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Generalized $*$-Lie ideal of $*$-prime ring

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Abstract: Let R be a $*$-prime ring with characteristic not 2, $\sigma,\tau : R \to R$ be two automorphisms, U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$, and a,b be in R. (i) If $a \in S_*(R)$ and $[U,a] = 0$, then $a \in Z(R)$ or $U \subset Z(R)$. (ii) If $a \in S_*(R)$ and $[U,a]_{\sigma,\tau} \subset C_{\sigma,\tau}$, then $a \in Z(R)$ or $U \subset Z(R)$. (iii) If $U \not\subset Z(R)$ and $U \not\subset C_{\sigma,\tau}$, then there exists a nonzero $*$-ideal M of R such that $[R,M]_{\sigma,\tau} \subset U$ but $[R,M]_{\sigma,\tau} \not\subset C_{\sigma,\tau}$. (iv) Let $U \not\subset Z(R)$ and $U \not\subset C_{\sigma,\tau}$. If $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.

Key words: $*$-prime ring, $*$-(σ,τ)-Lie ideal, (σ,τ)-derivation, derivation

1. Introduction

Let R be an associative ring with the center $Z(R)$. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. An involution $*$ of a ring R is an additive mapping satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x,\ y \in R$. A ring R equipped with an involution $*$ is said to be $*$-prime if $aRb = a^*Rb = 0$ or $aRb = aRb^* = 0$ implies that $a = 0$ or $b = 0$. R is said to be 2-torsion-free if whenever $2x = 0$ with $x \in R$ then $x = 0$. $S_*(R)$ will denote the set of symmetric and skew symmetric elements of R, i.e. $S_*(R) = \{x \in R \mid x^* = \pm x\}$. An ideal I of R is said to be a $*$-ideal if I is invariant under $*$, i.e. $I^* = I$. As usual the commutator $xy - yx$ will be denoted by $[x,y]$. An additive mapping $h : R \to R$ is called a derivation if $h(xy) = h(x)y + xh(y)$ holds for all $x,\ y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a,x]$ for $x \in R$ is a derivation, which is said to be an inner derivation determined by a. Let $\sigma,\ \tau$ be two mappings on R. Set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(r) = \tau(r)c \text{ for all } r \in R\}$ and it is known as the (σ,τ)-center of R. In particular, $C_{1,1} = Z(R)$ is the center of R where $1 : R \to R$ is the identity map. As usual, the (σ,τ)-commutator $x\sigma(y) - \tau(y)x$ will be denoted by $[x,y]_{\sigma,\tau}$. An additive mapping $d : R \to R$ is called an (σ,τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x,\ y \in R$. For a fixed $a \in R$, the mapping $d_a : R \to R$ is given by $d_a(x) = [a,x]_{\sigma,\tau}$ for $x \in R$ is called a (σ,τ)-inner derivation determined by a. The definition of a (σ,τ)-Lie ideal was given in [4] as follows: let U be an additive subgroup of R. Then: (i) U is a (σ,τ)-right Lie ideal of R if $[U,R]_{\sigma,\tau} \subset U$; (ii) U is a (σ,τ)-left Lie ideal of R if $[R,U]_{\sigma,\tau} \subset U$; (iii) if U is both a (σ,τ)-right Lie ideal and a (σ,τ)-left Lie ideal of R then U is a (σ,τ)-Lie ideal of R. A (σ,τ)-Lie ideal of R is said to be a $*$-(σ,τ)-Lie ideal if U is invariant under $*$, i.e. $U^* = U$. Every $*$-Lie ideal of R is a $*$-$(1,1)$-Lie ideal of R where $1 : R \to R$ is the identity map but every $*$-(σ,τ)-Lie ideal of R is in general not a $*$-Lie ideal of R.

