Some theorems for a new type of multivalued contractive maps on metric space

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Abstract: In this paper, taking into account the function θ, we introduce a new type of contraction for multivalued maps on metric space. This new concept includes many known contractions in the literature. We then present some fixed point results for closed and bounded set valued maps on complete metric space. Finally, we provide an example to show the significance of the investigation of this paper.

Key words: Fixed point, multivalued contraction, generalized multivalued θ-contraction

1. Introduction

Metric fixed point theory was started by the Banach contraction principle, which asserts that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, i.e. $d(Tx, Ty) \leq L d(x, y)$ for all $x, y \in X$, where $L \in [0, 1)$, then T has a unique fixed point in X. This principle has been extended and generalized in many directions (see [4, 5, 9, 15, 19]). Among all these, an attractive generalization given by Jleli and Samet [12] introduced a new type of contractive mapping. Throughout this study we shall call the contraction defined in [12] a θ-contraction. First, we recall some notions and some related results concerning θ-contraction.

Let $\theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

(θ_1) θ is nondecreasing;

(θ_2) For each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;

(θ_3) There exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$;

(θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by Θ and Ω the set of all functions θ satisfying (θ_1)–(θ_3) and (θ_1)–(θ_4), respectively. It is clear that $\Omega \subset \Theta$ and some examples of the functions belonging to class Ω are $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = e^{\sqrt{te^t}}$. If we define $\theta_3(t) = e^{\sqrt{t}}$ for $t \leq 1$, $\theta_3(t) = 9$ for $t > 1$, then $\theta_3 \in \Theta \backslash \Omega$.

Note that if θ is right continuous and satisfies (θ_1), then (θ_4) holds. Conversely, if (θ_4) holds, then θ is right continuous.

Let (X, d) be a metric space and $\theta \in \Theta$. Then $T : X \to X$ is said to be a θ-contraction if there exists $k \in (0, 1)$ such that

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$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{1.1}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

Choosing some appropriate functions for θ, such as $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = e^{\sqrt{te^t}}$, we can obtain some different types of nonequivalent contractions from (1.1). Considering this new concept, Jleli and Samet proved that every θ-contraction on a complete metric space has a unique fixed point. Some interesting papers are available related to θ-contractions in the literature (see [2, 11]).

2. Preliminaries

In this section, we give some notational and terminological conventions that will be used throughout this paper.

Let (X, d) be a metric space. It is well known that $H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}$$

is a metric on $\mathcal{CB}(X)$, which is called the Pompeiu–Hausdorff metric, where $\mathcal{CB}(X)$ is the class of all nonempty, closed, and bounded subsets of X and $D(x, B) = \inf \{d(x, y) : y \in B\}$. A fixed point of a multivalued mapping $T : X \to \mathcal{P}(X)$, which is the class of all nonempty subsets of X, is an element $x \in X$ such that $x \in Tx$. A function $f : X \to \mathbb{R}$ is lower semicontinuous if $x_n \to x$ implies $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

In 1969, Nadler [17] extended the Banach contraction principle to multivalued mappings and first initiated the study of fixed point results for multivalued linear contraction.

Theorem 1 (Nadler [17]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ a multivalued contraction; that is, there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

Later on, several studies were conducted on a variety of generalizations, extensions, and applications of this result of Nadler (see [1, 3, 6, 7, 13, 14, 16, 18]).

On the other hand, the concept of θ-contraction from the case of single valued mappings was extended to multivalued mappings by Hançer et al. [8] and they introduced the concept of the multivalued θ-contraction: let $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. If for all $x, y \in X$ with $H(Tx, Ty) > 0$, the contractive condition

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{2.1}$$

for some $k \in (0, 1)$ and $\theta \in \Theta$ is satisfied, then T is said to be a multivalued θ-contraction.

