Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 825 – 828

doi:10.3906/mat-1605-127

Product of arbitrary Fibonacci numbers with distance 1 to Fibonomial coefficient

Nurettin IRMAK*
Department of Mathematics, Faculty of Arts and Sciences, Ömer Halisdemir University, Niğde, Turkey

Received: 30.05.2016 • **Accepted/Published Online:** 21.09.2016 • **Final Version:** 25.07.2017

Abstract: In this paper, we solve completely the Diophantine equation

$$F_{n_1}F_{n_2}\dots F_{n_k} \pm 1 = \begin{bmatrix} m \\ t \end{bmatrix}_F \tag{1}$$

for $t = 1$ and $t = 2$ where $2 < n_1 < n_2 < \dots < n_k$ positive integers and $\begin{bmatrix} m \\ t \end{bmatrix}_F$ is the Fibonomial coefficient.

Key words: Diophantine equation, Fibonomial coefficient, Fibonacci number

1. Introduction

Let F_n denote the n^{th} term of the Fibonacci sequence. The first few terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13, \dots$. For $1 \leq k \leq m$, the Fibonomial coefficient $\begin{bmatrix} m \\ k \end{bmatrix}_F$ is defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} F_{m-2} \dots F_{m-k+1}}{F_1 F_2 \dots F_k}, \tag{2}$$

always taking the integer values.

In 1876, Brocard and independently Ramanujan posed the problem of finding all integral solutions of the Diophantine equation

$$n! + 1 = m^2,$$

which then became known as the Brocard–Ramanujan equation.

Recently, the Fibonacci version of the Brocard–Ramanujan equation has been studied by several authors. Marques [5] investigated the solutions of the Fibonacci version of the Brocard–Ramanujan Diophantine equation and showed that the Diophantine equation

$$F_n F_{n+1} \dots F_{n+k-1} + 1 = F_m^2 \tag{3}$$

has no solution in positive integers m and n. Although the idea of the proof is sufficient and correct, the solutions $F_4 + 1 = F_3^2$ and $F_6 + 1 = F_4^2$ are not been observed, as noted in [7]. Marques [4] then generalized equation (3) one step more and showed that the equation

$$F_n F_{n-1} \dots F_1 + 1 = F_m^t \tag{4}$$

*Correspondence: nirmak@nigde.edu.tr
2010 *AMS Mathematics Subject Classification:* 11D02; 11B39.

has at most finitely many solutions in positive integers n, m, where t is previously fixed. Moreover, it was proven that there is no solution of equation (4) in the same paper for $1 \leq t \leq 10$. Afterwards, Szalay [7] generalized the Diophantine equation (3) as

$$G_{n_1} G_{n_2} \ldots G_{n_k} + 1 = G_m^2,$$

where the binary recurrence $\{G_n\}$ is the Fibonacci sequence, the Lucas sequence, and the sequence of balancing numbers, respectively. In [6], Marques focused on the following Diophantine equation:

$$\begin{bmatrix} m \\ k \end{bmatrix}_F \pm 1 = F_n, \tag{5}$$

and he proved that there is no solution of equation (5) without $(m, k, n) = (3, 2, 4)$ and $(m, k, n) = (3, 2, 1)$, $(3, 2, 2)$ according to sign $+$ and $-$, respectively. Very recently, the author of this paper proved that the solutions of the equation

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F \pm 1 = F_n$$

are $(m, n) = (1, 3)$, $(3, 14)$ according to the sign $-$. If the sign is $+$, then there is no solution (see [3]).

In this paper, we will handle the following Diophantine equation:

$$F_{n_1} F_{n_2} \ldots F_{n_k} \pm 1 = \begin{bmatrix} m \\ t \end{bmatrix}_F, \tag{6}$$

for $t = 1$ and $t = 2$, respectively.

2. Auxiliary results

1. The sequence of the Lucas numbers is given by following recurrence:

$$L_{n+2} = L_{n+1} + L_n,$$

   with initial conditions $L_0 = 2$ and $L_1 = 1$ for $n \geq 2$.

2. $F_{2n} = L_n F_n$.

3. (**Binet formulas**) $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

4. $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ where $\alpha = \frac{1+\sqrt{5}}{2}$.

5. (**Primitive Divisor Theorem**) A primitive divisor p of F_n is a prime factor of F_n that does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor p of F_n exists whenever $n \geq 13$ (for more details, see [1, 2]). Shortly, we label this theorem as PDT.

