On λ-perfect maps

Mehrdad NAMDARI*, **Mohammad Ali SIAVOSHI**
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Abstract: λ-Perfect maps, a generalization of perfect maps (i.e. continuous closed maps with compact fibers) are presented. Using P_λ-spaces and the concept of λ-compactness some classical results regarding λ-perfect maps will be extended. In particular, we show that if the composition fg is a λ-perfect map where f, g are continuous maps with fg well-defined, then f, g are α-perfect and β-perfect, respectively, on appropriate spaces, where $\alpha, \beta \leq \lambda$.

Key words: λ-Compact, λ-perfect, P_λ-space, Lindelöf number

1. Introduction

A perfect map is a kind of continuous function between topological spaces. Perfect maps are weaker than homeomorphisms, but strong enough to preserve some topological properties such as local compactness that are not always preserved by continuous maps. Let X, Y be two topological spaces such that X is Hausdorff. A continuous map $f : X \to Y$ is said to be a perfect map provided that f is closed and surjective and each fiber $f^{-1}(y)$ is compact in X. In [3], the authors study a generalization of compactness, i.e. λ-compactness. Motivated by this concept we are led to generalize the concept of perfect mapping. In the current section we recall some preliminary definitions and related results such as λ-compact spaces and P_λ-spaces presented in [3]. In Section 2, we will introduce λ-perfect maps and generalize some classical results related to the perfect maps and finally we prove that if the composition of two continuous mappings is λ-perfect, then its components are μ-compact and β-compact, where $\mu, \beta \leq \lambda$. We extended some important well-known results regarding compactness properties to λ-compactness properties (namely, Theorem 2.2 is extended to Theorem 2.3). Moreover, some important known results regarding perfect maps are also extended accordingly.

The following definition, which is a natural generalization of compactness, has been considered by some authors earlier; see [3, 5].

Definition 1.1 *A topological space X (not necessarily Hausdorff) is said to be λ-compact whenever each open cover of X has an open subcover whose cardinality is less than λ, where λ is the least infinite cardinal number with this property. λ is called the compactness degree of X and we write $d_c(X) = \lambda$.*

We note that compact spaces, Lindelöf noncompact spaces are $\aleph_\circ$-compact, $\aleph_1$-compact spaces, respectively, and in general every topological space X is λ-compact for some infinite cardinal number λ; see [3] and also the concept of Lindelöf number in [1, p 193].

---

*Correspondence: namdari@ipm.ir

The following useful lemma, which is well known, is somehow a generalization of the fact that a continuous function $f : X \to Y$ takes compact sets in X to compact sets in Y; see [6].

Lemma 1.2 *Let* X, Y *be two topological spaces. If* $f : X \to Y$ *is a continuous function and* $A \subseteq X$ *is* λ*-compact, then* $d_c(f(A)) \leq \lambda$.

Proof   Let $f(A) \subseteq \bigcup_{i \in I} H_i$, where each H_i is open in Y. Clearly, $A \subseteq \bigcup_{i \in I} f^{-1}(H_i)$. By λ-compactness of A, there exists $J \subseteq I$ with $|J| < \lambda$ such that $A \subseteq \bigcup_{i \in J} f^{-1}(H_i)$. Hence $f(A) \subseteq \bigcup_{i \in J} H_i$, which means that $d_c(f(A) \leq \lambda$. $\qquad \square$

We also recall the following proposition, a proof of which can be found in [4].

Proposition 1.3 [4, Proposition 2.6] *Let* F *be a closed subset of a topological space* X; *then* $d_c(F) \leq d_c(X)$.

Next we are going to introduce a generalization of the concept of a P-space (i.e. pseudo-discrete space). P-spaces are very important in the contexts of rings of continuous functions, which are fully investigated by Gillman and Henriksen in [2].

Definition 1.4 *Let* X *be a topological space. The intersection of any family with cardinality less than* λ *of open subsets of* X *is called a* G_λ*-set. Obviously,* G_δ*-sets are precisely* $G_{\aleph_1}$*-sets.*

Definition 1.5 *The topological space* X *is said to be a* P_λ*-space whenever each* G_λ*-set in* X *is open. It is manifest that every arbitrary space is a* $P_{\aleph_0}$*-space and* $P_{\aleph_1}$*-spaces are precisely* P*-spaces. See Example 1.8 and Example 1.9 in* [3]

2. λ-Perfect maps

Lemma 2.1 *If* A *is a* β*-compact subspace of a space* X *and* y *is a point of a* P_λ*-space* Y *such that* $\beta \leq \lambda$ *then for every open set* $W \subseteq X \times Y$ *containing* $A \times \{y\}$ *there exist open sets* $U \subseteq X$ *and* $V \subseteq Y$ *such that* $A \times \{y\} \subseteq U \times V \subseteq W$.

