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When zero-divisor graphs are divisor graphs

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Abstract: Let R be a finite commutative principal ideal ring with unity. In this article, we prove that the zero-divisor graph $\Gamma(R)$ is a divisor graph if and only if R is a local ring or it is a product of two local rings with at least one of them having diameter less than 2. We also prove that $\Gamma(R)$ is a divisor graph if and only if $\Gamma(R[x])$ is a divisor graph if and only if $\Gamma(R[[x]])$ is a divisor graph.

Key words: Principal ideal ring, zero-divisor graph, divisor graph, polynomial ring, power series ring

1. Introduction

In this article, all rings are assumed to be finite commutative principal ideal rings with identity (abbreviated FCPI) and all graphs are assumed to be simple.

Let S be a nonempty set of positive integers and let G_S be the graph whose vertices are the elements of S. Two distinct vertices a, b are adjacent if $a|b$ or $b|a$. A graph G is called a divisor graph if there is a set of positive integers S such that $G \cong G_S$. For $S = \{1, 2, ..., n\}$, the length of a longest path in G_S is studied in [13, 17, 18]. In a directed graph G, a vertex is called a receiver if its out-degree is zero and its in-degree is positive. A transmitter is a vertex having positive out-degree and zero in-degree. A vertex t with positive in-degree and positive out-degree is transitive if whenever $u \to t$ and $t \to v$ are edges in G, then $u \to v$ is an edge in G. In [12], divisor graphs are investigated. Some results are listed below:

(1) No divisor graph contains an induced odd cycle of length 5 or more (Proposition 2.1).

(2) An induced subgraph of a divisor graph is a divisor graph (Proposition 2.2).

(3) Complete graphs and bipartite graphs are divisor graphs (Proposition 2.5 and Theorem 2.7).

(4) A graph G is a divisor graph if and only if there is an orientation D of G in which every vertex is a transmitter, a receiver, or transitive (Theorem 3.1).

Divisor graphs are also studied in [1, 2].

Another concept of interest in recent years is the concept of a zero-divisor graph, which was introduced by Beck in [11] and then studied by Anderson and Naseer in [3] in the context of coloring. The definition of

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zero-divisor graphs in its present form was given by Anderson and Livingston in [7, Theorem 2.3]. For other types of graphs associated with rings, see [4–6, 9, 10].

A zero-divisor graph of a commutative ring R is the graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R, with r, s adjacent if $r \neq s$ and $rs = 0$. In [7], Anderson and Livingston proved that the graph $\Gamma(R)$ is connected with diameter at most 3.

For each ring R, let $Z(R)$ be the set of all zero-divisors of R and $Reg(R) = R \setminus Z(R)$.

The following examples show that the concepts of divisor graphs and zero-divisor graphs are incomparable.

Example 1.1 *Every graph of order 4 is a divisor graph (see Theorem 3.2 in [15]). There is a graph of order 4 that is not isomorphic to any zero-divisor graph (see Example 2.1 (b) in [7]).*

Example 1.1 shows the existence of a divisor graph that is not a zero-divisor graph. This is on one hand. On the other hand, if we let R_j be a commutative ring with unity 1, and let $S = \prod_{j=1}^{5} R_j$, then $\Gamma(S)$ contains an induced cycle of length 5 as shown in Figure 1. Thus, $\Gamma(S)$ can not be a divisor graph.

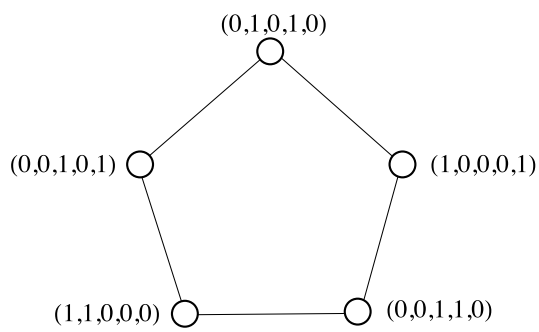

Figure 1. A five-cycle in $\Gamma(\prod_{j=1}^{5} R_j)$.

These examples motivate a question, which we answer affirmatively in this paper: can we characterize zero-divisor graphs that are divisor graphs?

2. When is $\Gamma(R)$ a divisor graph?

In our investigation, we start with the local rings case.