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Example. *As an example, set* $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$. *We define a map* $* : R \to R$ *as follows:* $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^* = \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}$. *Let* $\sigma \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $\tau \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}$ *be two endomorphisms of* R*. It is easy to check that* $U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}$ *is a* $*$-(σ, τ)*-Lie ideal of* R *but not a* $*$*-Lie ideal of* R*.*

In [3], Bergen et al. proved the following results for a nonzero Lie ideal U such that $U \not\subset Z(R)$ of a prime ring R with characteristic not 2: (i) there exists a nonzero ideal M of R such that $[M, R] \subset U$ but $[M, R] \not\subset Z(R)$; (ii) if $a, b \in R$ such that $aUb = 0$, then $a = 0$ or $b = 0$. In [2], Aydın and Kandamar generalized these results for a nonzero (σ, τ)-Lie ideal that is not included in $Z(R)$ and $C_{\sigma,\tau}$ of a prime ring. Oukhtite and Salhi [7] generalized these results, which were proved in [3] for a nonzero $*$-Lie ideal U such that $[U, U] \neq 0$ of a 2-torsion-free $*$-prime ring R. In this paper our main goal will be to extend the above results to a nonzero $*$-(σ, τ)-Lie ideal that is not included in $Z(R)$ and $C_{\sigma,\tau}$ of a $*$-prime ring with characteristic not 2.

Throughout the present paper R will be a $*$-prime ring, $Z(R)$ will be the center of R, $\sigma, \tau : R \to R$ will be two automorphisms, $C_{\sigma,\tau}$ will be the (σ, τ)-center of R, and $S_*(R)$ will be the set of symmetric and skew symmetric elements of R.

We will use the following basic commutator identities:

- $[x, yz] = y[x, z] + [x, y]z$,
- $[xy, z] = x[y, z] + [x, z]y$,
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$,
- $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau} y$,
- $[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau} \sigma(z)$,
- $\left[[x, y]_{\sigma,\tau}, z\right]_{\sigma,\tau} = \left[[x, z]_{\sigma,\tau}, y\right]_{\sigma,\tau} + [x, [y, z]]_{\sigma,\tau}$.

2. Results

For the proof of our results we need the following lemmas.

Lemma 2.1. *[1, Lemma 5] Let* R *be a ring and* U *be a nonzero* (σ, τ)*-left Lie ideal of* R *and* $T = \left\{ c \in R \mid [R, c]_{\sigma,\tau} \subset U \right\}$*. Then the following hold:*

i) T *is a subring of* R*.*

ii) If U *is also a* (σ, τ)*-right Lie ideal of* R *then* T *is the largest Lie ideal of* R *such that* $[R, T]_{\sigma,\tau} \subset U$ *and* $U \subset T$*.*

Lemma 2.2. *[2, Lemma 4] Let R be a ring and U be a nonzero (σ,τ)-left Lie ideal of R. Then*

$$R\left[T(U),\sigma(T(U))\right]\subset T(U) \text{ and } \left[T(U),\tau(T(U))\right]R\subset T(U).$$

Lemma 2.3. *[5, Lemma 4] Let R be a σ-prime ring with characteristic not 2, d be a derivation of R satisfying $d\sigma=\pm\sigma d$, and I be a nonzero σ-ideal of R. If $d^2(I)=0$ then $d=0$.*

Lemma 2.4. *[6, Theorem 2.2] Let I be a nonzero σ-ideal of a σ-prime ring R. If a,b in R are such that $aIb=aI\sigma(b)=0$ then $a=0$ or $b=0$.*

Lemma 2.5. *[8, 2.3. Lemma] Let I be a nonzero σ-ideal of a σ-prime ring R and $a\in R$. If $Ia=0$ (or $aI=0$) then $a=0$.*

Lemma 2.6. *[8, 2.8. Theorem] Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, and d be a nonzero (α,β)-derivation of R such that β commutes with σ. If $d(I)\subset C_{\alpha,\beta}$ then R is commutative.*

Lemma 2.7. *[8, 2.9. Lemma] Let R be a σ-prime ring with characteristic not 2, I be a nonzero σ-ideal of R, d be a (α,β)-derivation of R such that β commutes with σ, and h be a derivation of R satisfying $h\sigma=\pm\sigma h$. If $dh(I)=0$ and $h(I)\subset I$ then $d=0$ or $h=0$.*

Lemma 2.8. *Let U be a nonzero $*$-(σ,τ)-left Lie ideal of R such that τ commutes with $*$. If $U\subset C_{\sigma,\tau}$ then $U\subset Z(R)$.*