Proof   For every $x \in A$ the point (x, y) has a neighborhood of the form $U_x \times V_x$ contained in W. Clearly $A \times \{y\} \subseteq \bigcup_{x \in A} U_x \times V_x$. Since A is β-compact, there exists a family $\{x_i\}_{i \in I}$ with $|I| < \beta \leq \lambda$ such that $A \times \{y\} \subseteq \bigcup_{i \in I} U_{x_i} \times V_{x_i}$. Clearly the sets $U = \bigcup_{i \in I} U_{x_i}$ and $V = \bigcap_{i \in I} V_{x_i}$ have the required properties. $\square$

Let us recall the celebrated theorem of Kuratowski; see [1].

Theorem 2.2 *For a topological space* X *the following statements are equivalent:*
(i) The space X *is compact.*
(ii) For every topological space Y *the projection* $p : X \times Y \to Y$ *is closed.*
(iii) For every T_4*-space* Y *the projection* $p : X \times Y \to Y$ *is closed.*

Next we generalize the above theorem.

The implication (ii)$\Rightarrow$(i) in the following theorem can be obtained also as a consequence of Proposition 3 in [5].

Theorem 2.3 *For an infinite regular cardinal number* λ *and a topological space* X *the following conditions are equivalent.*

(i) The space X is β-compact for some $\beta \leq \lambda$.

(ii) For every P_λ-space, Y the projection $p : X \times Y \to Y$ is closed.

(iii) For every T_4, P_λ-space Y the projection $p : X \times Y \to Y$ is closed.

Proof (i)$\Rightarrow$(ii) Let X be a β-compact space and F be a closed subspace of $X \times Y$. Take a point $y \notin p(F)$; thus $X \times \{y\} \subseteq (X \times Y) \setminus F$. In view of lemma 2.1, we infer that y has a neighborhood V such that $(X \times V) \cap F = \emptyset$. Consequently, we have $p(F) \cap V = \emptyset$, which immediately shows that $Y \setminus p(F)$ is an open subset of Y; hence we are done.

(ii)$\Rightarrow$(iii) It is obvious.

(iii)$\Rightarrow$ (i) Suppose that there exists a family $\{F_s\}_{s \in S}$ of closed subsets of X with the λ-intersection property such that $\bigcap_{s \in S} F_s = \emptyset$ and proceed by contradiction. Take a point $y_0 \notin X$ and consider the set $Y = X \cup \{y_0\}$ with the the the topology T consisting of all subsets of X and of all sets of the form

$$\{y_0\} \cup (\bigcap_{i \in I} F_{s_i}) \cup K, \text{where} \quad s_i \in S, \quad |I| < \lambda \quad \text{and} \quad K \subseteq X.$$

Since $\bigcap_{s \in S} F_s = \emptyset$, then for every $x \in X$ there exists $s_x \in S$ such that $x \notin F_{s_x}$. Notice that the set

$$\{y_0\} \cup F_{s_x} \cup (X \setminus \{x\}) = Y \setminus \{x\}$$

belongs to T and since $X \in T$, $\{y_0\}$ is closed; therefore Y is a T_1-space.

Next we aim to show that Y is normal. Let A and B be two disjoint closed subsets of Y; hence at least one of them, say A, does not contain y_0 and so it is open. Therefore, A and $Y \setminus A$ are two disjoint open subsets of Y containing A and B, respectively. Hence Y is normal, and by regularity of λ it is a P_λ-space.

Now take $F = \overline{\{(x,x) : x \in X\}} \subseteq X \times Y$. By our hypothesis $p(F)$ is closed in Y. Since $\{y_0\}$ is not open, every open subset of Y that contains y_0 meets X; hence $\mathrm{cl}_Y X = Y$.

We note that if $x \in X$ then $(x,x) \in F$ and therefore $x = p(x,x) \in p(F)$; hence $X \subseteq p(F)$. Consequently,

$$y_0 \in Y = \mathrm{cl}_Y X \subseteq \mathrm{cl}_Y p(F) = p(F).$$

Therefore, there exists $x_0 \in X$ such that $(x_0, y_0) \in F$. For every neighborhood $U \subseteq X$ of x_0 and every $s \in S$, the set $U \times (\{y_0\} \cup F_s)$ is open in $X \times Y$. Hence

$$[U \times (\{y_0\} \cup F_s)] \cap \{(x,x) : x \in X\} \neq \emptyset,$$

and thus $U \cap F_s \neq \emptyset$, which means that $x_0 \in F_s$ for every $s \in S$. This implies that $\bigcap_{s \in S} F_s \neq \emptyset$, which is the desired contradiction. $\square$

