$$
$$z_2(\theta) = 0.2648446151\varepsilon + \varepsilon^2(0.1677793917(\cos(\theta) - \sin(\theta) - 1) + 0.8026476386\cos(\theta)\sin(\theta))$$

$$+\varepsilon^3(-0.02457903235 + 0.089929388\cos(\theta) + 4.67911741\sin(\theta) - 0.1184945825\cos^3(\theta) -$$

$$0.05314422625\cos^3(\theta)\sin(\theta) - 1.254794231\cos(\theta)\sin(\theta) - 0.07971633938\theta + 0.903451846\cos^4(\theta)$$

$$+0.4067826186\cos^4(\theta)\sin(\theta) + 0.203391309\cos^2(\theta)\sin(\theta) + 0.1184945825\sin^3(\theta)$$

$$-0.8503076200\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_3 can be written as

$$R_3(\theta) = 1.052534750\varepsilon + \varepsilon^2(1.477105867(1-\cos(\theta)) - 3.098703686\cos(\theta)\sin(\theta) -$$

$$0.7385529333\cos(\theta)\sin^2(\theta)) + \varepsilon^3(-2.522223413 + 6.024001794\cos^2(\theta) - 5.220460507\cos^3(\theta) -$$

$$10.28873442\cos^4(\theta) - 4.348657746\sin^3(\theta) + 11.2300639\cos(\theta) - 2.650524575\cos^2(\theta)\sin(\theta) -$$

$$0.8697315493\sin(\theta) - 5.218389295\sin(\theta)\cos^4(\theta) + 12.60707216\cos(\theta)\sin(\theta) +$$

$$0.7773526273\cos^6(\theta)) + O(\varepsilon^4)$$

$$z_3(\theta) = 0.7360098291\varepsilon + \varepsilon^2(0.7746759215(\cos(\theta) - \sin(\theta) - 1) + 2.215658800\cos(\theta)\sin(\theta))$$

$$+\varepsilon^3(2.715180445 - 1.602416196\cos(\theta) + 23.60963647\sin(\theta) - 1.520450913\cos^3(\theta) -$$

$$0.4076866637\cos^3(\theta)\sin(\theta) - 9.625935116\cos(\theta)\sin(\theta) - 0.61152999556\theta + 6.930673283\cos^4(\theta)$$

$$+1.865646306\cos^4(\theta)\sin(\theta) + 0.9328231528\cos^2(\theta)\sin(\theta) + 1.520450913\sin^3(\theta) - 6.522986619\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_4 can be written as

$$R_4(\theta) = 1.471578806\varepsilon + \varepsilon^2(2.887392243(1-\cos(\theta)) - 7.330045908\cos(\theta)\sin(\theta) -$$

$$1.443696121\cos(\theta)\sin^2(\theta)) + \varepsilon^3(-11.48326538 + 29.18147001\cos^2(\theta) - 14.26887216\cos^3(\theta)$$

$$-39.69828484\cos^4(\theta) - 14.38232027\sin^3(\theta) + 34.14443976\cos(\theta) - 5.769320391\cos^2(\theta)\sin(\theta) -$$

$$2.876464054\sin(\theta) - 17.25878433\sin(\theta)\cos^4(\theta) + 50.45688658\cos(\theta)\sin(\theta) + 2.12451261\cos^6(\theta)) + O(\varepsilon^4)$$

$$z_4(\theta) = 1.245269006\varepsilon + \varepsilon^2(1.832511477(\cos(\theta) - \sin(\theta) - 1) + 4.33108836\cos(\theta)\sin(\theta))$$

$$+\varepsilon^3(13.68610882 - 8.949198696\cos(\theta) + 61.69332737\sin(\theta) - 6.085252653\cos^3(\theta) -$$

$$1.348342526\cos^3(\theta)\sin(\theta) - 31.83586519\cos(\theta)\sin(\theta) - 2.0225137788\theta + 22.92182293\cos^4(\theta) +$$

$$5.098830275\cos^4(\theta)\sin(\theta) + 2.549415138\cos^2(\theta)\sin(\theta) + 6.085252653\sin^3(\theta) - 21.57348041\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_5 can be written as

$$R_5(\theta) = 0.3410699520\varepsilon + \varepsilon^2(0.1551049496(1-\cos(\theta)) + 0.1397906607\cos(\theta)\sin(\theta) -$$

$$0.07755247480\cos(\theta)\sin^2(\theta)) + \varepsilon^3(-0.03871339576 + 0.1970851503\cos^2(\theta) - 0.1887314966\cos^3(\theta)$$

$$-0.09697071785\cos^4(\theta) + 0.0635711919\sin^3(\theta) + 0.100879641\cos(\theta) - 0.2263003052\cos^2(\theta)\sin(\theta) +$$

$$0.0127142383\sin(\theta) + 0.0762854303\sin(\theta)\cos^4(\theta) + 0.0791778015\cos(\theta)\sin(\theta) + 0.0264508188\cos^6(\theta)) + O(\varepsilon^4)$$

$$z_5(\theta) = -0.1024648022\varepsilon + \varepsilon^2(-0.3494766517(\cos(\theta) - \sin(\theta) - 1)+$$
$$0.2326574244\cos(\theta)\sin(\theta)) + \varepsilon^3(0.10519866 - 0.08968978073\cos(\theta) - 0.7554675485\sin(\theta)-$$
$$0.009549081597\cos^3(\theta) + 0.00595979924\cos^3(\theta)\sin(\theta) + 0.140717482\cos(\theta)\sin(\theta)+$$
$$0.0089396988868\theta - 0.1013165872\cos^4(\theta) + 0.0634819652\cos^4(\theta)\sin(\theta) + 0.0317409826\cos^2(\theta)\sin(\theta)$$
$$+0.00954908159\sin^3(\theta) + 0.09535678792\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_6 can be written as

$$R_6(\theta) = 0.3391500426\varepsilon + \varepsilon^2(0.1533636685(1 - \cos(\theta)) + 0.1438888060\cos(\theta)\sin(\theta)-$$
$$0.07668183427\cos(\theta)\sin^2(\theta)) + \varepsilon^3(-0.04096715784 + 0.204935498\cos^2(\theta) - 0.1850863406\cos^3(\theta)-$$
$$0.1000566639\cos^4(\theta) + 0.06506652624\sin^3(\theta) + 0.0951680161\cos(\theta) - 0.2235388502\cos^2(\theta)\sin(\theta)+$$
$$0.0130133052\sin(\theta) + 0.0780798314\sin(\theta)\cos^4(\theta) + 0.08436313808\cos(\theta)\sin(\theta) + 0.02600664736\cos^6(\theta)) + O(\varepsilon^4)$$
$$z_6(\theta) = -0.1060657437\varepsilon + \varepsilon^2(-0.03597220149(\cos(\theta) - \sin(\theta) - 1)+$$
$$0.2300455028\cos(\theta)\sin(\theta)) + \varepsilon^3(0.1093096231 - 0.09303518750\cos(\theta) - 0.7798260976\sin(\theta)-$$
$$0.01017444881\cos^3(\theta) + 0.00609998683\cos^3(\theta)\sin(\theta) + 0.1440274669\cos(\theta)\sin(\theta) + 0.0091499802252\theta-$$
$$0.1036997762\cos^4(\theta) + 0.0624159536\cos^4(\theta)\sin(\theta) + 0.0312079768\cos^2(\theta)\sin(\theta)+$$
$$0.01017444881\sin^3(\theta) + 0.09759978936\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_7 can be written as

$$R_7(\theta) = 1.251863940\varepsilon + \varepsilon^2(2.089551099(1 - \cos(\theta)) + 5.410097940\cos(\theta)\sin(\theta)-$$
$$1.044775549\cos(\theta)\sin^2(\theta)) + \varepsilon^3(7.723199348 + 28.79056189\cos^2(\theta) + 3.105342788\cos^3(\theta)-$$
$$25.34233920\cos^4(\theta) + 9.03027536\sin^3(\theta) - 15.58468166\cos(\theta) - 2.231299112\cos^2(\theta)\sin(\theta)$$
$$+1.80605507\sin(\theta) + 10.83633044\sin(\theta)\cos^4(\theta) + 32.3105022\cos(\theta)\sin(\theta) + 1.30791683\cos^6(\theta)) + O(\varepsilon^4)$$
$$z_7(\theta) = -1.080408535\varepsilon + \varepsilon^2(-1.352524485(\cos(\theta) - \sin(\theta) - 1) + 3.134326648\cos(\theta)\sin(\theta))$$
$$+\varepsilon^3(15.52600685 - 10.78267454\cos(\theta) - 21.24396166\sin(\theta) - 3.896743992\cos^3(\theta)+$$
$$0.846588315\cos^3(\theta)\sin(\theta) + 19.9888907\cos(\theta)\sin(\theta) + 1.2698824473\theta - 14.39200136\cos^4(\theta)+$$
$$3.139000406\cos^4(\theta)\sin(\theta) + 1.569500203\cos^2(\theta)\sin(\theta) + 3.896743992\sin^3(\theta) + 13.54541305\cos^2(\theta)) + O(\varepsilon^4);$$

the limit cycle Γ_8 can be written as

$$R_8(\theta) = 1.730050696\varepsilon + \varepsilon^2(3.990767215(1 - \cos(\theta)) + 9.25302650\cos(\theta)\sin(\theta)-$$
$$1.995383607\cos(\theta)\sin^2(\theta)) + \varepsilon^3(21.96963076 + 58.74203840\cos^2(\theta) + 8.762149108\cos^3(\theta)$$
$$-54.66718410\cos^4(\theta) + 21.34427326\sin^3(\theta) - 38.25874897\cos(\theta) - 3.851723496\cos^2(\theta)\sin(\theta)+$$

$$4.26885465\sin(\theta)+25.61312792\sin(\theta)\cos^4(\theta)+68.39106506\cos(\theta)\sin(\theta)+3.452114799\cos^6(\theta))+O(\varepsilon^4)$$

$$z_8(\theta)=-1.337103376\varepsilon+\varepsilon^2(-2.313256626(\cos(\theta)-\sin(\theta)-1)+5.986150822\cos(\theta)\sin(\theta))$$

$$+\varepsilon^3(31.97347379-21.72427952\cos(\theta)-30.13531723\sin(\theta)-8.248168651\cos^3(\theta)+$$

$$2.001025618\cos^3(\theta)\sin(\theta)+47.2464382\cos(\theta)\sin(\theta)+3.0015384284\theta-34.01743551\cos^4(\theta)+$$

$$8.285075517\cos^4(\theta)\sin(\theta)+4.142537758\cos^2(\theta)\sin(\theta)+8.248168651\sin^3(\theta)+32.0164099\cos^2(\theta))+O(\varepsilon^4);$$