Theorem 2.1 *Let R be a local ring. Then $\Gamma(R)$ is a divisor graph.*

Proof Let M be the maximal ideal of R, and let $a \in R$ such that $M = aR$. Since R is finite, there exists $n \in \mathbb{N}$ such that $a^n = 0$ and $a^{n-1} \neq 0$. Any vertex in $\Gamma(R)$ has the form ua^i , where u is a unit in R and $i \in \{1, ..., n-1\}$. Two vertices ua^j and va^i are adjacent if $i + j \geq n$. Let $U = \{u_1, u_2, ..., u_m\}$ be the units of R. Then $M = \{u_i a^j : i \leq m,\ j \leq n\}$. With repeated elements deleted, we can construct an orientation for $\Gamma(R)$ for any adjacent vertices $u_{i_1} a^{j_1}$ and $u_{i_2} a^{j_2}$ as follows:

if $j_1 < j_2$, then the orientation is $u_{i_1} a^{j_1} \to u_{i_2} a^{j_2}$;

if $j_1 = j_2$ and $i_1 < i_2$, then the orientation is $u_{i_1}a^{j_1} \to u_{i_2}a^{j_2}$.

One can check easily that this is an orientation of $\Gamma(R)$ in which every vertex is transmitter, receiver, or transitive. Hence, using [12, Theorem 3.1], $\Gamma(R)$ is a divisor graph. In fact, in this orientation, for all $n_j < \lfloor \frac{n}{2} \rfloor$, $u_i a^{n_j}$ is transmitter and for all $n_j > \lceil \frac{n}{2} \rceil$ and $n_j \neq n-1$, $u_i a^{n_j}$ is transitive and $u_m a^{n-1}$ is the only receiver. When n is even, all vertices $u_i a^{\frac{n}{2}}$ are transitive except $u_1 a^{\frac{n}{2}}$, which is a transmitter. □

Before we proceed to the nonlocal rings case, we need the following lemma.

Lemma 2.2 *If S is a subring of R and $\Gamma(R)$ is a divisor graph, then so is $\Gamma(S)$ when it is nonempty.*

Proof The graph $\Gamma(S)$ is an induced subgraph of $\Gamma(R)$. Thus, it is a divisor graph when $\Gamma(R)$ is a divisor graph. □

Remark 2.3 *By the above lemma, if R is a ring, S is a direct summand of R that is not a field, and $\Gamma(S)$ is not a divisor graph, then $\Gamma(R)$ is not a divisor graph.*

Let us start the nonlocal rings case treatment. If R is a nonlocal ring, then R is a direct product of local rings (see [8, Theorem 8.7]). Clearly, if R is a product of two integral domains, then $\Gamma(R)$ is a complete bipartite graph, and so it is a divisor graph. We may assume first that one factor of R is a not an integral domain. By the above remark, the zero-divisor graph of this factor ring must be a divisor graph if $\Gamma(R)$ is a divisor graph

We begin with the following lemma.

Lemma 2.4 *A graph that contains the following induced subgraph (Figure 2) is not a divisor graph.*

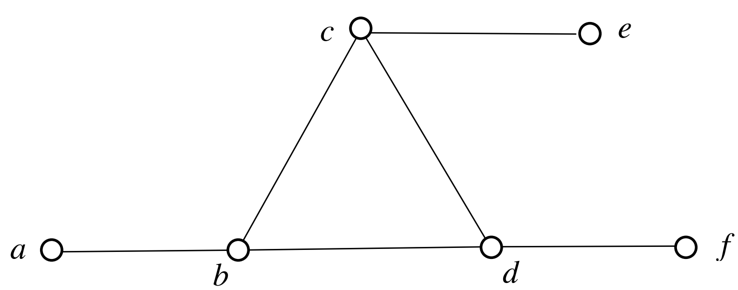

Figure 2.

Proof Let D be an orientation of the above graph in which every vertex is a receiver, a transmitter, or transitive. We may assume without loss of generality that $e \to c$ in D. Thus, we will have the following digraph (Figure 3):

To complete the orientation we must have either $b \to a$ or $a \to b$. Both cases are impossible because in either case the vertex b is neither a receiver nor a transmitter nor transitive. □

Using Lemma 2.4, we deduce the following Theorem.

Theorem 2.5 *If a ring R is a product of 3 nontrivial rings, then $\Gamma(R)$ is not a divisor graph.*

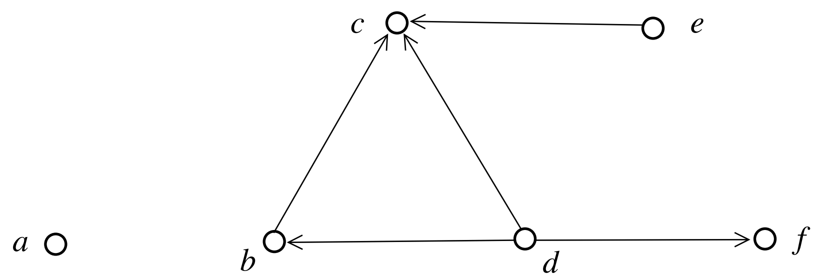

Figure 3. The orientation D.