Definition 2.4 *Let X and Y be two topological spaces. A continuous map $f : X \to Y$ is said to be λ-perfect whenever f is closed and surjective and λ is the least infinite cardinal number such that for every $y \in Y$ the compactness degree of $f^{-1}(y)$ is less than or equal to λ.*

Proposition 2.5 *Let Y be a P_λ-space and X be a β-compact space such that $\beta \leq \lambda$. Then the projection $p : X \times Y \to Y$ is β-perfect.*

Proof By Theorem 2.3 p is closed and we notice that $p^{-1}(y) = X \times \{y\}$ is β-compact for every $y \in Y$. $\square$

The following theorem is an extension of [1, Theorem 3.7.2]

Theorem 2.6 *Let* λ *and* β *be two regular cardinal numbers and* $f : X \to Y$ *be a* λ-*perfect mapping. Then for every* β-*compact subspace* $Z \subseteq Y$ *the inverse image* $f^{-1}(Z)$ *is* μ-*compact for some* $\mu \leq \max\{\lambda, \beta\}$.

Proof Let $\{U_s\}_{s \in S}$ be a family of open subsets of X whose union contains $f^{-1}(Z)$, K be the family of all subsets of S with cardinality less than λ, and $U_T = \bigcup_{s \in T} U_s$ for some $T \in K$. For each $z \in Z$, $d_c(f^{-1}(z)) \leq \lambda$ and thus is contained in the set U_T for some $T \in K$; it follows that $z \in Y \setminus f(X \setminus U_T)$ and thus

$$Z \subseteq \bigcup_{T \in K} (Y \setminus f(X \setminus U_T)).$$

Since f is closed, we infer that for each $T \in K$ the set $Y \setminus f(X \setminus U_T)$ is open. Hence there exists $I \subseteq K$ with $|I| < \beta$ such that $Z \subseteq \bigcup_{T \in I}(Y \setminus f(X \setminus U_T))$. Thus

$$f^{-1}(Z) \subseteq \bigcup_{T \in I} f^{-1}(Y \setminus f(X \setminus U_T)) = \bigcup_{T \in I} (X \setminus f^{-1}f(X \setminus U_T)) \subseteq$$

$$\bigcup_{T \in I} (X \setminus (X \setminus U_T)) = \bigcup_{T \in I} U_T = \bigcup_{s \in S_0} U_s,$$

where $S_0 = \bigcup_{T \in I} T$. Since λ and β are regular cardinals, we infer that $|S_0| < \max\{\lambda, \beta\}$, i.e. $f^{-1}(Z)$ is μ-compact for some $\mu \leq \max\{\lambda, \beta\}$. $\square$

The following corollary is now immediate.

Corollary 2.7 *Let* $f : X \to Y$ *be a* λ-*perfect mapping. Then for every* λ-*compact subspace* $Z \subseteq Y$ *the inverse image* $f^{-1}(Z)$ *is* λ-*compact as well.*

Proof By the previous theorem $f^{-1}(Z)$ is μ-compact for some $\mu \leq \lambda$, and by lemma 1.2, $Z = f(f^{-1}(Z))$ is γ-compact for some $\gamma \leq \mu$. However, by our hypothesis $\gamma = \lambda$; hence, $\mu = \lambda$. $\square$

Corollary 2.8 *Let* $g : X \to Z$ *be a* λ-*perfect and* $f : Z \to Y$ *be a* β-*perfect mapping. Then the composition* $fg : X \to Y$ *is* γ-*perfect for some* $\gamma \leq \max\{\lambda, \beta\}$.

Proof Let $y \in Y$. Since f is β-perfect, $f^{-1}(\{y\})$ is μ-compact, where $\mu \leq \beta$. Now by Theorem 2.6 and the equality $(fg)^{-1}(\{y\}) = g^{-1}(f^{-1}(\{y\}))$ the proof is complete. $\square$

It is well known that in every regular space two disjoint closed sets, one of which is compact, are contained in two disjoint open sets. Next we have an extension of this fact.

Theorem 2.9 *Let* A *be a* β-*compact subspace of a regular* P_λ-*space* X, *for some* $\beta \leq \lambda$. *Then for every closed set* B *disjoint from* A *there exist open sets* $U, V \subseteq X$ *such that* $A \subseteq U$, $B \subseteq V$, *and* $U \cap V = \emptyset$. *If, moreover,* $d_c(B) = \alpha \leq \lambda$ *then it suffices to assume that* X *is a Hausdorff* P_λ-*space.*

Proof Since X is regular, for every $x \in A$ there exist two disjoint open sets $U_x, V_x \subseteq X$ such that $x \in U_x, B \subseteq V_x$. Clearly $A \subseteq \bigcup_{x \in A} U_x$; therefore there exists $J \subseteq A$ with $|J| \leq \beta$ such that $A \subseteq \bigcup_{x \in J} U_x$. Obviously the sets $U = \bigcup_{x \in J} U_x$ and $V = \bigcap_{x \in J} V_x$ have the required properties. Now let we assume that