the limit cycle Γ_9 can be written as

$$R_9(\theta)=5.837133551\varepsilon+\varepsilon^2(45.42950412(1-\cos(\theta))+56.81012816\cos(\theta)\sin(\theta)-$$

$$22.71475206\cos(\theta)\sin^2(\theta))+\varepsilon^3(620.5033991+609.7168989\cos^2(\theta)+237.3652471\cos^3(\theta)-$$

$$751.7903327\cos^4(\theta)+442.1444068\sin^3(\theta)-848.3842537\cos(\theta)-23.92981550\cos^2(\theta)\sin(\theta)+$$

$$88.42888136\sin(\theta)+530.5732881\sin(\theta)\cos^4(\theta)+764.086440\cos(\theta)\sin(\theta)+132.5890413\cos^6(\theta))+O(\varepsilon^4)$$

$$z_9(\theta)=-2.433134674\varepsilon+\varepsilon^2(-14.20253204(\cos(\theta)-\sin(\theta)-1)+68.14425618\cos(\theta)\sin(\theta))$$

$$+\varepsilon^3(353.0288407-219.4266741\cos(\theta)+168.374540\sin(\theta)-92.15112845\cos^3(\theta)+$$

$$41.45103814\cos^3(\theta)\sin(\theta)+978.7050671\cos(\theta)\sin(\theta)+62.17655572\theta-704.6676483\cos^4(\theta)+$$

$$318.2136992\cos^4(\theta)\sin(\theta)+159.1068496\cos^2(\theta)\sin(\theta)+92.15112845\sin^3(\theta)+663.216610\cos^2(\theta))+O(\varepsilon^4);$$

and finally the limit cycle Γ_{10} can be written as

$$R_{10}(\theta)=15.26399619\varepsilon+\varepsilon^2(310.65277(1-\cos(\theta))+175.8511264\cos(\theta)\sin(\theta)-$$

$$155.3263865\cos(\theta)\sin^2(\theta))+\varepsilon^3(5955.469872+2201.770044\cos^2(\theta)+2006.95599\cos^3(\theta)-$$

$$5582.270975\cos^4(\theta)+3578.921231\sin^3(\theta)-6952.826310\cos(\theta)-108.0870363\cos^2(\theta)\sin(\theta)+$$

$$715.7842462\sin(\theta)+4294.705477\sin(\theta)\cos^4(\theta)+2799.70739\cos(\theta)\sin(\theta)+2370.901371\cos^6(\theta))+O(\varepsilon^4)$$

$$z_{10}(\theta)=-2.880161987\varepsilon+\varepsilon^2(-43.96278160(\cos(\theta)-\sin(\theta)-1)+465.979159\cos(\theta)\sin(\theta))$$

$$+\varepsilon^3(1461.029430-787.8524117\cos(\theta)+5334.98338\sin(\theta)-337.6531531\cos^3(\theta)+$$

$$335.5238654\cos^3(\theta)\sin(\theta)+7922.091267\cos(\theta)\sin(\theta)+503.2857984\theta-5703.905712\cos^4(\theta)+$$

$$5690.163291\cos^4(\theta)\sin(\theta)+2845.081646\cos^2(\theta)\sin(\theta)+337.6531531\sin^3(\theta)+5368.381846\cos^2(\theta))+O(\varepsilon^4).$$

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Appendix

$$F_{11} = \frac{1}{b}[a_{200}\rho^2 cos^3(\theta)+\rho(a_1+a_{101}\xi)cos^2(\theta)+(a_{110}+b_{200})\rho^2 sin(\theta)cos^2(\theta)+a_{002}\xi^2 cos(\theta)+(a_{020}+b_{110})\rho^2 sin^2(\theta)cos(\theta)+(a_{011}+b_{101})\rho\xi sin(\theta)cos(\theta)+b_{020}\rho^2 sin^3(\theta)+\rho(a_1+b_{011}\xi)sin^2(\theta)+b_{002}\xi^2 sin(\theta)],$$

$$F_{21} = \frac{1}{b^2}[a_2 b\rho + A_{002} b\xi^2 cos(\theta) + (\xi(A_{101}b\rho^2 - a_{002}b_{002}\xi^3)cos^2(\theta))/\rho + (A_{200}b\rho^2 - a_1 b_{002}\xi^2 - a_{101}b_{002}\xi^3 - a_{002}b_{101}\xi^3)cos^3(\theta) - \rho\xi(a_1 b_{101}+a_{200}b_{002}\xi+a_{101}b_{101}\xi+a_{002}b_{200}\xi)cos^4(\theta) - \rho^2(a_1 b_{200}+a_{200}b_{101}\xi+a_{101}b_{200}\xi)cos^5(\theta) - a_{200}b_{200}\rho^3 cos^6(\theta)+bB_{002}\xi^2 sin(\theta)+(\xi(A_{011}b\rho^2+bB_{101}\rho^2+a_{002}^2\xi^3-b_{002}^2\xi^3)cos(\theta)sin(\theta))/\rho+(A_{110}b\rho^2+bB_{200}\rho^2+a_{002}a_1\xi^2+2a_{002}a_{101}\xi^3-a_{011}b_{002}\xi^3-a_{002}b_{011}\xi^3-2b_{002}b_{101}\xi^3)cos^2(\theta)sin(\theta)+\rho\xi(a_1a_{101}-a_1b_{011}+a_{101}^2\xi+2a_{002}a_{200}\xi-a_{110}b_{002}\xi-a_{101}b_{011}\xi-a_{011}b_{101}\xi-b_{101}^2\xi-a_{002}b_{110}\xi-2b_{002}b_{200}\xi)cos^3(\theta)sin(\theta)+\rho^2(a_1a_{200}-a_1b_{110}+2a_{101}a_{200}\xi-a_{200}b_{011}\xi-a_{110}b_{101}\xi-a_{101}b_{110}\xi-a_{011}b_{200}\xi-2b_{101}b_{200}\xi)cos^4(\theta)sin(\theta)+(a_{200}^2-a_{200}b_{110}-a_{110}b_{200}-b_{200}^2)\rho^3cos^5(\theta)sin(\theta)+(\xi(bB_{011}\rho^2+a_{002}b_{002}\xi^3)sin^2(\theta))/\rho+(A_{020}b\rho^2+bB_{110}\rho^2-a_1b_{002}\xi^2+2a_{002}a_{011}\xi^3+a_{101}b_{002}\xi^3-2b_{002}b_{011}\xi^3+a_{002}b_{101}\xi^3)cos(\theta)sin^2(\theta)+\rho\xi(a_{011}a_1-a_1b_{101}+2a_{011}a_{101}\xi+2a_{002}a_{110}\xi-a_{020}b_{002}\xi+a_{200}b_{002}\xi-a_{011}b_{011}\xi-a_{002}b_{020}\xi+a_{101}b_{101}\xi-2b_{011}b_{101}\xi-2b_{002}b_{110}\xi+a_{002}b_{200}\xi)cos^2(\theta)sin^2(\theta)+\rho^2(a_1a_{110}-a_1b_{020}-a_1b_{200}+2a_{101}a_{110}\xi+2a_{011}a_{200}\xi-a_{110}b_{011}\xi-a_{101}b_{020}\xi-a_{020}b_{101}\xi+a_{200}b_{101}\xi-a_{011}b_{110}\xi-2b_{101}b_{110}\xi+a_{101}b_{200}\xi-2b_{011}b_{200}\xi)cos^3(\theta)sin^2(\theta)+(2a_{110}a_{200}-a_{200}b_{020}-a_{110}b_{110}-a_{020}b_{200}+a_{200}b_{200}-2b_{110}b_{200})\rho^3cos^4(\theta)sin^2(\theta)+(bB_{020}\rho^2+a_{002}a_1\xi^2+a_{011}b_{002}\xi^3+a_{002}b_{011}\xi^3)sin^3(\theta)+\rho\xi(a_1a_{101}-a_1b_{011}+a_{011}^2\xi+2a_{002}a_{020}\xi+a_{110}b_{002}\xi+a_{101}b_{011}\xi-b_{011}^2\xi-2b_{002}b_{020}\xi+a_{011}b_{101}\xi+a_{002}b_{110}\xi)cos(\theta)sin^3(\theta)+\rho^2(a_{020}a_1+a_1a_{200}-a_1b_{110}+2a_{020}a_{101}\xi+2a_{011}a_{110}\xi-a_{020}b_{011}\xi+a_{200}b_{011}\xi-a_{011}b_{020}\xi+a_{110}b_{101}\xi-2b_{020}b_{101}\xi+a_{101}b_{110}\xi-2b_{011}b_{110}\xi+a_{011}b_{200}\xi)cos^2(\theta)sin^3(\theta)+(a_{110}^2+2a_{020}a_{200}-a_{110}b_{020}-a_{020}b_{110}+a_{200}b_{110}-b_{110}^2+a_{110}b_{200}-2b_{020}b_{200})\rho^3cos^3(\theta)sin^3(\theta)+\rho\xi(a_{011}a_1+a_{020}b_{002}\xi+a_{011}b_{011}\xi+a_{002}b_{020}\xi)sin^4(\theta)+\rho^2(a_1a_{110}-a_1b_{020}+2a_{011}a_{020}\xi+a_{110}b_{011}\xi+a_{101}b_{020}\xi-2b_{011}b_{020}\xi+a_{020}b_{101}\xi+a_{011}b_{110}\xi)cos(\theta)sin^4(\theta)+(2a_{020}a_{110}-a_{020}b_{020}+a_{200}b_{020}+a_{110}b_{110}-2b_{020}b_{110}+a_{020}b_{200})\rho^3cos^2(\theta)sin^4(\theta)+(a_{110}b_{020}-b_{020}^2+a_{020}b_{110})\rho^3cos(\theta)sin^5(\theta)+\rho^2(a_{020}a_1+a_{020}b_{011}\xi+a_{011}b_{020}\xi+a_{020}^2\rho cos(\theta))sin^5(\theta)+a_{020}b_{020}\rho^3sin^6(\theta)],$$