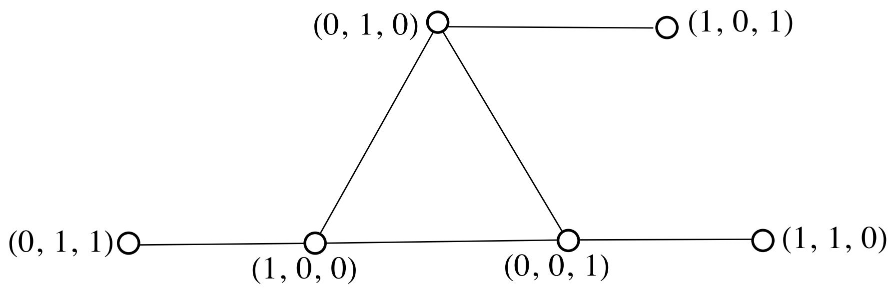

Figure 4.

Proof Assume that $R = R_1 \times R_2 \times R_3$. Then we have an induced subgraph of $\Gamma(R)$ (Figure 4):

By Lemma 2.4, $\Gamma(R)$ is not a divisor graph. □

By Theorems 2.5 and 2.1, we need to consider the case of the product of two local rings only to finish our investigation. Our discussion will be based on the fact that diameters of zero-divisor graphs can not exceed 3 (see [7]). Note that if R is a local FCPI ring, then $diam(\Gamma(R)) \leq 2$; similarly, if $R = R_1 \times R_2$ is a product of two fields, then $diam(\Gamma(R)) = 2$ since the distance between $(a, 0)$ and $(b, 0)$ is 2 for any two distinct nonzero elements a, b in R_1, while if $R = R_1 \times R_2$ and a is a nonzero zero-divisor of R_1, then the distance between $(a, 1)$ and $(1, 0)$ is clearly greater than 2. It is shown in [7] that for a finite ring R, $diam(\Gamma(R)) = 1$ if and only if $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with maximal ideal M such that $M^2 = \{0\}$. Finally, $diam(\Gamma(R)) = 0$ if either R is an integral domain or $R = \mathbb{Z}_4$ or $R = \mathbb{Z}_2[x]/(x^2)$, i.e. $|Z(R)| \leq 2$.

Theorem 2.6 *Let $R = R_1 \times R_2$ such that $diam(\Gamma(R_2)) = 3$. Then $\Gamma(R)$ is not a divisor graph.*

Proof We already assumed that our rings are finite principal ideal rings. If R_2 is local, then $diam(\Gamma(R_2)) \leq 2$. Thus, R_2 cannot be local. Hence, R_2 is a product of two nontrivial rings. Therefore, by Theorem 2.5, $\Gamma(R)$ is not a divisor graph. □

If G is a divisor graph, then there is a one-to-one function $f : V(G) \to \mathbb{N}$ such that v is adjacent to u in G if and only if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$. This function is called a divisor labeling of G; see [12]. We use labeling functions in the proofs of Theorems 2.7, 2.8, and 2.9.

Theorem 2.7 *Let $R = R_1 \times R_2$ such that $diam(\Gamma(R_1)) = diam(\Gamma(R_2)) = 0$. Then $\Gamma(R)$ is a divisor graph.*

Proof We have three cases.

Case 1: $Z(R_1) = \{0, a\}$ and $Z(R_2) = \{0, b\}$.

In this case $Reg(R_1) = \{x_1, x_2\}$ and $Reg(R_2) = \{y_1, y_2\}$; however, we will consider the more general case where $Reg(R_1) = \{x_1, x_2, ..., x_n\}$ and $Reg(R_2) = \{y_1, y_2, ..., y_m\}$ in order to make the treatment of case 1 simpler (being a subgraph of case 2). Figure 5 illustrates the orientation we are going to construct.

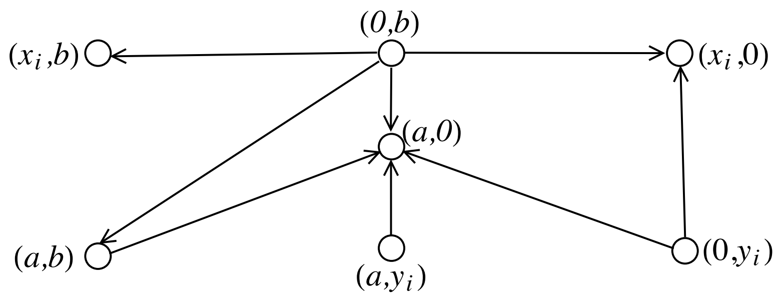

Figure 5.