Considering the Pompeiu–Hausdorff metric H, they established a fixed point result for mappings of this type on complete metric spaces as follows:

Theorem 2 ([8]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$, which is the class of all nonempty compact subsets of X, be a multivalued mapping. If T is a multivalued θ-contraction, then T has a fixed point.*

In [8], they gave an example (Example 2.4 of [8]) that shows that in Theorem 2 we cannot take $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$ under the same conditions and they showed that it is possible by adding the weak condition (θ_4) on the function θ.

Theorem 3 ([8]) *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. If T is a multivalued θ-contraction with $\theta \in \Omega$, then T has a fixed point.*

3. The results

Our main results are based on the following new concept.

Let (X, d) be a metric space, $\theta \in \Theta$, and $T : X \to \mathcal{CB}(X)$ be given a mapping. Then we say that T is a generalized multivalued θ-contraction if there exists a constant $k \in (0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(M(x, y))]^k , \tag{3.1}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$, where

$$M(x, y) = \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2} [D(x, Ty) + D(y, Tx)] \right\}.$$

Before we give our main results, we recall some notations and related lemmas concerning multivalued mappings as follows: let X and Y be two topological spaces. Then a multivalued mapping $T : X \to \mathcal{P}(Y)$ is said to be upper semicontinuous (lower semicontinuous) if the inverse image of a closed set (open set) is closed (open). A multivalued mapping is continuous if it is both upper and lower semicontinuous.

Lemma 1 ([10]) *Let (X, d) be a metric space and $T : X \to \mathcal{P}(X)$ be an upper semicontinuous mapping such that Tx is closed for all $x \in X$. If $x_n \to x_0$, $y_n \to y_0$ and $y_n \in Tx_n$, then $y_0 \in Tx_0$.*

By the agency of the concept of generalized multivalued θ-contraction, we will give the following theorem. This result is relevant in mapping $T : X \to \mathcal{CB}(X)$.

Theorem 4 *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a generalized multivalued θ-contraction with $\theta \in \Omega$. If T is upper semicontinuous or θ is continuous, then T has a fixed point.*

Proof Suppose that T has no fixed point. Then $D(x, Tx) > 0$ for all $x \in X$. Let $x_0 \in X$ be an arbitrary point. Since Tx is nonempty for all $x \in X$, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. On the other hand, from $0 < D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$ and (θ_1), we obtain

$$
\begin{aligned}
\theta(D(x_1, Tx_1)) &\leq \theta(H(Tx_0, Tx_1)) \\
&\leq [\theta(M(x_0, x_1))]^k \\
&= \left[\theta \left(\max \left\{ \begin{array}{c} d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \\ \frac{1}{2} [D(x_0, Tx_1) + D(x_1, Tx_0)] \end{array} \right\} \right) \right]^k \\
&\leq [\theta(\max \{ d(x_0, x_1), D(x_1, Tx_1) \})]^k . \tag{3.2}
\end{aligned}
$$

If $\max \{ d(x_0, x_1), D(x_1, Tx_1) \} = D(x_1, Tx_1)$, from (3.2), we get

$$\theta(D(x_1, Tx_1)) \leq [\theta(D(x_1, Tx_1)]^k < \theta(D(x_1, Tx_1),$$

which is a contradiction. Thus, $\max\{d(x_0, x_1), D(x_1, Tx_1)\} = d(x_0, x_1)$, and then

$$\theta(D(x_1, Tx_1)) \leq [\theta(d(x_0, x_1))]^k. \tag{3.3}$$

From (θ_4), we know that

$$\theta(D(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y)),$$

and so, from (3.3), we have

$$\inf_{y \in Tx_1} \theta(d(x_1, y)) \leq [\theta(d(x_0, x_1))]^k < [\theta(d(x_0, x_1))]^s, \tag{3.4}$$

where $s \in (k, 1)$. Then, from (3.4), there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq [\theta(d(x_0, x_1))]^s.$$