$$F_{31} = \frac{1}{b^3}[a_3b^2\rho+(b^2A'_{002}-b(a_1B_{002}+a_2b_{002}))\xi^2\cos(\theta)+((b^2A'_{101}-b(a_1B_{101}+a_2b_{101}))\rho\xi-b(a_{002}B_{002}+A_{002}b_{002})\xi^4/\rho+a_1b_{002}^2\xi^4/\rho)\cos^2(\theta)+(b^2A'_{200}\rho^2-b(a_1B_{200}+a_2b_{200})\rho^2-b(a_{101}B_{002}+a_{002}B_{101}+A_{101}b_{002}+A_{002}b_{101})\xi^3+2a_1b_{101}b_{002}\xi^3+a_{002}b_{002}^2\xi^6\rho^2/)\cos^3(\theta)+(((a_1b_{101}^2+2a_1b_{200}b_{002})-b(a_{200}B_{002}+a_{101}B_{101}+a_{002}B_{200}+A_{200}b_{002}+A_{101}b_{101}+A_{002}b_{200}))\rho\xi^2+(a_{101}b_{002}^2+2a_{002}b_{101}b_{002})\times\xi^5/\rho)\cos^4(\theta)+((2a_1b_{200}b_{101}-b(a_{200}B_{101}+a_{101}B_{200}+b_{101}A_{200}+A_{101}b_{200}))\rho^2\xi+(a_{200}b_{002}^2+2b_{002}(a_{101}b_{101}+b_{200}a_{002})+a_{002}b_{101}^2)\xi^4)\cos^5(\theta)+((a_1b_{200}^2-b(a_{200}B_{200}+A_{200}b_{200}))\rho^3+(a_{101}b_{101}^2+2(a_{200}b_{101}b_{002}+a_{101}b_{200}b_{002}+a_{002}b_{200}b_{101}))\rho\xi^3)\cos^6(\theta)+(a_{200}b_{101}^2+a_{002}b_{200}^2+2b_{200}(a_{200}b_{002}+a_{101}b_{101}))\rho^2\xi^2\cos^7(\theta)+(a_{101}b_{200}^2+2a_{200}b_{200}b_{101})\rho^3\xi\cos^8(\theta)+a_{200}b_{200}^2\rho^4\cos^9(\theta)+(b^2B'_{002}+b(a_1A_{002}+a_2a_{002}))\xi^2\sin(\theta)+((b^2(A'_{011}+B'_{101})+b(a_1(A_{101}-B_{011})+a_2(a_{101}-b_{011})))\rho\xi+2(ba_{002}A_{002}-bb_{002}B_{002}-a_1a_{002}b_{002})\xi^4/\rho)\cos(\theta)\sin(\theta)+((b^2(A'_{110}+B'_{200})+b(a_1(A_{200}-B_{110})+a_2(a_{200}-b_{110})))\rho^2+b(A_{002}(2a_{101}-b_{011})-B_{002}(2b_{101}+a_{011})+a_{002}(2A_{101}-B_{011})-b_{002}(2B_{101}+A_{011})+a_2(a_{200}-b_{110}))\xi^3-2a_1b_{101}a_{002}\xi^3+2a_1b_{002}(b_{011}-a_{101})\xi^3+(b_{002}^3-2b_{002}a_{002}^2)\xi^6/\rho^2)\cos^2(\theta)\sin(\theta)+((b(-B_{002}(a_{110}+2b_{200})+A_{002}(2a_{200}-b_{110})-B_{101}(2b_{101}+a_{011})+a_{101}(2A_{101}-B_{011})+a_{002}(2A_{200}-B_{110})-b_{002}(2B_{200}+A_{110})-A_{101}b_{011}-A_{011}b_{101})+2a_1(b_{002}(b_{110}-a_{200})-b_{200}a_{002}+b_{101}(b_{011}-a_{101})))\rho\xi^2+(b_{002}^2(a_{011}+b_{101})+2b_{101}(b_{002}^2-a_{002}^2)+2a_{002}b_{002}(b_{011}-2a_{101}))\xi^5/\rho)\cos^3(\theta)\sin(\theta)+((b(a_{200}(2A_{101}-B_{011})+a_{101}(2A_{200}-B_{110})-b_{101}(2B_{200}+A_{110})-b_{200}(A_{011}+2B_{101})-b_{011}A_{200}-a_{110}B_{101}-A_{101}b_{110})+2a_1(b_{200}(b_{011}-a_{101})+b_{101}(b_{110}-a_{200})))\rho^2\xi+(b_{002}^2(a_{110}+b_{200})+b_{002}(b_{101}^2-2a_{101}^2)+2b_{200}(b_{002}^2-a_{002}^2)+2(b_{101}b_{002}(a_{011}+b_{101})+a_{002}b_{002}(b_{110}-2a_{200})+a_{002}b_{101}(b_{011}-2a_{101})+a_{101}b_{011}b_{002}))\xi^4)\cos^4(\theta)\sin(\theta)+((b(a_{200}(2A_{200}-B_{110})-b_{200}(2B_{200}+A_{110})-a_{110}B_{200}-b_{110}A_{200})+2a_1b_{200}(b_{110}-$$

$a_{200}))\rho^3+(b_{101}^2(a_{011}+b_{101})-2b_{101}a_{101}^2+2(b_{101}b_{002}(a_{110}+b_{200})+b_{200}b_{002}(a_{011}+2b_{101})+a_{002}b_{101}(b_{110}-2a_{200})+$
$a_{101}b_{002}(b_{110}-2a_{200})+a_{002}b_{200}(b_{011}-2a_{101})+b_{011}(a_{200}b_{002}++a_{101}b_{101})))\rho\xi^3)\cos^5(\theta)\sin(\theta)+(b_{101}^2(a_{110}+$
$b_{200})+b_{002}b_{200}^2+2(-b_{002}a_{200}^2-b_{200}a_{101}^2+b_{200}b_{002}(a_{110}+b_{200})+b_{200}b_{101}(a_{011}+b_{101})+a_{002}b_{200}(b_{110}-2a_{200})+$
$a_{101}b_{101}(b_{110}-2a_{200})+a_{200}(b_{110}b_{002}+b_{101}b_{011})+a_{101}b_{200}b_{011}))\rho^2\xi^2\cos^6(\theta)\sin(\theta)+(b_{200}^2(a_{011}+b_{101})+2(-b_{101}a_{200}^2+$
$b_{200}b_{101}(a_{110}+b_{200})+a_{200}b_{200}(b_{011}-2a_{101})+b_{110}(a_{200}b_{101}+a_{101}b_{200})))\rho^3\xi\cos^7(\theta)\sin(\theta)+(b_{200}^2(a_{110}+b_{200})+$
$2b_{200}(a_{200}b_{110}-a_{200}^2))\rho^4\cos^8(\theta)\sin(\theta)+((b^2B'_{011}+b(a_1A_{011}+a_2a_{011}))\rho\xi+(a_1a_{002}^2+b(b_{002}A_{002}+B_{002}a_{002}))\xi^4/\rho)\sin^2(\theta)+$
$((b^2(A'_{020}+B'_{110})+ba_1(A_{110}-B_{020})+ba_2(a_{110}-b_{020}))\rho^2+(b(B_{002}(a_{101}-2b_{011})+A_{002}(2a_{011}+b_{101})+b_{002}(A_{101}-$
$2B_{011})+a_{002}(2A_{011}+B_{101}))+2a_1(a_{002}(a_{101}-b_{011})-2a_{011}b_{002}))\xi^3+(a_{002}^3-2a_{002}b_{002}^2)\xi^6/\rho^2)\cos(\theta)\sin^2(\theta)+$
$((b(A_{002}(2a_{110}+b_{200}-b_{020})+B_{002}(a_{200}-2b_{110}-a_{020})-B_{011}(a_{011}+2b_{101})+B_{101}(a_{101}-2b_{011})+a_{002}(2A_{110}+$
$B_{200}-B_{020})+A_{101}(2a_{011}+b_{101})+A_{011}(2a_{101}-b_{011})+b_{002}(A_{200}-A_{020}-2B_{110}))+(a_1(b_{011}^2+a_{101}^2)+2a_1(a_{002}(a_{200}-$
$b_{110})+b_{002}(b_{020}-a_{110})-b_{101}a_{011}-a_{101}b_{011})))\rho\xi^2+(a_{101}a_{002}^2+b_{011}b_{002}^2+2(b_{002}^2(b_{011}-a_{101})+a_{002}^2(a_{101}-b_{011})-$
$2a_{002}b_{002}(a_{011}+b_{101})))\xi^5/\rho)\cos^2(\theta)\sin^2(\theta)+((b(a_{110}(2A_{101}-B_{011})+b_{200}(A_{101}-2B_{011})+a_{200}(2A_{011}+B_{101})-$
$B_{101}(a_{020}+2b_{110})+a_{101}(2A_{110}-B_{020}+B_{200})+A_{200}(2a_{011}+b_{101})-b_{011}(2B_{200}+A_{110})-B_{110}(2b_{101}+a_{011})-$
$A_{020}b_{101}-A_{011}b_{110}-A_{101}b_{020})+2a_1(b_{110}(b_{011}-a_{101})+a_{200}(a_{101}-b_{011})+b_{101}(b_{020}-a_{101})-b_{200}a_{011}))\rho^2\xi+$
$(3a_{002}a_{101}^2+3a_{200}a_{002}^2+a_{002}b_{011}^2+b_{002}^2(a_{020}+b_{110})+2(b_{002}^2(b_{110}-a_{200})-b_{110}a_{002}^2-b_{002}a_{002}(a_{110}+b_{200})-$
$a_{002}b_{101}(a_{011}+b_{101})-b_{002}a_{101}(a_{011}+b_{101})+b_{011}b_{002}(a_{011}+b_{101})-a_{101}b_{002}(a_{011}+b_{101})+a_{002}b_{002}(b_{020}-$
$a_{110}-b_{200})-a_{002}(2a_{101}b_{011}+b_{101}a_{011})+2b_{002}b_{101}b_{011}))\xi^4)\cos^3(\theta)\sin^2(\theta)+((b(a_{200}(2A_{110}-B_{020}+B_{200})+$
$b_{200}(A_{200}-2B_{110}-A_{020})+a_{110}(2A_{200}-B_{110})-b_{110}(2B_{200}+A_{110})-a_{020}B_{200}-b_{020}A_{200}))+(a_1(b_{110}^2+a_{200}^2)+$
$2a_1(b_{200}(b_{020}-a_{110})-b_{110}a_{200})))\rho^3+(a_{101}^3+a_{101}b_{011}^2+b_{011}b_{101}^2+2(-b_{011}a_{101}^2+a_{200}a_{002}(3a_{101}-2b_{011})-$
$a_{200}b_{002}(2a_{011}+2b_{101})-b_{101}a_{002}(2a_{110}+b_{200}-b_{020})-a_{101}b_{002}(2a_{110}+2b_{200}-b_{020})+b_{011}b_{002}(a_{110}+3b_{200})+$
$b_{101}b_{002}(a_{020}+2b_{110})+b_{110}a_{002}(b_{011}-2a_{101})+b_{110}b_{002}(a_{011}+b_{101})-b_{200}a_{002}(2a_{011}+b_{101})-b_{101}a_{101}(a_{011}+$
$b_{101})+b_{101}b_{011}(a_{011}+b_{101})-a_{101}b_{101}a_{011})\rho\xi^3)\cos^4(\theta)\sin^2(\theta)+(a_{200}(b_{011}^2+a_{101}^2)+a_{002}(a_{200}^2+b_{110}^2)+b_{101}^2(a_{020}+$
$b_{110})+2(a_{002}a_{200}^2+a_{200}a_{101}^2-b_{110}a_{101}^2-a_{200}a_{002}b_{110}+a_{200}b_{002}(b_{020}-2a_{110}-2b_{200})-a_{200}b_{101}(2a_{011}+b_{101})+$
$b_{002}b_{110}(a_{110}+2b_{200})-a_{002}b_{200}(2a_{110}+b_{200}-b_{020})+a_{101}b_{011}(b_{110}-2a_{200})-b_{101}a_{101}(2a_{110}+b_{200}-b_{020})+$
$b_{101}b_{011}(a_{110}+2b_{200})+b_{200}b_{002}(a_{020}+b_{110})+b_{200}b_{011}(a_{011}+b_{101})+b_{110}b_{101}(a_{011}+b_{101})-b_{200}a_{101}(2a_{011}+$
$b_{101})))\rho^2\xi^2\cos^5(\theta)\sin^2(\theta)+(a_{101}(a_{200}^2+b_{110}^2)+b_{011}b_{200}^2+2(a_{200}^2(a_{101}-b_{011})-a_{200}b_{200}a_{011}+a_{200}b_{110}(b_{011}-$
$2a_{101})+a_{200}b_{101}(b_{020}-2a_{110}-b_{200})+b_{200}b_{011}(a_{110}+b_{200})+b_{110}b_{101}(a_{110}+b_{200})-b_{200}a_{101}(2a_{110}+b_{200}-$
$b_{020})+b_{200}b_{101}(a_{020}+b_{110})-a_{200}b_{200}(a_{011}+b_{101})+b_{110}b_{200}(a_{011}+b_{101})))\rho^3\xi\cos^6(\theta)\sin^2(\theta)+(a_{200}^3+a_{200}b_{110}^2+$
$b_{200}^2(a_{020}+b_{110})+2(a_{200}b_{200}(b_{020}-2a_{110}-b_{200})-b_{110}a_{200}^2+b_{200}b_{110}(a_{110}+b_{200})))\rho^4\cos^7(\theta)\sin^2(\theta)+((b^2B'_{020}+$
$b(a_1A_{020}+a_2a_{020}))\rho^2+(b(b_{011}A_{002}+b_{002}A_{011}+a_{011}B_{002}+B_{011}a_{002})+2a_1a_{011}a_{002})\xi^3+b_{002}a_{002}^2\xi^6/\rho^2)\sin^3(\theta)+$
$((b(A_{002}(b_{110}+2a_{020})+B_{002}(a_{110}-2b_{020})+B_{011}(a_{101}-2b_{011})+a_{002}(2A_{020}+B_{110})+b_{002}(A_{110}-2B_{020})+$
$A_{011}(b_{101}+2a_{011})+b_{011}A_{101}+a_{011}B_{101})+(2a_1(a_{002}(a_{110}-b_{020})+a_{011}(a_{101}-b_{011})-a_{020}b_{002})))\rho\xi^2+(a_{002}^2(a_{011}+$
$b_{101})+2(a_{011}(a_{002}^2+b_{002}^2)+b_{002}a_{002}(a_{101}-2b_{011})))\xi^5/\rho)\cos(\theta)\sin^3(\theta)+((b(A_{011}(2a_{110}+b_{200}-b_{020})+B_{011}(a_{200}-$
$2b_{110}-a_{020})+B_{101}(a_{110}-2b_{020})+A_{101}(2a_{020}+b_{110})+A_{110}(2a_{011}+b_{101})+A_{020}(2a_{101}-b_{011})-B_{020}(2b_{101}+$
$a_{011})+B_{110}(a_{101}-2b_{011})+A_{200}b_{011}+a_{011}B_{200})+(2a_1(a_{011}(a_{200}-b_{110})+a_{110}(a_{101}-b_{011})+b_{020}(b_{011}-a_{101})-$
$a_{020}b_{101})))\rho^2\xi+(b_{002}(3b_{011}^2+a_{101}^2)+b_{020}(3b_{002}^2-2a_{002}^2)+a_{002}^2(a_{110}+b_{200})+2(a_{110}(a_{002}^2-b_{002}^2)-a_{002}b_{002}(2a_{020}+$
$2b_{110}+a_{200})+a_{002}a_{011}(2a_{101}-b_{011})-a_{011}b_{002}(a_{011}+2b_{101})-b_{011}a_{002}(a_{011}+2b_{101})+a_{101}a_{002}(a_{011}+b_{101})-$
$2b_{002}a_{101}b_{011}))\xi^4)\cos^2(\theta)\sin^3(\theta)+((b(A_{110}(2a_{110}+b_{200}-b_{020})-B_{020}(2b_{200}+a_{110})+A_{020}(2a_{200}-b_{110})+$
$A_{200}(2a_{020}+b_{110})+B_{200}(a_{110}-2b_{020})+B_{110}(a_{200}-a_{020}-2b_{110}))+2a_1(b_{020}(b_{110}-a_{200})+a_{110}(a_{200}-b_{110})-$