Let $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n, l_1, l_2, ..., l_m, s_1, s_2, ..., s_m$ be distinct odd primes. Define the function f: $V(\Gamma(R)) \to \mathbb{N}$ such that

$$f(x,y) = \begin{cases} 2 & (x,y) = (0,b) \\ 4 & (x,y) = (a,b) \\ 8 \times \prod_{j=1}^{m} l_j \times \prod_{j=1}^{m} s_j & (x,y) = (a,0) \\ 2p_i & (x,y) = (x_i,b) \\ 2q_i \times \prod_{j=1}^{m} l_j & (x,y) = (x_i,0) \\ l_j & (x,y) = (0,y_j) \\ s_j & (x,y) = (a,y_j). \end{cases}$$

Then f is a one-to-one function such that $(x,y)(\alpha,\beta) = (0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence, $\Gamma(R)$ is a divisor graph.

Case 2: R_1 is an integral domain and $Z(R_2) = \{0, b\}$.

In this case we may view $\Gamma(R)$ as an induced subgraph of the divisor graph in case 1 by deleting $(a,0), (a,b)$, and (a, y_j) for each j. Thus, $\Gamma(R)$ is a divisor graph.

Case 3: R_1 and R_2 are integral domains.

In this case, $\Gamma(R)$ is a complete bipartite graph (see [15, Theorem 3.1]), and so it is a divisor graph. □

Theorem 2.8 *Let $R = R_1 \times R_2$ such that $diam(R_1) = 0$ and $diam(R_2) = 1$. Then $\Gamma(R)$ is a divisor graph.*

Proof We have two cases regarding the ring R_1.

Case 1: $Z(R_1) = \{0, a\}$.

In this case, $Reg(R_1) = \{x_1, x_2\}$ and $Reg(R_2) = \{y_1, y_2\}$; however, we will consider the more general case where $Reg(R_1) = \{x_1, x_2, ..., x_n\}$ and $Reg(R_2) = \{y_1, y_2, ..., y_m\}$ in order to make the treatment simpler. Assume that $Z(R_2) = \{0, z_1, z_2, ..., z_m\}$. Figure 6 illustrates the orientation we are going to construct.

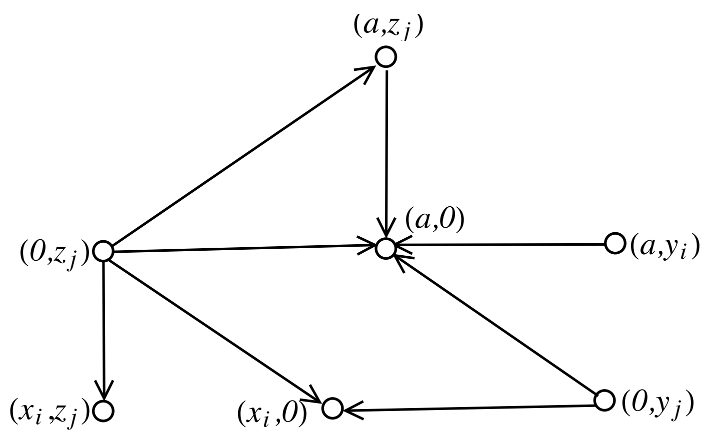

Figure 6.

Let $q_1, q_2, ..., q_k, l_1, l_2, ...l_k, s_1, s_2, ..., s_n, r_{1,1}, r_{1,2}, ...r_{n,m}$ $q_1, q_2, ..., q_k, s_1, s_2, ..., s_n$ be distinct odd primes greater than 3. Define the function $f : V(\Gamma(R)) \to \mathbb{N}$ by

$$f(x,y) = \begin{cases} 2^m \times 3^j & (x,y) = (a, z_j) \\ 2^j & (x,y) = (0, z_j) \\ 2^m \times 3^m \times \prod_{i=1}^{k} l_i \times \prod_{j=1}^{k} q_j & (x,y) = (a, 0) \\ l_j & (x,y) = (a, y_j) \\ 2^m \times r_{i,j} & (x,y) = (x_i, z_j) \\ 2^m \times s_i \times \prod_{j=1}^{k} q_j & (x,y) = (x_i, 0) \\ q_j & (x,y) = (0, y_j). \end{cases}.$$

Then f is a one-to-one function such that $(x,y)(\alpha,\beta) = (0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence, $\Gamma(R)$ is a divisor graph.

Case 2: R_1 is an integral domain.