$d_c(B) = \alpha \leq \lambda$ and X is only a Hausdorff P_λ-space. For every $x \in B$ we consider the sets A and $\{x\}$ and clearly similar to the first part we will obtain the disjoint open sets U_x and V_x such that $x \in V_x$ and $A \subseteq U_x$. Clearly $B \subseteq \bigcup_{x \in B} V_x$ □

Lemma 2.10 *Suppose that X is a Hausdorff P_λ-space and $f : X \to Y$ is a β-perfect mapping with $\beta \leq \lambda$. Then f has no continuous extension to any Hausdorff space Z such that $X \subsetneq Z$ and $\mathrm{cl}X = Z$.*

Proof Without loss of generality, we can assume that $Z = X \cup \{x\}$. Now if $F : Z \to Y$ is a continuous extension of f then $d_c(f^{-1}(F(x))) \leq \beta$ and clearly $f^{-1}(F(x))$ does not contain x. Therefore, by the second part of Theorem 2.9 there exist open sets $U, V \subseteq Z$ such that $x \in U, f^{-1}(F(x)) \subseteq V$, and $U \cap V = \emptyset$. Clearly the sets $(Z \setminus V) \cap X = X \setminus V$, $f(X \setminus V)$, and $F^{-1}(f(X \setminus V))$ are closed. Thus we have

$$cl_z(X \setminus V) \subseteq F^{-1}(f(X \setminus V)) = f^{-1}(f(X \setminus V)) \subseteq X.$$

Since $x \notin cl_z V$, we have $cl_z V \subseteq X$; therefore $cl_z X = cl_z(X \setminus V) \cup cl_z V \subseteq X$. Hence X is not dense in Z. □

Proposition 2.11 *If the composition gf of continuous mappings $f : X \to Y$ and $g : Y \to Z$, where Y is Hausdorff and f is surjective, is λ-perfect, then the mappings g, f are β-perfect and μ-perfect respectively for some $\mu, \beta \leq \lambda$.*

Proof Since for every $z \in Z$ we have $d_c((gf)^{-1}(z)) \leq \lambda$, it follows from lemma 1.2 that the compactness degree of the set $g^{-1}(z) = f[(gf)^{-1}(z)]$ is less than or equal to λ. The fact that g is a closed mapping follows from [1, Proposition 2.1.3] and thus the mapping g is μ-perfect for some $\mu \leq \lambda$.

We note that for every $y \in Y$, $f^{-1}(y)$ is a closed subset of $(gf)^{-1}(g(y))$; hence $d_c(f^{-1}(y)) \leq d_c(gf)^{-1}(g(y)) \leq \lambda$. Thus to complete the proof it suffices to show that f is closed. To this end, we consider the arbitrary closed subset $F \subseteq X$ and the mapping $h = (gf)|_F$. For every $z \in Z, h^{-1}(z) = (gf)^{-1}(z) \cap F$ is a closed subset of $(gf)^{-1}(z)$. Hence $d_c(h^{-1}(z)) \leq d_c((gf)^{-1}(z)) \leq \lambda$. This means that h is α-perfect for some $\alpha \leq \lambda$. Consequently, by the first part of our proof, the restriction $g|_{f(F)}$ is θ-perfect for some $\theta \leq \alpha$. Clearly $g|_{f(F)}$ can be continuously extended to $\overline{f(F)}$ (note that the restriction of g to $f(F)$ is the continuous extension of $g|_{f(F)}$ to $\overline{f(F)}$). It follows from the previous lemma that $f(F) = \overline{f(F)}$ and therefore f is a closed mapping. □

Acknowledgments

References

[1] Engelking R. General Topology. Berlin, Germany: Heldermann Verlag, 1989.

[2] Gillman L, Henriksen M. Concerning rings of continuous functions. T Am Math Soc 1954; 77: 340-362.

[3] Karamzadeh OAS, Namdari M, Siavoshi MA. On λ-compact spaces. Math Slovaca 2013; 63: 1371-1380.

[4] Namdari M, Siavoshi MA. A generalization of compact spaces. JP J Geom Topol 2011; 11: 259-270.

[5] Neira Clara M. The Kuratowski-Mrówka characterization and weak forms of compactness. Rev Colomb Mat 2009; 43: 9-17.

[6] Stephenson RM. Initially κ-compact and related spaces. In: Kunen K, Vaughan J, editor. Handbook of Set-Theoretic Topology. Amsterdam, the Netherlands: Elsevier Science Publishers BV, 1984, pp. 605-632.