Therefore, continuing recursively, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^s, \tag{3.5}$$

for all $n \in \mathbb{N}$. Denote $c_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. Then $c_n > 0$ for all $n \in \mathbb{N}$ and, using (3.5), we have

$$\theta(c_n) \leq [\theta(c_{n-1})]^s \leq [\theta(c_{n-2})]^{s^2} \leq \cdots \leq [\theta(c_1)]^{s^{n-1}},$$

i.e.

$$1 < \theta(c_n) \leq [\theta(c_1)]^{s^{n-1}} \tag{3.6}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.6), we obtain

$$\lim_{n \to \infty} \theta(c_n) = 1. \tag{3.7}$$

From (θ_2), we get $\lim_{n \to \infty} c_n = 0^+$ and so from (θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n \to \infty} \frac{\theta(c_n) - 1}{(c_n)^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left| \frac{\theta(c_n) - 1}{(c_n)^r} - l \right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(c_n) - 1}{(c_n)^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$n(c_n)^r \leq An[\theta(c_n) - 1],$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(c_n) - 1}{(c_n)^r} \geq B.$$

This implies that, for all $n \geq n_0$,

$$n \left[c_n\right]^r \leq An \left[\theta(c_n) - 1\right],$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n \left[c_n\right]^r \leq An \left[\theta(c_n) - 1\right],$$

for all $n \geq n_0$. Using (3.6), we obtain

$$n \left[c_n\right]^r \leq An \left[\left[\theta(c_1)\right]^{s^{n-1}} - 1\right],$$

for all $n \geq n_0$. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} n \left[c_n\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n \left[c_n\right]^r \leq 1$ for all $n \geq n_1$, so we have, for all $n \geq n_1$,

$$c_n \leq \frac{1}{n^{1/r}}. \tag{3.8}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (3.8), we have

$$
\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&= c_n + c_{n+1} + \cdots + c_{m-1} \\
&= \sum_{i=n}^{m-1} c_i \leq \sum_{i=n}^{\infty} c_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}
$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$; that is, $\lim_{n \to \infty} x_n = z$.

Now suppose that T is upper semicontinuous. Then, from Lemma 1, we have $z \in Tz$, which is in contrast to our assumption.

Now suppose θ is continuous. In this case, since $z \notin Tz$, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$. Since $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$, then we have

$$
\begin{aligned}
\theta(D(x_{n_k+1}, Tz)) &\leq \theta(H(Tx_{n_k+1}, Tz)) \\
&\leq \left[\theta(M(x_{n_k}, z))\right]^k \\
&\leq \left[\theta\left(\max\left\{ \begin{array}{c} d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), D(z, Tz), \\ \frac{1}{2}\left[D(x_{n_k}, Tz) + D(z, Tx_{n_k+1})\right] \end{array} \right\}\right)\right]^k.
\end{aligned}
$$

Letting limit $n \to \infty$ in the above and using the continuity of θ, we have $\theta(D(z, Tz)) \leq [\theta(D(z, Tz))]^k$, which is a contradiction.

Therefore, T has a fixed point in X. Thereby, this completes the proof. □

Now we give a significant example showing that T is generalized multivalued θ-contraction, but it is not only a generalized multivalued contraction but also a multivalued θ-contraction.

Example 1 *Let* $X = [0, 1] \cup \{2, 3, \cdots\}$ *and*

$$d(x, y) = \begin{cases} 0 & , & if \ x = y \\ |x - y| & , & if \ x, y \in [0, 1] \\ x + y & , & if \ one \ of \ x, y \notin [0, 1] \end{cases}.$$

Then (X, d) *is a complete metric space. Define the mapping* $T : X \to CB(X)$ *by*

$$Tx = \begin{cases} \{\frac{3}{4}\} & , & x = 0 \\ \{1\} & & x \in (0, 1] \\ \{1, x - 1\} & , & x \in \{2, 3, ...\} \end{cases}.$$

For $y = 1$ *and* $x > 2$, *since* $H(Tx, Ty) = x$ *and* $M(x, y) = x + 1$, *we get*

$$\lim_{x \to \infty} \frac{H(Tx, Ty)}{M(x, y)} = \lim_{x \to \infty} \frac{x}{x + 1} = 1.$$