In this case, we may view $\Gamma(R)$ as an induced subgraph of the divisor graph in case 1 through deleting $(a,0), (a,z_j)$, and (a,y_j) for each j. Thus, $\Gamma(R)$ is a divisor graph. □

Theorem 2.9 *Let $R = R_1 \times R_2$ such that $diam(R_1) = diam(R_2) = 1$. Then $\Gamma(R)$ is a divisor graph.*

Proof Assume that $Z(R_1) = \{0, x_1, x_2, ..., x_n\}$, $Reg(R_1) = \{y_1, y_2, ..., y_m\}$, $Z(R_2) = \{0, z_1, z_2, ..., z_k\}$, and $Reg(R_2) = \{w_1, w_2, ..., w_l\}$. Note that the subgraph with vertices $(Z(R_1) \times Z(R_2)) \setminus \{(0,0)\}$ is complete in $\Gamma(R)$. Let $p_1, p_2, ..., p_m$, $q_1, q_2, ..., q_l$, $r_{1,1}, r_{1,2}, ..., r_{n,l}$, $s_{1,1}, s_{1,2}, ..., s_{m,k}$ be distinct odd primes greater than 5 and $N_{1,1} < N_{1,2} < ... < N_{n,k}$ be an ascending chain of positive distinct integers. Figure 7 is an orientation that makes $\Gamma(R)$ a divisor graph.

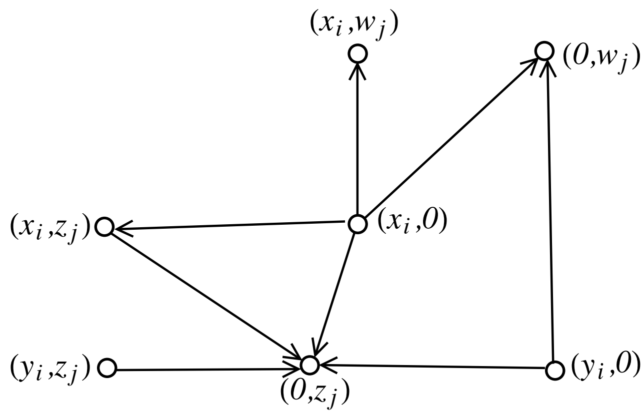

Figure 7.

Now define the function $f : V(\Gamma(R)) \to \mathbb{N}$ such that

$$f(x,y) = \begin{cases} 2^i & (x,y) = (x_i,0) \\ 2^n \times s_{i,j} & (x,y) = (x_i,w_j) \\ p_i & (x,y) = (y_i,0) \\ r_{i,j} & (x,y) = (y_i,z_j) \\ 2^n \times q_j \times \prod p_i & (x,y) = (0,w_j) \\ 2^n \times 3^{N_{i,j}} & (x,y) = (x_i,z_j) \\ 2^n \times 3^{N_{n,k}} \times 5^j \times \prod p_i \times \prod r_{i,j} & (x,y) = (0,z_j) \end{cases}.$$

Then f is a one-to-one function such that $(x,y)(a,b) = (0,0)$ if and only if $f(x,y)$ divides $f(a,b)$ or $f(a,b)$ divides $f(x,y)$. Hence, $\Gamma(R)$ is a divisor graph. □

Theorem 2.10 *Let $R = R_1 \times R_2$ such that $diam(R_1) = 0$ and $diam(R_2) = 2$. Then $\Gamma(R)$ is a divisor graph.*

Proof Since R_2 is a finite local ring, $Z(R_2)$ is generated by a nilpotent, say $Z(R_2) = xR_2$ with $x^l = 0$ but $x^{l-1} \neq 0$. We have two cases:

Case 1: $Z(R_1) = \{0, a\}$.

Let $Reg(R_1) = \{u_1, u_2, ..., u_n\}$, $Reg(R_2) = \{v_1, v_2, ..., v_m\}$, and $Z(R_2) = \{0, v_1x, v_2x, ..., v_mx^{l-1}\}$. If $(v_ix^j)(v_sx^r) = 0$, then $j + r \geq l$. If $j > r$, then we set $(0, v_ix^j) \to (0, v_sx^r)$ and $(a, v_sx^r) \to (a, v_ix^j)$. If $j = r$, then we set $(0, v_ix^j) \to (0, v_sx^r)$ and $(a, v_sx^r) \to (a, v_ix^j)$ if $i > s$. Now we have the following: the sets $\{(0, v_i) : i \in \{1, ..., m\}\}$, $\{(a, v_j) : j \in \{1, ..., m\}\}$, $\{(u_i, 0) : i \in \{1, ..., n\}\}$, and $\{(u_i, v_jx^r) : i \in \{i, ..., n\}, j \in \{1, ..., m\}\}$ form discrete subgraphs. The vertices $(0, v_j), (a, v_j)$ are transmitters for each j. The vertices $(a, 0), (u_i, 0), (u_i, v_sx^r)$ are receivers for each i, s, r. Now assume that we have a path