$$
b_{200}a_{020}))\rho^3 + (b_{011}^2(a_{011}+3b_{101}) + a_{101}^2(3a_{011}+b_{101}) + 2(a_{200}a_{002}(3a_{011}+b_{101}) - a_{011}b_{002}(a_{110}+2b_{200}) -
b_{011}a_{002}(2a_{110}+2b_{200}-b_{020}) + a_{101}a_{002}(3a_{110}+b_{200}-2b_{020}) - b_{101}a_{002}(2a_{020}+b_{110}) - a_{101}b_{002}(2a_{020}+2b_{110} -
a_{200}) + b_{011}b_{002}(a_{020}+3b_{110}-2a_{200}) + b_{002}b_{020}(a_{011}+3b_{101}) - b_{011}a_{101}(2a_{011}+2b_{101}) - a_{110}b_{002}(a_{011}+2b_{101}) -
b_{110}a_{002}(2a_{011}+b_{101}) - b_{101}a_{011}(a_{011}+b_{101}))))\rho\xi^3)\cos^3(\theta)\sin^3(\theta) + (b_{002}(a_{200}^2+b_{110}^2) + b_{011}^2(a_{110}+3b_{200}) +
a_{101}^2(3a_{110}+b_{200}) + b_{020}(b_{101}^2-2a_{101}^2) + 2(-a_{200}b_{002}(2a_{020}+2b_{110}) + a_{200}a_{002}(3a_{110}-2b_{020}+b_{200}) + a_{200}a_{011}(2a_{101} -
b_{011}) - b_{110}a_{002}(2a_{110}+b_{200}+b_{020}) - a_{110}b_{002}(a_{110}+2b_{200}) + b_{020}b_{002}(a_{110}+3b_{200}) - a_{011}b_{101}(a_{110}+b_{200}) -
a_{101}b_{011}(2a_{110}+2b_{200}-b_{020}) - b_{200}a_{002}(2a_{020}+b_{110}) - b_{101}a_{101}(2a_{020}+b_{110}) + b_{101}b_{011}(a_{020}+2b_{110}-a_{200}) +
b_{110}b_{002}(a_{020}+b_{110}) - b_{110}a_{101}(2a_{011}+b_{101}) + (a_{011}+b_{101})(a_{200}a_{101}+b_{020}b_{101}-a_{200}b_{011}+b_{110}b_{011}-b_{200}a_{011} -
a_{110}b_{101})))\rho^2\xi^2\cos^4(\theta)\sin^3(\theta) + (b_{110}^2(a_{011}+b_{101}) + a_{200}^2(3a_{011}+b_{101}) + 2(-a_{200}b_{110}(2a_{011}+b_{101}) + a_{200}b_{011}(b_{020} -
2a_{110}-2b_{200}) + b_{200}b_{020}(a_{011}+2b_{101}) + b_{200}b_{011}(a_{020}+2b_{110}) - a_{200}b_{101}(2a_{020}+b_{110}) + a_{200}a_{101}(3a_{110}-2b_{020} +
b_{200}) - b_{200}a_{011}(a_{110}+b_{200}) - b_{110}a_{101}(2a_{110}+b_{200}-b_{020}) + b_{101}b_{020}(a_{110}+b_{200}) - a_{110}b_{101}(a_{110}+b_{200}) +
b_{110}b_{011}(a_{110}+b_{200}) + b_{110}b_{101}(a_{020}+b_{110}) - b_{200}a_{101}(a_{020}+b_{110}-a_{020}) - b_{200}a_{110}(a_{011}+b_{101})))\rho^3\xi\cos^5(\theta)\sin^3(\theta) +
(b_{020}(b_{200}^2-2a_{200}^2) + b_{110}^2(a_{110}+b_{200}) + a_{200}^2(3a_{110}+b_{200}) + 2(-a_{200}b_{200}(2a_{020}+b_{110}) + a_{200}b_{110}(b_{020}-2a_{110}-b_{200}) +
b_{200}b_{110}(a_{020}+b_{110}) + (a_{110}+b_{200}^2)(b_{020}-a_{110})))\rho^4\cos^6(\theta)\sin^3(\theta) + ((b(b_{020}A_{002}+b_{011}A_{011}+b_{002}A_{020}+a_{002}B_{020} +
B_{002}a_{020}+B_{011}a_{011}) + a_1a_{011}^2 + 2a_1a_{002}a_{020})\rho\xi^2 + (b_{011}a_{002}^2 + 2a_{002}a_{011}b_{002})\xi^5/\rho)\sin^4(\theta) + ((b(B_{011}(a_{110} -
2b_{020}) + A_{011}(2a_{020}+b_{110}) + b_{011}(A_{110}-2B_{020}) + A_{020}(2a_{011}+b_{101}) + b_{020}A_{101} + B_{110}a_{011} + B_{101}a_{020} +
a_{101}B_{020}) + 2a_1(a_{011}(a_{110}-b_{020}) + a_{020}(a_{101}-b_{011})))\rho^2\xi + (a_{002}^2(a_{020}+b_{110}) + a_{002}(a_{011}^2-b_{011}^2) + 2a_{020}(a_{002}^2 -
b_{002}^2) + 2(a_{011}a_{002}(a_{011}+b_{101}) + a_{002}b_{002}(a_{110}-2b_{020}) + a_{011}b_{002}(a_{101}-2b_{011}) + b_{011}a_{101}a_{002}))\xi^4)\cos(\theta)\sin^4(\theta) +
((b(A_{020}(b_{200}+2a_{110}-b_{020}) + A_{110}(2a_{020}+b_{110}) + b_{020}(A_{200}-2B_{110}) + B_{020}(a_{200}-2b_{110}-a_{020}) + a_{110}B_{110} +
B_{200}a_{020}) + (a_1(a_{110}^2+b_{020}^2) + 2a_1(a_{020}(a_{200}-b_{110}) - a_{110}b_{020}))\rho^3 + (b_{011}^3 + a_{101}(a_{011}^2-2b_{011}^2) + b_{011}a_{101}^2 +
2(a_{011}a_{002}(2a_{110}+b_{200}-b_{020}) - a_{011}b_{002}(a_{020}+b_{110}-a_{200}) - b_{011}a_{002}(2a_{020}+2b_{110}-a_{200}) + a_{101}a_{002}(3a_{020} +
b_{110}) - b_{020}a_{002}(a_{011}+2b_{101}) + b_{020}b_{002}(3b_{011}-2a_{101}) - b_{011}a_{011}(a_{011}+2b_{101}) + a_{101}a_{011}(a_{011}+b_{101}) - a_{020}b_{002}(a_{011} +
2b_{101}) + a_{110}a_{002}(a_{011}+b_{101}) + a_{110}b_{002}(a_{101}-2b_{011}) - b_{002}b_{110}a_{011}))\rho\xi^3)\cos^2(\theta)\sin^4(\theta) + (a_{200}(a_{011}^2-2b_{011}^2) +
a_{002}(a_{110}^2+b_{020}^2) + (a_{020}+b_{110})(b_{011}^2+a_{101}^2) + 2(a_{020}a_{101}^2 + b_{110}b_{011}^2 + a_{200}a_{002}(3a_{020}+b_{110}) - a_{020}b_{002}(a_{110} +
2b_{200}) + a_{002}a_{110}(a_{110}+b_{200}-b_{020}) - a_{002}b_{020}(a_{110}+2b_{200}) - b_{011}a_{011}(a_{110}+2b_{200}) + a_{011}a_{101}(2a_{110}+b_{200} -
b_{020}) - b_{110}a_{002}(2a_{020}+b_{110}) - a_{110}b_{002}(a_{020}+2b_{110}-a_{200}) + b_{002}b_{020}(a_{020}+3b_{110}-2a_{200}) - b_{101}a_{011}(a_{020} +
b_{110}) - a_{101}b_{011}(2a_{020}+2b_{110}-a_{200}) + b_{020}b_{101}(2b_{011}-a_{101}) - b_{011}a_{110}(a_{011}+2b_{101}) + (a_{011}+b_{101})(a_{200}a_{011} +
b_{020}b_{011} - b_{110}a_{011} - b_{020}a_{101} - a_{020}b_{101} + a_{110}a_{101})))\rho^2\xi^2\cos^3(\theta)\sin^4(\theta) + (a_{101}(a_{110}^2+b_{020}^2) + b_{011}(a_{200}^2+b_{110}^2) +
2(a_{200}a_{011}(2a_{110}-b_{020}+b_{200}) + a_{200}a_{020}(2a_{101}-b_{011}) + b_{020}b_{101}(a_{020}+2b_{110}-a_{200}) + b_{020}b_{200}(2b_{011}-a_{101}) -
a_{101}b_{110}(2a_{020}+b_{110}) - b_{020}a_{101}(2a_{110}+b_{200}) + b_{110}b_{011}(a_{020}+b_{110}-a_{200}) - b_{011}a_{110}(a_{110}+2b_{200}) + (a_{110} +
b_{200})(b_{020}b_{011} - b_{110}a_{011} - a_{020}b_{101} + a_{110}a_{101}) + (a_{020}+b_{110})(a_{200}a_{101} - b_{200}a_{011} - a_{110}b_{101} - a_{200}b_{011}) + (a_{011} +
b_{101})(b_{110}b_{020} - a_{200}b_{020} + a_{200}a_{110} - b_{110}a_{110} - b_{200}a_{020})))\rho^3\xi\cos^4(\theta)\sin^4(\theta) + (a_{200}(a_{110}^2+b_{020}^2) + (a_{020} +
b_{110})(a_{200}^2+b_{110}^2) + 2(a_{020}a_{200}^2 - b_{110}a_{200}(2a_{020}+b_{110}) + b_{200}b_{020}(a_{020}+2b_{110}-a_{200}) + a_{200}a_{110}(a_{110}+b_{200}-b_{020}) -
a_{110}b_{200}(a_{020}+b_{110}) + (a_{110}+b_{200})(b_{110}b_{020} - a_{110}b_{110} - a_{020}b_{200} - a_{200}b_{020})))\rho^4\cos^5(\theta)\sin^4(\theta) + ((2a_1a_{020}a_{011} +
b(b_{020}A_{011}+b_{011}A_{020}+B_{020}a_{011}+B_{011}a_{020}))\rho^2\xi + (b_{020}a_{002}^2 + b_{002}a_{011}^2 + 2a_{002}(b_{011}a_{011}+b_{002}a_{020}))\xi^4)\sin^5(\theta) +
((2a_1a_{020}(a_{110}-b_{020}) + b(b_{020}(A_{110}-2B_{020}) + a_{020}(2A_{020}+B_{110}) + a_{110}B_{020} + b_{110}A_{020}))\rho^3 + (a_{011}^2(a_{011} +
b_{101}) + 2(-a_{011}b_{011}^2 + a_{011}a_{002}(2a_{020}+b_{110}) + a_{020}a_{002}(a_{011}+b_{101}) - 2b_{020}b_{002}a_{011} + a_{002}b_{020}(a_{101}-2b_{011}) +
b_{011}a_{101}a_{011} - 2b_{002}a_{020}b_{011} + b_{011}a_{110}a_{002} + b_{002}(a_{110}a_{011}+a_{020}a_{101})))\rho\xi^3)\cos(\theta)\sin^5(\theta) + (a_{011}^2(a_{110}+b_{200}) +
b_{020}(a_{101}^2+3b_{011}^2) + b_{002}(3b_{020}^2+a_{110}^2) - 2a_{110}b_{011}^2 + 2(a_{002}a_{020}(2a_{110}+b_{200}-b_{020}) - a_{020}b_{002}(a_{020}+2b_{110}-a_{200}) +
$$