Proof For any $u\in U$, $r\in R$, we have $[r\sigma(u),u]_{\sigma,\tau}=[r,u]_{\sigma,\tau}\sigma(u)\in U$. From the hypothesis, it holds that $\left[[r,u]_{\sigma,\tau}\sigma(u),s\right]_{\sigma,\tau}=0$ for all $u\in U$, $r,s\in R$. Expanding this equation and using the hypothesis

$$\begin{aligned}0&=\left[[r,u]_{\sigma,\tau}\sigma(u),s\right]_{\sigma,\tau}=[r,u]_{\sigma,\tau}\sigma([u,s])+\left[[r,u]_{\sigma,\tau},s\right]_{\sigma,\tau}\sigma(u)\\&=[r,u]_{\sigma,\tau}\sigma([u,s]),\end{aligned}$$

so it implies that

$$[r,u]_{\sigma,\tau}\sigma([u,s])=0,\ \forall r,s\in R,\ u\in U.$$

In this equation, taking $\tau(r)k$ instead of r where $k\in R$, it follows that

$$\tau([r,u])R\sigma([u,s])=0,\ \forall r,s\in R,\ u\in U. \tag{2.1}$$

Assume that $u\in U\cap S_*(R)$. In (2.1), replacing r by r^* and using that $\tau*=*\tau$, we get

$$\tau^*([r,u])R\sigma([u,s])=0,\ \forall r,s\in R.$$

Thus,

$$\tau([r,u])R\sigma([u,s])=\tau^*([r,u])R\sigma([u,s])=0,\ \forall r,s\in R$$

is obtained. By the $*$-primeness of R, we have

$$u\in Z(R),\ \forall u\in U\cap S_*(R).$$

Thus, it holds that

$$U \cap S_{*}(R) \subset Z(R). \tag{2.2}$$

Now, for all $u \in U$, we know that $u - u^{*} \in U \cap S_{*}(R)$. From (2.2), it implies $u - u^{*} \in Z(R)$. This means that $[u, r] = [u^{*}, r]$ for all $r \in R$. In (2.1), taking r^{*} instead of r, and using that $\tau * = * \tau$ and $[u, r] = [u^{*}, r]$ for all $r \in R$, we get

$$\tau^{*}([r, u]) R \sigma([u, s]) = 0, \ \forall r, s \in R, \ u \in U.$$

By the $*$-primeness of R, we have

$$u \in Z(R), \ \forall u \in U,$$

which implies that

$$U \subset Z(R).$$

Lemma 2.9. *Let U be a nonzero $*$-(σ, τ)-left Lie ideal of R such that τ commutes with $*$ and $a \in R$. If $Ua = 0$, then $a = 0$ or $U \subset Z(R)$.*

Proof Since U is a $*$-(σ,τ)-left Lie ideal of R, we know that $[r, u]_{\sigma,\tau} a = 0$ for all $r \in R, u \in U$. Replacing r by rs where $s \in R$ in the last equality, we get $[rs, u]_{\sigma,\tau} a = 0$ for all $r, s \in R, u \in U$. Expanding this equation,

$$0 = [rs, u]_{\sigma,\tau} a = r [s, u]_{\sigma,\tau} a + [r, \tau(u)] sa,$$

and using the hypothesis, we have

$$[r, \tau(u)] Ra = 0, \ \forall r \in R, u \in U.$$

In the last equation, taking r^{*}, u^{*} instead of r, u respectively and using that τ commutes with $*$, we get

$$([r, \tau(u)])^{*} Ra = 0, \ \forall r \in R, u \in U.$$

Thus,

$$[r, \tau(u)] Ra = ([r, \tau(u)])^{*} Ra = 0, \ \forall r \in R, u \in U$$

is obtained. From the $*$-primeness of R, it yields

$$a = 0 \text{ or } [R, \tau(U)] = 0.$$

Since τ is an automorphism, we arrive at $a = 0$ or $U \subset Z(R)$.

Lemma 2.10. *Let U be a nonzero $*$-(σ, τ)-left Lie ideal of R such that τ commutes with $*$. If $a \in R$ and $[U, a] = 0$, then $[\sigma(U), a] = 0$.*

Proof Since U is a $*$-(σ,τ)-left Lie ideal of R, we know that $[r\sigma(u), u]_{\sigma,\tau} = [r, u]_{\sigma,\tau} \sigma(u) \in U$ for all $u \in U$, $r \in R$. From the hypothesis, it holds that

$$\left[[r, u]_{\sigma,\tau} \sigma(u), a\right] = 0.$$

Expanding this equation,

$$0 = [r, u]_{\sigma,\tau} [\sigma(u), a] + \left[[r, u]_{\sigma,\tau}, a\right] \sigma(u),$$

and using the hypothesis,

$$[r,u]_{\sigma,\tau}\left[\sigma\left(u\right),a\right]=0,\ \forall u\in U,r\in R$$

is obtained. In the last equation, replacing r by $\tau\left(s\right)r$ where $s\in R,$ it implies

$$\tau\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=0,\ \forall u\in U,s\in R. \tag{2.3}$$