$(0, v_1x^{r_1}) \to (0, v_2x^{r_2}) \to (\alpha, v_3x^{r_3})$, where $\alpha \in \{0, a, u_1, u_2, ..., u_n\}$. Then $r_1 + r_2 \geq l$ and $r_2 + r_3 \geq l$ and since $r_1 \geq r_2$, we must have $r_1 + r_3 \geq l$. Thus, by our construction we must have $(0, v_1x^{r_1}) \to (\alpha, v_3x^{r_3})$, and so the vertex $(0, v_2x^{r_2})$ is transitive. Similarly, if we have $(\alpha, v_1x^{r_1}) \to (a, v_2x^{r_2}) \to (a, v_3x^{r_3})$, where $\alpha \in \{0, a\}$, then $r_1 + r_2 \geq l$ and $r_2 + r_3 \geq l$, and since $r_2 \leq r_3$, we must have $r_1 + r_3 \geq l$. By our construction we must therefore have $(\alpha, v_1x^{r_1}) \to (a, v_3x^{r_3})$. Thus, $(a, v_2x^{r_2})$ is a transitive vertex. Hence, $\Gamma(R)$ is a divisor graph.

Case 2: R_1 is an integral domain.

In this case we may view $\Gamma(R)$ as an induced subgraph of the graph in Case 1 by deleting the vertices $(a, 0), (a, v_j)$ and (a, v_ix^j) for each i, j . Thus, $\Gamma(R)$ is a divisor graph. □

Theorem 2.11 *Let* $R = R_1 \times R_2$ *such that* $diam(\Gamma(R_1)) = 1$ *and* $diam(\Gamma(R_2)) = 2$. *Then* $\Gamma(R)$ *is a divisor graph.*

Proof Since R_2 is a finite local ring, $Z(R_2)$ is generated by a nilpotent, say $Z(R_2) = xR_2$ with $x^l = 0$ but $x^{l-1} \neq 0$. Let $Z(R_1) = \{0, x_1, x_2, ..., x_k\}$, $Reg(R_1) = \{u_1, u_2, ..., u_n\}$, $Reg(R_2) = \{v_1, v_2, ..., v_m\}$, and $Z(R_2) = \{0, v_1x, v_2x, ..., v_mx^{l-1}\}$. Consider the following orientation: if $(v_ix^j)(v_sx^r) = 0$, then $j + r \geq l$. If $j > r$, then we set $(0, v_ix^j) \to (0, v_sx^r)$ and $(x_\alpha, v_sx^r) \to (x_\beta, v_ix^j)$; if $j = r$, then we set $(0, v_ix^j) \to (0, v_sx^r)$ and $(x_\alpha, v_sx^r) \to (x_\beta, v_ix^j)$ if $i > s$, $(x_{\alpha,0}) \to (x_\beta, 0)$ whenever $\alpha < \beta$; and finally we set $(0, v_sx^r) \to (0, v_ix^j)$ and $(x_\alpha, v_ix^j) \to (x_\beta, v_sx^r)$ if $j = r$ and $i < s$. In this orientation, the vertices $(0, v_j), (x_i, v_j)$ are transmitters for each i, j. The vertices $(u_i, 0)$ and (u_i, v_sx^r) are receivers for each i, s, r and the vertices $(x_i, 0)$ are all transitive except $(x_k, 0)$, which is a receiver. If $(0, v_1x^{r_1}) \to (0, v_2x^{r_2}) \to (\alpha, v_3x^{r_3})$, where $\alpha \in \{0, x_1, ..., x_k, u_1, ..u_n\}$, then $r_1 + r_2 \geq l$ and $r_2 + r_3 \geq l$, and so $r_1 + r_3 \geq l$, since $r_1 \geq r_2$. According to this orientation, $(0, v_1x^{r_1}) \to (\alpha, v_3x^{r_3})$. If $(0, v_1x^{r_1}) \to (0, v_2x^{r_2}) \to (x_i, 0)$ or $(0, v_1x^{r_1}) \to (0, v_2x^{r_2}) \to (u_i, 0)$, then $(0, v_1x^{r_1}) \to (x_i, 0)$ and $(0, v_1x^{r_1}) \to (u_i, 0)$. Thus, $(0, v_2x^{r_2})$ is transitive. If $(\alpha, v_1x^{r_1}) \to (x_i, v_2x^{r_2}) \to (x_j, v_3x^{r_3})$, $\alpha \in \{0, x_1, ..., x_k\}$, then $r_1 + r_2 \geq l$ and $r_2 + r_3 \geq l$, and so $r_1 + r_3 \geq l$, since $r_2 \leq r_3$. According to this orientation, $(\alpha, v_1x^{r_1}) \to (x_j, v_3x^{r_3})$. Finally, if $(\alpha, v_1x^{r_1}) \to (x_i, v_2x^{r_2}) \to (x_j, 0)$, where $\alpha \in \{0, x_1, ..., x_k\}$, then $(\alpha, v_1x^{r_1}) \to (x_j, 0)$. Thus, $(x_i, v_2x^{r_2})$ is transitive. Hence, $\Gamma(R)$ is a divisor graph. □