$a_{110}a_{002}(a_{020}+b_{110})-a_{002}b_{020}(a_{020}+2b_{110}-a_{200})-a_{011}b_{011}(a_{020}+2b_{110}-a_{200})+a_{101}a_{011}(2a_{020}+b_{110})-2b_{020}a_{110}b_{002}-b_{020}a_{011}(2b_{101}+a_{011})-2b_{011}b_{020}a_{101}+a_{020}a_{101}(a_{011}+b_{101})+a_{110}a_{011}(a_{011}+b_{101})-b_{011}a_{020}(a_{011}+2b_{101})+b_{011}a_{110}a_{101}))\rho^2\xi^2\cos^2(\theta)\sin^5(\theta)+((a_{011}+b_{101})(a_{110}^2+b_{020}^2)+2(b_{101}b_{020}^2+a_{200}a_{011}(2a_{020}+b_{110})+b_{020}b_{110}(2b_{011}-a_{101})-b_{020}a_{011}(a_{110}+2b_{200})+a_{101}a_{020}(2a_{110}+b_{200}-b_{020})+b_{020}a_{200}(a_{101}-2b_{011})-b_{020}a_{110}(a_{011}+2b_{101})-b_{011}a_{020}(a_{110}+2b_{200})-b_{011}a_{110}(a_{020}+2b_{110}-a_{200})+(a_{011}+b_{101})(a_{200}a_{020}-a_{020}b_{110})+a_{110}a_{011}(a_{110}+b_{200})+(a_{020}+b_{110})(a_{110}a_{101}-b_{110}a_{011}+b_{011}b_{020}-a_{020}b_{101}-a_{101}b_{020}))\rho^3\xi\cos^3(\theta)\sin^5(\theta)+((a_{110}+b_{200})(a_{110}^2+b_{020}^2)+b_{020}(a_{200}^2+b_{110}^2)+2(b_{200}b_{020}^2+a_{200}a_{020}(2a_{110}+b_{200}-b_{020})-a_{110}b_{020}(a_{110}+2b_{200})-a_{020}b_{110}(a_{110}+b_{200})+b_{110}b_{020}(a_{020}+b_{110}-a_{200})+(a_{020}+b_{110})(a_{110}a_{200}-a_{020}b_{200}-b_{110}a_{110}-a_{200}b_{020})))\rho^4\cos^4(\theta)\sin^5(\theta)+((a_1a_{020}^2+b(a_{020}B_{020}+b_{020}A_{020}))\rho^3+(b_{011}a_{011}^2+2(a_{011}b_{020}a_{002}+a_{002}a_{020}b_{011}+b_{002}a_{020}a_{011}))\rho\xi^3)\sin^6(\theta)+(a_{011}^2(a_{020}+b_{110})+a_{002}(a_{020}^2-2b_{020}^2)-2a_{020}b_{011}^2+2(a_{002}a_{020}(a_{020}+b_{110})+a_{020}a_{011}(a_{011}+b_{101})+a_{020}b_{002}(a_{110}-2b_{020})+a_{002}a_{110}b_{020}+b_{020}a_{011}(a_{101}-2b_{011})+b_{011}(a_{101}a_{020}+a_{110}a_{011})))\rho^2\xi^2\cos(\theta)\sin^6(\theta)+(b_{011}(a_{110}^2+3b_{020}^2)+a_{101}(a_{020}^2-2b_{020}^2)+2(a_{011}a_{020}(a_{110}+b_{200})-b_{020}a_{011}(a_{020}+b_{110}-a_{200})+(a_{020}+b_{110})(a_{020}a_{101}+a_{110}a_{011}-a_{020}b_{011})+b_{020}a_{110}(a_{101}-b_{011})-b_{020}a_{020}(a_{011}+2b_{101})-b_{020}b_{110}a_{011}+a_{110}a_{020}(a_{011}+b_{101})+a_{020}b_{011}(a_{200}-b_{110})-b_{011}a_{110}b_{020}))\rho^3\xi\cos^2(\theta)\sin^6(\theta)+(a_{200}(a_{020}^2-2b_{020}^2)+2b_{110}b_{020}^2+(a_{020}+b_{110})(a_{110}^2+b_{020}^2)+2(-a_{020}b_{020}(a_{110}+2b_{200})-b_{020}a_{110}(a_{020}+2b_{110}-a_{200})+a_{020}a_{110}(a_{110}+b_{200})+(a_{020}+b_{110})(a_{200}a_{020}-a_{020}b_{110})))\rho^4\cos^3(\theta)\sin^6(\theta)+(b_{020}a_{011}^2+b_{002}a_{020}^2+2a_{020}(a_{002}b_{020}+a_{011}b_{011}))\rho^2\xi^2\sin^7(\theta)+(a_{020}^2(a_{011}+b_{101})+2(-a_{011}b_{020}^2+a_{011}a_{020}(a_{020}+b_{110})+b_{020}a_{020}(a_{101}-b_{011})+a_{110}a_{011}b_{020}+a_{020}b_{011}(a_{110}-b_{020})))\rho^3\xi\cos(\theta)\sin^7(\theta)+(b_{020}^3+b_{020}a_{110}^2-2a_{110}b_{020}^2+a_{020}^2(a_{110}+b_{200})+2(a_{110}a_{020}(a_{020}+b_{110})-a_{020}b_{020}(a_{020}+2b_{110}-a_{200})))\rho^4\cos^2(\theta)\sin^7(\theta)+(b_{011}a_{020}^2+2a_{020}a_{011}b_{020})\rho^3\xi\sin^8(\theta)+(a_{020}^2(a_{020}+b_{110})-2a_{020}b_{020}^2+2a_{110}a_{020}b_{020})\rho^4\cos(\theta)\sin^8(\theta)+a_{020}^2b_{020}\rho^4\sin^9(\theta)],$