6. The factorization of $F_n \pm 1$ depends on the class of n modul 4, namely the identities for the case sign $+$

$$\begin{aligned} F_{4l}+1 &= F_{2l-1}L_{2l+1} \\ F_{4l+1}+1 &= F_{2l+1}L_{2l} \\ F_{4l+2}+1 &= F_{2l+2}L_{2l} \\ F_{4l+3}+1 &= F_{2l+1}L_{2l+2} \end{aligned}$$

hold. Similarly, the identities

$$\begin{aligned} F_{4l}-1 &= F_{2l+1}L_{2l-1} \\ F_{4l+1}-1 &= F_{2l}L_{2l+1} \\ F_{4l+2}-1 &= F_{2l}L_{2l+2} \\ F_{4l+3}-1 &= F_{2l+2}L_{2l+1} \end{aligned}$$

hold for the case $-$. The above identities can be proven by using Binet formulas for Fibonacci and Lucas numbers.

7. For $n \in \mathbb{Z}^+$, then

$$F_nF_{n-1} - F_{n+1}F_{n-2} = (-1)^n$$

holds.

3. The case $t = 1$

Theorem 1 *The solutions of the Diophantine equation*

$$F_{n_1}F_{n_2}\dots F_{n_k} \pm 1 = F_m$$

in positive integers k, m *and* $2 < n_1 < n_2 < \dots < n_k$ *are* $F_3+1 = F_4$, $F_4-1 = F_3$, $F_3-1 = F_2$, *and* $F_3F_4-1 = F_5$

Proof We focus on the Diophantine equation (6) with the case $+$. For the case $-$, we can follow in a similar way. Suppose that $F_{4l}-1 = F_{2l+1}L_{2l-1}$ holds. Equation (6) turns into

$$F_{n_1}F_{n_2}\dots F_{n_k} = F_{2l+1}L_{2l-1}.$$

By identity 2, we have that

$$F_{n_1}F_{n_2}\dots F_{n_k}F_{2l-1} = F_{2l+1}F_{4l-2}.$$

In the sequel, assume that $l \geq 6$. By the PDT, there exists prime divisors p of F_{2l+1} and F_{4l-1} that do not divide $\prod_{j=1}^{2l} F_j$ and $\prod_{j=1}^{4l-2} F_j$, respectively, since $4l-1 \geq 2l+1 \geq 13$. Together with identity 5, the equations $4l-1 = n_k$ and $2n+1 = n_{k-1}$ must hold by equation $F_{n_1}F_{n_2}\dots F_{n_k}F_{2l-1} = F_{2l+1}F_{4l-2}$. We get $F_{n_1}F_{n_2}\dots F_{n_{k-2}}F_{2l-1} = 1$, which is not possible since $F_{2l-1} \geq 1$. The solutions for the case $l \leq 5$ are given in Theorem 1. The remaining cases can be proven in a similar way. □

4. The case $t = 2$

Theorem 2 *The Diophantine equation*

$$F_{n_1}F_{n_2}\dots F_{n_k} + (-1)^n = \begin{bmatrix} n \\ 2 \end{bmatrix}_F \tag{7}$$

has an infinite family of solutions given by

$$F_{n_1}F_{n_2} = F_{n+1}F_{n-2}, n \geq 2.$$

Proof Assume that $n \geq 15$. Let $k = 1$ in (7). By identity 7, since we have $F_{n_1} = F_{n+1}F_{n-2}$, then the equation $n_1 = n + 1$ must hold by the PDT. Then $F_{n-2} = 1$ contradicts the fact $n \geq 15$. If $k = 2$, then $F_{n_1}F_{n_2} = F_{n+1}F_{n-2}$ gives that $n_2 = n - 2$ and $n_1 = n + 1$. If $k \geq 3$, then $F_{n_1}F_{n_2}\dots F_{n_k} = F_{n+1}F_{n-2}$ yields that $F_{n_1}F_{n_2}\dots F_{n_{k-2}} = 1$ since $n_k = n + 1$ and $n_{k-1} = n - 2$ by the PDT. However, this is not possible since $n_1 > 2$. Checking the eligible possibilities for the case $n \leq 14$, there is no solution of the Diophantine equation (7). □

References

[1] Bilu Y, Hanrot G, Voutier PM. Existence of primitive divisors of Lucas and Lehmer numbers. J Reine Angew Math 2011; 539: 75-122.

[2] Carmichael RD. On the numerical factors of the arithmetics forms $\alpha^n \pm \beta^n$. Ann Math 1913; 2: 30-70.

[3] Irmak N. Sum of the Fibonomial coefficients at most one away from Fibonacci numbers. Mathematical Reports 2016; 18: 567-571.

[4] Marques D. The Fibonacci version of the Brocard-Ramanujan Diophantine equation. Far East J Math Sci 2011; 56: 219-224.

[5] Marques D. The Fibonacci version of the Brocard-Ramanujan Diophantine equation. Portugal Math 2011; 68: 185-189.

[6] Marques D. Fibonomial coefficients at most one away from Fibonacci numbers. Demonstratio Math 2012; 45: 25-28.

[7] Szalay L. Diophantine equations with binary recurrences associated to Brocard-Ramanujan problem. Portugal Math 2012; 69: 213-220.