Theorem 2.12 *Let* $R = R_1 \times R_2$ *such that* $diam(\Gamma(R_1)) = diam(\Gamma(R_2)) = 2$. *Then* $\Gamma(R)$ *is not a divisor graph.*

Proof Since $diam(\Gamma(R_1)) = diam(\Gamma(R_2)) = 2$, there are distinct vertices a, b, c, x, y, and z such that $a - b - c$ is a path in $\Gamma(R_1)$ and $x - y - z$ is a path in $\Gamma(R_2)$ with $ac \neq 0$ and $xz \neq 0$. Then we will have the following induced subgraph (Figure 8):

Thus, by Lemma 2.4, $\Gamma(R)$ is not a divisor graph. □

We collect the above results in the following:

Conclusion 2.13 *Let* R *be a finite commutative principal ideal ring with unity. Then* $\Gamma(R)$ *is a divisor graph if and only if* R *is a local ring or it is a product of two local rings with at least one of them having diameter less than* 2.

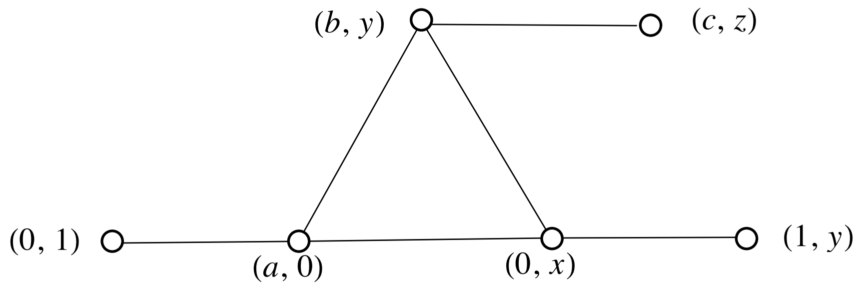

Figure 8.

3. Some extensions

In this section we study when the zero-divisor graphs of some extensions of a ring R are divisor graphs.

It is shown in [16] that if $f(x) \in Z(R[x])$, then there exists a nonzero constant $c \in R \setminus \{0\}$ such that $cf(x) = 0$. A similar result is proved in [14] for $Z(R[[x]])$ when R is a Noetherian ring.

Theorem 3.1 *If R is a finite commutative principal ideal ring with unity, $T = R[x]$ or $R[[x]]$, then for each $f \in T$, there exists $c_f \in R$, $f_1 \in T \setminus Z(T)$ such that $f = c_f f_1$.*