$$F_{12}=\frac{1}{b}[c_{200}cos^2(\theta)\rho^2+c_{020}sin^2(\theta)\rho^2+c_{110}cos(\theta)sin(\theta)\rho^2+c_{101}\xi cos(\theta)\rho+c_{011}\xi sin(\theta)\rho+c_{002}\xi^2+c_1\xi],$$

$F_{22}=\frac{1}{b^2}[b\xi(c_2+C_{002}\xi)+(\xi(bC_{101}\rho^2-b_{002}c_1\xi^2-b_{002}c_{002}\xi^3)cos(\theta))/\rho+(bC_{200}\rho^2-b_{101}c_1\xi^2-b_{101}c_{002}\xi^3-b_{002}c_{101}\xi^3)cos^2(\theta)-\rho\xi(b_{200}c_1+b_{200}c_{002}\xi+b_{101}c_{101}\xi+b_{002}c_{200}\xi)cos^3(\theta)-(b_{200}c_{101}+b_{101}c_{200})\rho^2\xi cos^4(\theta)-b_{200}c_{200}\rho^3cos^5(\theta)+(\xi(bC_{011}\rho^2+a_{002}c_1\xi^2+a_{002}c_{002}\xi^3)sin(\theta))/\rho+(bC_{110}\rho^2+a_{101}c_1\xi^2-b_{011}c_1\xi^2+a_{101}c_{002}\xi^3-b_{011}c_{002}\xi^3-b_{002}c_{011}\xi^3+a_{002}c_{101}\xi^3)cos(\theta)sin(\theta)+\rho\xi(a_{200}c_1-b_{110}c_1+a_{200}c_{002}\xi-b_{110}c_{002}\xi-b_{101}c_{011}\xi+a_{101}c_{101}\xi-b_{011}c_{101}\xi-b_{002}c_{110}\xi+a_{002}c_{200}\xi)cos^2(\theta)sin(\theta)-(b_{200}c_{011}-a_{200}c_{101}+b_{110}c_{101}+b_{101}c_{110}-a_{101}c_{200}+b_{011}c_{200})\rho^2\xi cos^3(\theta)sin(\theta)-(b_{200}c_{110}-a_{200}c_{200}+b_{110}c_{200})\rho^3cos^4(\theta)sin(\theta)+(bC_{020}\rho^2+a_{011}c_1\xi^2+a_{011}c_{002}\xi^3+a_{002}c_{011}\xi^3)sin^2(\theta)+\rho\xi(a_{110}c_1-b_{020}c_1+a_{110}c_{002}\xi-b_{020}c_{002}\xi+a_{101}c_{011}\xi-b_{011}c_{011}\xi-b_{002}c_{020}\xi+a_{011}c_{101}\xi+a_{002}c_{110}\xi)cos(\theta)sin^2(\theta)+(a_{200}c_{011}-b_{110}c_{011}-b_{101}c_{020}+a_{110}c_{101}-b_{020}c_{101}+a_{101}c_{110}-b_{011}c_{110}+a_{011}c_{200})\rho^2\xi cos^2(\theta)sin^2(\theta)-(b_{200}c_{020}-a_{200}c_{110}+b_{110}c_{110}-a_{110}c_{200}+b_{020}c_{200})\rho^3cos^3(\theta)sin^2(\theta)+\rho\xi(a_{020}c_1+a_{020}c_{002}\xi+a_{011}c_{011}\xi+a_{002}c_{020}\xi)sin^3(\theta)+(a_{110}c_{011}-b_{020}c_{011}+a_{101}c_{020}-b_{011}c_{020}+a_{020}c_{101}+a_{011}c_{110})\rho^2\xi cos(\theta)sin^3(\theta)+(a_{200}c_{020}-b_{110}c_{020}+a_{110}c_{110}-b_{020}c_{110}+a_{020}c_{200})\rho^3cos^2(\theta)sin^3(\theta)+(a_{020}c_{011}+a_{011}c_{020})\rho^2\xi sin^4(\theta)+(a_{110}c_{020}-b_{020}c_{020}+a_{020}c_{110})\rho^3cos(\theta)sin^4(\theta)+a_{020}c_{020}\rho^3sin^5(\theta)],$

$F_{32}=\frac{1}{b^3}[b^2\xi(c_3+C'_{002}\xi)+(b^2C'_{101}\rho\xi-b((c_1B_{002}+c_2b_{002})\xi^3/\rho+(c_{002}B_{002}+C_{002}b_{002})\xi^4/\rho))\cos(\theta)+(b^2C'_{200}\rho^2-b((c_1B_{101}+c_2b_{101})\xi^2+(c_{002}B_{101}+c_{101}B_{002}+C_{002}b_{101}+C_{101}b_{002})\xi^3)+c_1b_{002}^2\xi^5/\rho^2+c_{002}b_{002}^2\xi^6/\rho^2)\cos^2(\theta)+(-b((c_1B_{200}+c_2b_{200})\rho\xi+(c_{002}B_{200}+c_{200}B_{002}+c_{101}B_{101}+C_{002}b_{200}+C_{200}b_{002}+C_{101}b_{101})\rho\xi^2)+2c_1b_{101}b_{002}\xi^4/\rho+2c_{002}b_{101}b_{002}\xi^5/\rho+c_{101}b_{002}^2\xi^5/\rho)\cos^3(\theta)+$