Then we cannot find $\lambda \in (0, 1)$ *satisfying*

$$H(Tx, Ty) \leq \lambda M(x, y).$$

Also, since $H(T0, T\frac{1}{4}) = \frac{1}{4} = d(0, \frac{1}{4})$, *then for all* $\theta \in \Theta$, *which is continuous, and any* $k \in (0, 1)$, *we have*

$$\theta(H(Tx, Ty)) = \theta(\frac{1}{4}) > \left[\theta(\frac{1}{4})\right]^k = [\theta(d(x, y))]^k.$$

Therefore, T *is not a multivalued* θ-*contraction mapping. Then Theorem 3 cannot be applied to this example.*

Now we claim that T *is a generalized multivalued* θ-*contraction with* $\theta(t) = e^{\sqrt{te^t}}$ *and* $k = e^{-\frac{1}{8}}$. *To see* (3.1), *we have to show that*

$$\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx, Ty) - M(x, y)} \leq e^{-\frac{1}{4}}$$

for all $x, y \in X$ *with* $H(Tx, Ty) > 0$. *Note that* $H(Tx, Ty) > 0$ *if and only if* $(x, y) \notin \triangle \cup \{(1, 2), (2, 1)\} \cup (0, 1) \times (0, 1)$, *where* $\triangle = \{(x, x) : x \in X\}$. *Now, without loss of generality, we may assume* $x > y$ *in the following four cases:*

Case 1. For $x, y \in [0, 1]$, *then we have* $y = 0$ *and* $x \neq 0$. *Therefore, since*

$$H(Tx, Ty) = \frac{1}{4} \leq \frac{1}{3}\frac{3}{4} \leq \frac{1}{3}d(0, T0) = \frac{1}{3}d(y, Ty) \leq \frac{1}{3}M(x, y)$$

we obtain

$$\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx, Ty) - M(x, y)} \leq \frac{\frac{1}{3}M(x, y)}{M(x, y)} e^{-\frac{2}{3}M(Tx, Ty)} \leq \frac{1}{3} < e^{-\frac{1}{4}}.$$

Case 2. For $y \in 0$ and $x \in \{2, 3, ...\}$, since $H(Tx, Ty) = x - \frac{1}{4}$ and $d(x, y) = x$, we have

$$
\begin{aligned}
\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx, Ty) - M(x, y)} &\leq \frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx, Ty) - d(x, y)} \\
&\leq \frac{x - \frac{1}{4}}{x} e^{-\frac{1}{4}} \leq e^{-\frac{1}{4}}.
\end{aligned}
$$

Case 3. For $y \in (0, 1]$ and $x \in \{2, 3, ...\}$, since $H(Tx, Ty) = x$ and $D(x, Tx) = x + 1$, we have

$$
\begin{aligned}
\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx, Ty) - M(x, y)} &\leq \frac{H(Tx, Ty)}{d(x, Tx)} e^{H(Tx, Ty) - d(x, Tx)} \\
&\leq \frac{x}{x + 1} e^{-1} \leq e^{-1} < e^{-\frac{1}{4}}.
\end{aligned}
$$

Case 4. For $x, y \in \{2, 3, ...\}$, since $H(Tx, Ty) = x + y - 2$ and $d(x, y) = x + y$, we have

$$
\begin{aligned}
\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx, Ty) - M(x, y)} &\leq \frac{H(Tx, Ty)}{d(x, y)} e^{H(Tx, Ty) - d(x, y)} \\
&\leq \frac{x + y - 2}{x + y} e^{-2} < e^{-2} < e^{-\frac{1}{4}}.
\end{aligned}
$$

This shows that T is a generalized multivalued θ-contraction. Thus, all conditions of Theorem 4 are satisfied and so T has a fixed point.

The following result is relevant in mapping $T : X \to \mathcal{K}(X)$. Here we can remove the condition (θ_4) on the function θ.