Proof If $f \in Reg(T)$, then $c_f = 1$ and $f = f_1$. We may thus assume that $f \in Z(T)$. As a first step we assume that R is a local ring with maximal ideal $M = aR$. If $f = b_0 + b_1 x + ... \in Z(T)$, then it follows from the results of [14, 16] that $b_i \in Z(R) = M$ for each i, and so $b_i = \beta_i a^{k_i}$, with $k_i \geq 1$ and β_i is a unit in R for each i. Let $k_N = Min\{k_0, k_1, k_2, ...\}$ and let $c_f = a^{k_N}$. Then $f = c_f f_1$, where $f_1 = \beta_0 a^{k_0 - k_N} + \beta_1 a^{k_1 - k_N} x + ...$ Therefore, $f_1 \in T \setminus Z(T)$ since the coefficient of x^N is β_N, which is a unit in R. Now assume that $R = \prod_{j=1}^{m} R_j$ with R_j being a local ring for each j, and let $f = b_0 + b_1 x + ... \in Z(T)$. Then it follows from the results of [14, 16] that $b_i \in Z(R)$ for each i. Moreover, $b_i = (\beta_{i,1} a_1^{k_{i,1}}, \beta_{i,2} a_2^{k_{i,2}}, ..., \beta_{i,m} a_m^{k_{i,m}})$, where $\beta_{i,j}$ is a unit in R_j, $a_j R_j = M_j$ is the unique maximal ideal in R_j, and for at least one j, we have $k_{i,j} \geq 1$ since $b_i \in Z(R)$. Let $a = (\alpha_1, \alpha_2, ...) \in R \setminus \{0\}$ such that $af = 0$. It is clear that for each i, if $k_{i,j} = 0$ for some j, then $\alpha_j = 0$. Since $a \neq 0$, it follows that there exists at least one j_0 such that $\alpha_{j_0} \neq 0$, and so $k_{i,j_0} \geq 1$ for each i. Let $k_{N,j} = Min\{k_{0,j}, k_{1,j}, k_{2,j}, ...\}$ for each j, and let $c_f = (a_1^{k_{N,1}}, a_2^{k_{N,2}}, ..., a_m^{k_{N,m}})$. Then $c_f \in Z(R)$, since $k_{N,j_0} \geq 1$. Let $f_1 = d_0 + d_1 x + d_2 x^2 + ...$, where $d_i = (\beta_{i,1} a_1^{k_{i,1} - k_{N,1}}, \beta_{i,2} a_2^{k_{i,2} - k_{N,2}}, ..., \beta_{i,m} a_m^{k_{i,m} - k_{N,m}})$. If $b = (\gamma_1, \gamma_2, ..., \gamma_m) \in R$ with $bf_1 = 0$, then $\gamma_j = 0$ for each j, since $0 = \gamma_j \beta_{N,j} a_j^{k_{N,j} - k_{N,j}} = \gamma_j \beta_{N,j}$, and $\beta_{N,j}$ is a unit in R_j. Hence $f_1 \in T \setminus Z(T)$ and $f = c_f f_1$. □

Definition 3.2 *Let R be a finite principal ideal ring with unity and let T denote $R[x]$ or $R[[x]]$ and $f \in Z(T)$. The constant c_f (mentioned in Theorem 3.1) is called an annihilating content of f.*

Corollary 3.3 *If R is a finite commutative principal ideal ring with unity, $T = R[x]$ or $R[[x]]$, then for each $f, g \in Z(T)$, $fg = 0$ if and only if $c_f c_g = 0$.*

Theorem 3.4 *Let R be a finite commutative principal ideal ring with unity, $T = R[x]$ or $R[[x]]$. Then $\Gamma(R)$ is a divisor graph if and only if $\Gamma(T)$ is.*

Proof Assume that $\Gamma(R)$ is a divisor graph with orientation D. If $f(x), g(x) \in T$, with $f(x)g(x) = 0$, then $c_f c_g = 0$. The following defines an orientation L on $\Gamma(T)$:

if $c_f \rightarrow c_g$ in D, then let $f(x) \rightarrow g(x)$ in L;

if $c_g \rightarrow c_f$ in D, then let $g(x) \rightarrow f(x)$ in L.

Let us see why this is actually an orientation. Let $f(x) = c_f f_1$ be a vertex in $\Gamma(T)$. If c_f is transmitter (receiver) in D, then $f(x)$ is clearly a transmitter (receiver) in L. Assume that c_f is transitive in D and assume that $h(x) \rightarrow f(x)$, $f(x) \rightarrow g(x)$ in L. Then $h(x)f(x) = f(x)g(x) = 0$. This implies that $c_h c_f = c_f c_g = 0$ and so $c_h \rightarrow c_f, c_f \rightarrow c_g$ in D. Hence, $c_h \rightarrow c_g$ in D since c_f is transitive in D. Thus, $c_h c_g = 0$. Hence, $h(x)g(x) = 0$ and $h(x) \rightarrow g(x)$ in L, and so $f(x)$ is transitive in L. Therefore, $\Gamma(T)$ is a divisor graph.

The converse follows immediately from Lemma 2.2. □

4. Conclusion and questions

In this article we proved that if R is a finite commutative principal ideal ring with unity, then $\Gamma(R)$ is a divisor graph if and only if R is local, R is a product of two integral domains, R is a product of two local rings such that at least one of them is not an integral domain, or R is a product of two local rings such that at least one of them has diameter less than 2, equals $\mathbb{Z}_4$, equals $Z[x]/(x^2)$, or its unique maximal ideal M satisfies $M^2 = \{0\}$. One may ask the following questions:

(1) Can we generalize the results of this article to any finite ring? What about when the ring is Noetherian or Artinian?

(2) When is the complement zero-divisor graph $\overline{\Gamma(R)}$ a divisor graph? What about the line graph $L(\Gamma(R))$ and its complement graph $\overline{L(\Gamma(R))}$?

(3) What divisor graphs can be realized as zero-divisor graphs?

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