$$(-b(c_{200}B_{101} + c_{101}B_{200} + C_{200}b_{101} + C_{101}b_{200})\rho^2\xi + 2c_1b_{200}b_{002}\xi^3 + c_1b_{101}^2\xi^3 + (2c_{002}b_{200}b_{002} + c_{002}b_{101}^2 +$$
$$c_{200}b_{002}^2+2c_{101}b_{101}b_{002})\xi^4)\cos^4(\theta)+(-bc_{200}B_{200}\rho^3-bC_{200}b_{200}\rho^3+2c_1b_{200}b_{101}\rho\xi^2+(2c_{002}b_{200}b_{101}+2c_{200}b_{101}b_{002}+$$
$$2c_{101}b_{200}b_{002}+c_{101}b_{101}^2)\rho\xi^3)\cos^5(\theta)+(c_1b_{200}^2\rho^2\xi+(c_{002}b_{200}^2+2c_{200}b_{200}b_{002}+c_{200}b_{101}^2+2c_{101}b_{200}b_{101})\rho^2\xi^2)\cos^6(\theta)+$$
$$(2c_{200}b_{200}b_{101} + c_{101}b_{200}^2)\rho^3\xi\cos^7(\theta) + c_{200}b_{200}^2\rho^4\cos^8(\theta) + (b^2C'_{011}\rho\xi + b((c_1A_{002} + c_2a_{002})\xi^3/\rho + (c_{002}A_{002} +$$
$$C_{002}a_{002})\xi^4/\rho))\sin(\theta) + (b^2C'_{110}\rho^2 + b((c_1(A_{101} - B_{011}) - c_2(b_{011} - a101))\xi^2 + (c_{002}(A_{101} - B_{011}) + c_{101}A_{002} -$$
$$c_{011}B_{002}-C_{002}(b_{011}-a_{101})+C_{101}a_{002}-C_{011}b_{002})\xi^3)-2b_{002}a_{002}(c_1\xi^5/\rho^2+c_{002}\xi^6/\rho^2))\cos(\theta)\sin(\theta)+(-b(c_1(B_{110}-$$
$$A_{200}) + c_2(b_{110} - a_{200}))\rho\xi + b(-c_{002}(B_{110} - A_{200}) + c_{200}A_{002} - c_{110}B_{002} - c_{101}(B_{011} - A_{101}) - c_{011}B_{101} -$$
$$C_{002}(b_{110}-a_{200})-C_{101}(b_{011}-a_{101})-C_{110}b_{002}+C_{200}a_{002}-C_{011}b_{101})\rho\xi^2-2c_1(b_{101}a_{002}+a_{101}b_{002}-b_{011}b_{002})\xi^4/\rho-$$
$$2c_{002}(b_{101}a_{002}+a_{101}b_{002}-b_{011}b_{002})\xi^5/\rho-2c_{101}b_{002}a_{002}\xi^5/\rho+c_{011}b_{002}^2\xi^5/\rho)\cos^2(\theta)\sin(\theta)+(b(c_{200}(A_{101}-B_{011})-$$
$$c_{110}B_{101} - c_{101}(B_{110} - A_{200}) - c_{011}B_{200} - C_{110}b_{101} - C_{011}b_{200} - C_{101}(b_{110} - a_{200}) - C_{200}(b_{011} - a101))\rho^2\xi +$$
$$2(c_1b_{002}(b_{110} - a_{200}) - c_1b_{200}a_{002} - c_1b_{101}a_{101} + c_1b_{101}b_{011})\xi^3 + (2(c_{002}(b_{002}(b_{110} - a_{200}) - b_{200}a_{002} - b_{101}a_{101} +$$
$$b_{101}b_{011}) - c_{200}b_{002}a_{002} - c_{101}b_{101}a_{002} - c_{101}a_{101}b_{002} + c_{101}b_{011}b_{002} + c_{011}b_{101}b_{002}) + c_{110}b_{002}^2)\xi^4) \times \cos^3(\theta)\sin(\theta)+$$
$$(-b(c_{200}(B_{110}-A_{200})+c_{110}B_{200}+C_{200}(b_{110}-a_{200})+C_{110}b_{200})\rho^3+2c_1(b_{200}(b_{011}-a_{101})+b_{101}(b_{110}-a_{200}))\rho\xi^2+$$
$$(2(c_{002}b_{200}(b_{011} - a101) + c_{002}b_{101}(b_{110} - a_{200}) - c_{200}b_{101}a_{002} - c_{200}a_{101}b_{002} + c_{200}b_{011}b_{002} + c_{110}b_{101}b_{002} -$$
$$c_{101}b_{200}a_{002}+c_{101}b_{002}(b_{110}-a_{200})+c_{101}b_{101}(b_{011}-a_{101})+c_{011}b_{200}b_{002})+c_{011}b_{101}^2)\rho\xi^3)\cos^4(\theta)\sin(\theta)+(2c_1b_{200}(b_{110}-$$
$$a_{200})\rho^2\xi + (2(c_{002}b_{200}(b_{110} - a_{200}) + c_{200}b_{002}(b_{110} - a_{200}) - c_{200}a_{002}b_{200} + c_{200}b_{101}(b_{011} - a_{101}) + c_{110}b_{200}b_{002} +$$
$$c_{101}b_{200}(b_{011}-a_{101})+c_{101}b_{110}b_{101}-c_{101}a_{200}b_{101}+c_{011}b_{200}b_{101})+c_{110}b_{101}^2)\rho^2\xi^2)\cos^5(\theta)\sin(\theta)+(2(c_{200}b_{200}(b_{011}-$$
$$a_{101})+c_{200}b_{101}(b_{110}-a_{200})+c_{110}b_{200}b_{101}+c_{101}b_{200}(b_{110}-a_{200}))+c_{011}b_{200}^2)\rho^3\xi\cos^6(\theta)\sin(\theta)+(2c_{200}b_{200}(b_{110}-$$
$$a_{200})+c_{110}b_{200}^2)\cos^7(\theta)\sin(\theta)+(b^2\acute{C}_{020}\rho^2+b((c_1A_{011}+c_2a_{011})\xi^2+(bc_{002}A_{011}+c_{011}A_{002}+C_{002}a_{011}+C_{011}a_{002})\xi^3)+$$
$$c_1a_{002}^2\xi^5/\rho^2 + c_{002}a_{002}^2\xi^6/\rho^2)\sin^2(\theta) + (-b(c_1(B_{020} - A_{110}) + c_2(b_{020} - a_{110}))\rho\xi + b(-c_{002}(B_{020} - A_{110}) +$$
$$c_{110}A_{002} - c_{020}B_{002} + c_{101}A_{011} - c_{011}(B_{011} - A_{101}) - C_{002}(b_{020} - a_{110}) + C_{110}a_{002} - C_{020}b_{002} + C_{101}a_{011} -$$
$$C_{011}(b_{011} - a_{101}))\rho\xi^2 + 2((c_1a_{011}b_{002} - c_1b_{011}a_{002} + c_1a_{101}a_{002})\xi^4/\rho + (c_{002}a_{002}(a_{101} - b_{011}) - c_{002}a_{011}b_{002} -$$
$$c_{011}b_{002}a_{002})\xi^5/\rho) + c_{101}a_{002}^2\xi^5/\rho) \times \cos(\theta)\sin^2(\theta) + (b(c_{200}A_{011} - c_{110}(B_{011} - A_{101}) - c_{020}B_{101} - c_{101}(B_{020} -$$
$$A_{110}) - c_{011}(B_{110} - A_{200}) + C_{200}a_{011} - C_{101}(b_{020} - a_{110}) - C_{011}(b_{110} - a_{200}) - C_{020}b_{101} - C_{110}(b_{011} - a_{101}))\rho^2\xi +$$
$$c_1(2a_{002}(a_{200} - b_{110}) - 2a_{110}b_{002} - 2b_{101}a_{011} - 2a_{101}b_{011} + b_{011}^2 + a_{101}^2 + 2b_{020}b_{002})\xi^3 + (2c_{002}a_{002}(a_{200} - b_{110}) -$$
$$2c_{002}a_{110}b_{002} - 2c_{002}b_{101}a_{011} - 2c_{002}a_{101}b_{011} + c_{002}b_{011}^2 + c_{002}a_{101}^2 + 2c_{002}b_{020}b_{002} + c_{200}a_{002}^2 - 2c_{110}b_{002}a_{002} +$$
$$c_{020}b_{002}^2-2c_{101}a_{011}b_{002}-2c_{101}b_{011}a_{002}+2c_{101}a_{002}a_{101}-2c_{011}b_{101}a_{002}-2c_{011}a_{101}b_{002}+2c_{011}b_{011}b_{002})\xi^4)\cos^2(\theta)\sin^2(\theta)+$$
$$(-b(c_{200}(B_{020} - A_{110}) + c_{110}(B_{110} - A_{200}) + c_{020}B_{200} + C_{200}(b_{020} - a_{110}) + C_{020}b_{200} + C_{110}(b_{110} - a_{200}))\rho^3 +$$
$$2c_1(b_{110}(b_{011} - a_{101}) + a_{200}(a_{101} - b_{011}) - b_{200}a_{011} + b_{020}b_{101} - a_{110}b_{101})\rho\xi^2 + (-2c_{002}b_{110}a_{101} + 2c_{002}a_{200}a_{101} -$$
$$2c_{002}a_{200}b_{011} + 2c_{002}b_{110}b_{011} - 2c_{002}b_{200}a_{011} + 2c_{002}b_{020}b_{101} - 2c_{002}a_{110}b_{101} - 2c_{200}a_{011}b_{002} + 2c_{200}a_{002}(a_{101} -$$
$$b_{011})-2c_{110}b_{101}a_{002}-2c_{110}a_{101}b_{002}+2c_{110}b_{011}b_{002}+2c_{020}b_{101}b_{002}+2c_{101}b_{002}(b_{020}-a_{110})+2c_{101}a_{002}(a_{200}-b_{110})-$$
$$2c_{101}b_{101}a_{011} - 2c_{101}a_{101}b_{011} + c_{101}b_{011}^2 - 2c_{011}b_{200}a_{002} + 2c_{011}b_{110}b_{002} + c_{101}a_{101}^2 - 2c_{011}a_{200}b_{002} - 2c_{011}b_{101}a_{101} +$$
$$2c_{011}b_{101}b_{011})\rho\xi^3)\cos^3(\theta)\sin^2(\theta) + (c_1(2b_{200}(b_{020} - a_{110}) + b_{110}^2 + a_{200}^2 - 2b_{110}a_{200})\rho^2\xi + (2c_{002}b_{200}(b_{020} - a_{110}) +$$
$$c_{002}b_{110}^2+c_{002}a_{200}^2-2c_{002}b_{110}a_{200}+2c_{200}a_{002}(a_{200}-b_{110})-2c_{200}a_{110}b_{002}-2c_{200}b_{101}a_{011}-2c_{200}a_{101}b_{011}+c_{200}b_{011}^2+$$
$$c_{200}a_{101}^2 + 2c_{200}b_{020}b_{002} - 2c_{110}a_{200}b_{002} + 2c_{110}b_{110}b_{002} - 2c_{110}b_{200}a_{002} + 2c_{110}b_{101}(b_{011} - a_{101}) + 2c_{020}b_{200}b_{002} -$$
$$2c_{101}b_{200}a_{011}-2c_{101}b_{110}a_{101}+c_{020}b_{101}^2+2c_{101}b_{110}b_{011}-2c_{101}a_{200}b_{011}-2c_{101}a_{110}b_{101}+2c_{101}b_{020}b_{101}+2c_{101}a_{200}a_{101}-$$
$$2c_{011}b_{200}a_{101} + 2c_{011}b_{110}b_{101} + 2c_{011}b_{200}b_{011} - 2c_{011}a_{200}b_{101})\rho^2\xi^2)\cos^4(\theta)\sin^2(\theta) + (2c_{200}b_{110}(b_{011} - a_{101}) +$$
$$2c_{200}a_{200}(a_{101}-b_{011})-2c_{200}b_{200}a_{011}+2c_{200}b_{020}b_{101}-2c_{200}a_{110}b_{101}+2c_{110}b_{200}(b_{011}-a_{101})+2c_{110}b_{101}(b_{110}-a_{200})+$$
$$2c_{020}b_{200}b_{101}+2c_{101}b_{200}(b_{020}-a_{110})-2c_{101}b_{110}a_{200}+c_{101}b_{110}^2+c_{101}a_{200}^2+2c_{011}b_{200}(b_{110}-a_{200}))\rho^3\xi\cos^5(\theta)\sin^2(\theta)+$$