Theorem 5 *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a generalized multivalued θ-contraction. If T is upper semicontinuous or θ is continuous, then T has a fixed point.*

Proof   As in the proof of Theorem 4, we get

$$
\begin{aligned}
\theta(D(x_1, Tx_1)) &\leq \theta(H(Tx_0, Tx_1)) \\
&\leq [\theta(M(x_0, x_1))]^k \\
&\vdots \\
&\leq [\theta(d(x_1, x_0))]^k. \tag{3.9}
\end{aligned}
$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = D(x_1, Tx_1)$. From (3.9),

$$
\theta(d(x_1, x_2)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_1, x_0))]^k.
$$

By induction, we obtain a sequence $\{x_n\}$ in X with the property that $x_{n+1} \in Tx_n$, and

$$
\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_n, x_{n-1}))]^k,
$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 4.   □

From Theorem 4, we obtain the following corollaries.

Corollary 1 *Let* (X, d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be given a mapping that satisfies*

$$\theta(H(Tx, Ty)) \le \left[\theta(ad(x, y) + bD(x, Tx) + cD(y, Ty), e\left[D(x, Ty) + D(y, Tx)\right])\right]^k, \qquad (3.10)$$

for all $x, y \in X$ *with* $H(Tx, Ty) > 0$, *where* $k \in (0, 1)$, $a, b, c \ge 0$, *and* $a + b + c + 2e < 1$. *If* T *is upper semicontinuous or* θ *is continuous, then* T *has a fixed point.*

Proof For all $x, y \in X$ with $H(Tx, Ty) > 0$, we have

$$ad(x, y) + bD(x, Tx) + cD(y, Ty), e\left[D(x, Ty) + D(y, Tx)\right]$$

$$\le \quad (a + b + c + 2e) \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}\left[D(x, Ty) + D(y, Tx)\right]\right\}$$

$$\le \quad M(x, y).$$

Then by (θ_1) we see that (3.1) is a consequence of (3.10). This completes the proof. $\qquad \square$

Remark 1 *Since every multivalued* θ*-contraction is multivalued nonexpansive and every multivalued nonexpansive mapping is upper semicontinuous, then* T *is upper semicontinuous. Since (3.10) is a consequence of (2.1), we get Theorem 3 from Corollary 1.*

Corollary 2 *Let* (X, d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be given a mapping that satisfies*

$$\frac{H(Tx, Ty)(H(Tx, Ty) + 1)}{M(x, y)(M(x, y) + 1)} \le k^2 < 1,$$

for all $x, y \in X$ *with* $H(Tx, Ty) > 0$, *where* $k \in [0, 1)$. *Then* T *has a fixed point.*

Proof By taking $\theta(t) = e^{\sqrt{t^2 + t}} \in \Theta$, we obtain the corollary from Theorem 4. $\qquad \square$

Corollary 3 *Let* (X, d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be given a mapping that satisfies*

$$H(Tx, Ty) \le ad(x, y) + bD(x, Tx) + cD(y, Ty)$$

for all $x, y \in X$, *where* $a, b, c \ge 0$ *and* $a + b + c < 1$. *Then* T *has a fixed point.*

Proof If $\theta(t) = e^{\sqrt{t}}$ and $k = \sqrt{a + b + c}$, since $H(Tx, Ty) \le (a + b + c)M(x, y)$, from Theorem 4, then the corollary is proved. $\qquad \square$

Corollary 4 *Let* (X, d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be given a mapping that satisfies*

$$H(Tx, Ty) \le \lambda \max\left\{D(x, Tx), D(y, Ty)\right\}$$

for all $x, y \in X$, *where* $\lambda \in [0, 1)$. *Then* T *has a fixed point.*

Proof If $\theta(t) = e^{\sqrt{t}}$ and $k = \sqrt{\lambda}$, since $H(Tx, Ty) \le \lambda M(x, y)$, from Theorem 4, then the corollary is proved. $\qquad \square$

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