$(2c_{200}b_{200}(b_{020}-a_{110})+c_{200}b_{110}^2-2c_{200}b_{110}a_{200}+c_{200}a_{200}^2+2c_{110}b_{200}(b_{110}-a_{200})+c_{020}b_{200}^2)\rho^4\cos^6(\theta)\sin^2(\theta)+(b((c_1A_{020}+c_2a_{020})\rho\xi+(c_{002}A_{020}+c_{020}A_{002}+c_{011}A_{011}+C_{002}a_{020}+C_{020}a_{002}+C_{011}a_{011}))\rho\xi^2+2c_1a_{011}a_{002}\xi^4/\rho+(2c_{002}a_{011}a_{002}+c_{011}a_{002}^2)\xi^5/\rho)\sin^3(\theta)+(b(c_{110}A_{011}-c_{011}(B_{020}-A_{110})+c_{101}A_{020}-c_{020}(B_{011}-A_{101})+C_{110}a_{011}-C_{020}(b_{011}-a_{101})-C_{011}(b_{020}-a_{110})+C_{101}a_{020})\rho^2\xi+2c_1(a_{011}(a_{101}-b_{011})-a_{020}b_{002}+a_{002}(a_{110}-b_{020}))\xi^3+(2c_{002}a_{011}(a_{101}-b_{011})-2c_{002}a_{020}b_{002}+2c_{002}a_{110}a_{002}-2c_{002}b_{020}a_{002}+c_{110}a_{002}^2-2c_{020}b_{002}a_{002}+2c_{101}a_{011}a_{002}-2c_{011}a_{011}b_{002}+2c_{011}a_{002}(a_{101}-b_{011}))\xi^4)\cos(\theta)\sin^3(\theta)+(b(c_{200}A_{020}-c_{110}(B_{020}-A_{110})-c_{020}(B_{110}-A_{200})-C_{110}(b_{020}-a_{110})-C_{020}(b_{110}-a_{200})+C_{200}a_{020})\rho^3+2c_1(-b_{110}a_{011}+a_{200}a_{011}+b_{020}b_{011}-a_{110}b_{011}-a_{020}b_{101}+a_{110}a_{101}-b_{020}a_{101})\rho\xi^2+2(c_{002}a_{011}(a_{200}-b_{110})+c_{002}b_{011}(b_{020}-a_{110})-c_{002}a_{020}b_{101}+c_{002}a_{101}(a_{110}-b_{020})+c_{200}a_{011}a_{002}-c_{110}(a_{011}b_{002}+b_{011}a_{002})+c_{110}a_{101}a_{002}-c_{020}b_{101}a_{002}+c_{020}b_{002}(b_{011}-a101)+c_{101}a_{002}(a_{110}-b_{020})-c_{101}a_{020}b_{002}+c_{011}a_{002}(a_{200}-b_{110})+c_{101}a_{011}(a_{101}-b_{011})+c_{011}b_{002}(b_{020}-a_{110})-c_{011}b_{101}a_{011}-c_{011}a_{101}b_{011})\rho\xi^3+c_{011}(b_{011}^2+a_{101}^2)\rho\xi^3)\cos^2(\theta)\sin^3(\theta)+(2c_1(b_{020}(b_{110}-a_{200})+a_{110}(a_{200}-b_{110})-b_{200}a_{020})\rho^2\xi+(2c_{002}(b_{020}(b_{110}-a_{200})+a_{110}(a_{200}-b_{110})-b_{200}a_{020})+2c_{200}a_{011}(a_{101}-b_{011})-2c_{200}a_{020}b_{002}+2c_{200}a_{110}a_{002}-2c_{200}b_{020}a_{002}+2c_{110}a_{002}(a_{200}-b_{110})+2c_{110}b_{002}(b_{020}-a_{110})-2c_{110}b_{101}a_{011}-2c_{110}a_{101}b_{011}+c_{110}(b_{011}^2+a_{101}^2)-2c_{020}b_{200}a_{002}+2c_{020}(b_{002}(b_{110}-a_{200})+b_{101}(b_{011}-a_{101}))+2c_{101}(b_{020}(b_{011}-a_{101})+a_{110}(a_{101}-b_{011}))+2c_{101}(a_{011}(a_{200}-b_{110})-a_{020}b_{101})+2c_{011}(b_{110}(b_{011}-a_{101})-b_{200}a_{011}+a_{200}(a_{101}-b_{011})-a_{110}b_{101}+b_{020}b_{101}))\rho^2\xi^2)\cos^3(\theta)\sin^3(\theta)+(2c_{200}(a_{011}(a_{200}-b_{110})+b_{020}(b_{011}-a_{101})+a_{110}(a_{101}-b_{011})-a_{020}b_{101})+2c_{110}(b_{110}(b_{011}-a_{101})+a_{200}(a_{101}-b_{011})+b_{101}(b_{020}-a_{110})-b_{200}a_{011})+2c_{020}(b_{200}(b_{011}-a_{101})+b_{101}(b_{110}-a_{200}))+2c_{101}(-b_{200}a_{020}-b_{110}a_{110}+b_{110}b_{020}+a_{200}(a_{110}-b_{020}))+c_{011}(2b_{200}(b_{020}-a_{110})-2b_{110}a_{200}+b_{110}^2+a_{200}^2))\rho^3\xi\cos^4(\theta)\sin^3(\theta)+(2c_{200}(-b_{200}a_{020}+b_{110}(b_{020}-a_{110})+a_{200}(a_{110}-b_{020}))+c_{110}(2b_{200}(b_{020}-a_{110})+b_{110}^2+a_{200}^2-2a_{200}b_{110})+2c_{020}b_{200}(b_{110}-a_{200}))\rho^4\cos^5(\theta)\sin^3(\theta)+(b(c_{020}A_{011}+c_{011}A_{020}+C_{011}a_{020}+C_{020}a_{011})\rho^2\xi+c_1(2a_{020}a_{002}+a_{011}^2)\xi^3+(2c_{002}a_{020}a_{002}+c_{002}a_{011}^2+c_{020}a_{002}^2+2c_{011}a_{011}a_{002})\xi^4)\sin^4+(b(c_{110}A_{020}-c_{020}(B_{020}-A_{110})-C_{020}(b_{020}-a_{110})+C_{110}a_{020})\rho^3+2c_1(a_{011}(a_{110}-b_{020})+a_{020}(a_{101}-b_{011}))\rho\xi^2+2(-c_{002}b_{020}a_{011}+c_{002}a_{020}(a_{101}-b_{011})+a_{011}(c_{002}a_{110}+c_{110}a_{002}-c_{020}b_{002})+a_{002}(c_{020}(a_{101}-b_{011})+c_{101}a_{020})+c_{011}(-b_{020}a_{002}-a_{020}b_{002}+a_{110}a_{002}+a_{011}(a_{101}-b_{011})))\rho\xi^3+c_{101}a_{011}^2\rho\xi^3)\cos(\theta)\sin^4(\theta)+(c_1(2a_{020}\times(a_{200}-b_{110})+b_{020}^2+a_{110}^2-2b_{020}a_{110})\rho^2\xi+(c_{002}(2a_{020}(a_{200}-b_{110})+b_{020}^2+a_{110}^2-2b_{020}a_{110})+c_{200}(2a_{020}a_{002}+a_{011}^2)+2c_{110}(-b_{020}a_{002}+a_{011}(a_{101}-b_{011})-a_{020}b_{002}+a_{110}a_{002})+c_{020}(2a_{002}(a_{200}-b_{110})+2b_{002}(b_{020}-a_{110})-2b_{101}a_{011}-2a_{101}b_{011}+b_{011}^2+a_{101}^2)+2c_{101}(a_{020}(a_{101}-b_{011})+a_{011}(a_{110}-b_{020}))+2c_{011}(a_{011}(a_{200}-b_{110})+b_{020}(b_{011}-a_{101})+a_{110}(a_{101}-b_{011})-a_{020}b_{101}))\rho^2\xi^2)\cos^2(\theta)\sin^4(\theta)+(c_{101}(a_{110}^2+b_{020}^2)+2(c_{200}(a_{011}(a_{110}-b_{020})+a_{020}(a_{101}-b_{011}))+c_{020}b_{110}(b_{011}-a_{101})+c_{020}a_{200}(a_{101}-b_{011})+c_{011}b_{110}(b_{020}-a_{110})+c_{110}a_{011}(a_{200}-b_{110})+c_{110}b_{020}(b_{011}-a_{101})+c_{020}b_{101}(b_{020}-a_{110})+c_{101}a_{020}(a_{200}-b_{110})+c_{011}a_{200}(a_{110}-b_{020})+c_{110}a_{110}(a_{101}-b_{011})-c_{020}b_{200}a_{011}-a_{020}b_{101}c_{110}-a_{110}b_{020}c_{101}-a_{020}b_{200}c_{011}))\rho^3\xi\cos^3(\theta)\sin^4(\theta)+(c_{200}(a_{110}^2+b_{020}^2)+c_{020}(a_{200}^2+b_{110}^2)+2(c_{200}a_{020}(a_{200}-b_{110})-a_{110}b_{020}c_{200}+b_{110}c_{110}(b_{020}-a_{110})+a_{200}c_{110}(a_{110}-b_{020})-a_{020}b_{200}c_{110}+c_{020}b_{200}(b_{020}-a_{110})-a_{200}b_{110}c_{020}))\rho^4\cos^4(\theta)\sin^4(\theta)+(b(c_{020}A_{020}+C_{020}a_{020})\rho^3+2c_1a_{020}a_{011}\rho\xi^2+(c_{011}a_{011}^2+2(c_{002}a_{020}a_{011}+a_{011}a_{002}c_{020}+a_{002}a_{020}c_{011}))\rho\xi^3)\sin^5(\theta)+(2c_1a_{020}(a_{110}-b_{020})\rho^2\xi+((c_{110}a_{011}^2+2(c_{002}a_{020}(a_{110}-b_{020})+c_{020}a_{002}(a_{110}-b_{020})+a_{011}c_{020}(a_{101}-b_{011})+a_{020}c_{011}(a_{101}-b_{011})+a_{011}c_{011}(a_{110}-b_{020})+c_{110}a_{020}a_{002}-a_{020}b_{002}c_{020}+a_{020}a_{011}c_{101}))\rho^2\xi^2))\times\cos(\theta)\sin^5(\theta)+(c_{011}(a_{110}^2+b_{020}^2)+2(c_{110}a_{020}(a_{101}-b_{011})+c_{020}a_{110}(a_{101}-b_{011})+c_{101}a_{020}(a_{110}-b_{020})+c_{011}a_{020}(a_{200}-b_{110})+c_{110}a_{011}(a_{110}-b_{020})++c_{020}b_{020}(b_{011}-a_{101})+c_{020}a_{011}(a_{200}-b_{110})+a_{020}a_{011}c_{200}-c_{020}a_{020}b_{101}-a_{110}b_{020}c_{011}))\rho^3\xi\cos^2(\theta)\sin^5(\theta)+(c_{110}(a_{110}^2+b_{020}^2)+2(c_{200}a_{020}(a_{110}-b_{020})+a_{020}c_{110}(a_{200}-b_{110})+c_{020}a_{200}(a_{110}-$

$b_{020}) + c_{020}b_{110}(b_{020} - a_{110}) - a_{110}c_{110}b_{020} - a_{020}b_{200}c_{020}))\rho^4 \cos^3(\theta)\sin^5(\theta) + (c_1 a_{020}^2\rho^2\xi + (c_{002}a_{020}^2 + c_{020}a_{011}^2 + 2a_{020}(c_{011}a_{011} + c_{020}a_{002}))\rho^2\xi^2) \times \sin^6(\theta) + (c_{101}a_{020}^2 + 2(c_{011}a_{020}(a_{110} - b_{020}) + c_{020}a_{011}(a_{110} - b_{020}) + a_{020}c_{020}(a_{101} - b_{011}) + c_{110}a_{020}a_{011}))\rho^3\xi\cos(\theta)\sin^6(\theta) + (c_{200}a_{020}^2 + c_{020}(a_{110}^2 + b_{020}^2) + c_{020}a_{020}(a_{200} - b_{110}) + c_{110}a_{020}(a_{110} - b_{020}) - c_{020}a_{110}b_{020})\rho^4\cos^2(\theta)\sin^6(\theta) + (c_{011}a_{020}^2 + 2a_{020}a_{011}c_{020})\rho^3\xi\sin^7(\theta) + (c_{110}a_{020}^2 + 2a_{020}c_{020} \times (a_{110} - b_{020}))\rho^4\cos(\theta)\sin^7(\theta) + c_{020}a_{020}^2\rho^4\sin^8(\theta)].$