Suppose that $u\in U\cap S_{*}\left(R\right).$ Taking s^{*} instead of s in (2.3) and using that τ commutes with $*$, it follows that

$$\tau^{*}\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=0,\ \forall s\in R,u\in U\cap S_{*}\left(R\right).$$

Thus,

$$\tau\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=\tau^{*}\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=0,\ \forall s\in R,u\in U\cap S_{*}\left(R\right).$$

Since R is a $*$-prime ring and τ is an automorphism, we obtain $u\in Z\left(R\right)$ or $\left[\sigma\left(u\right),a\right]=0$ for all $u\in U\cap S_{*}\left(R\right).$ That is,

$$\left[\sigma\left(u\right),a\right]=0,\ \forall u\in U\cap S_{*}\left(R\right). \tag{2.4}$$

Now, suppose that $u\in U.$ In this case, we know $u-u^{*}\in U\cap S_{*}\left(R\right).$ It follows that $\left[\sigma\left(u-u^{*}\right),a\right]=0$ from (2.4). Thus, we have

$$\left[\sigma\left(u\right),a\right]=\left[\sigma\left(u^{*}\right),a\right],\ \forall u\in U. \tag{2.5}$$

In (2.3), replacing s,u by s^{*},u^{*}, respectively, and using that τ commutes with $*$ and (2.5), we get

$$\tau^{*}\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=0,\ \forall u\in U,s\in R.$$

Thus,

$$\tau\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=\tau^{*}\left([s,u]\right)R\left[\sigma\left(u\right),a\right]=0,\ \forall u\in U,s\in R.$$

Since R is a $*$-prime ring and τ is an automorphism, it implies that $u\in Z\left(R\right)$ or $\left[\sigma\left(u\right),a\right]=0$ for all $u\in U.$ This means

$$\left[\sigma\left(U\right),a\right]=0.$$

Theorem 2.11. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $a\in S_{*}\left(R\right)$ and $[U,a]=0$, then $a\in Z\left(R\right)$ or $U\subset Z\left(R\right).$*

Proof Let

$$T\left(U\right)=\left\{c\in R\mid [R,c]_{\sigma,\tau}\subset U\right\}.$$

Since U is a (σ,τ)-Lie ideal of $R,$ from Lemma 2.1, $T\left(U\right)$ is a Lie ideal of R such that $U\subset T\left(U\right).$ Hence, it follows that

$$[R,U]\subset [R,T\left(U\right)]\subset T\left(U\right).$$

From the definition of $T\left(U\right),$ we have

$$[R,[R,U]]_{\sigma,\tau}\subset [R,T\left(U\right)]_{\sigma,\tau}\subset U.$$

From the hypothesis, we have $\left[[R,[R,U]]_{\sigma,\tau},a\right]=0.$ For any $r,s\in R$ and $u\in U,$ it holds that

$$\left[[r,[s,u]]_{\sigma,\tau},a\right]=0. \tag{2.6}$$

Replacing r by ra in (2.6) and expanding by using (2.6),

$$\begin{aligned}0&=\left[[ra,[s,u]]_{\sigma,\tau},a\right]=[r[a,\sigma([s,u])],a]+\left[[r,[s,u]]_{\sigma,\tau}a,a\right]\\&=r[[a,\sigma([s,u])],a]+[r,a][a,\sigma([s,u])]+[r,[s,u]]_{\sigma,\tau}[a,a]\\&\quad+\left[[r,[s,u]]_{\sigma,\tau},a\right]a\\&=r[[a,\sigma([s,u])],a]+[r,a][a,\sigma([s,u])]\end{aligned}$$

is obtained. It follows that

$$r[[a,\sigma([s,u])],a]+[r,a][a,\sigma([s,u])]=0,\ \forall r,s\in R,u\in U. \tag{2.7}$$

In (2.7), taking rm instead of r where $m\in R$ and expanding by using (2.7),

$$\begin{aligned}0&=rm[[a,\sigma([s,u])],a]+[rm,a][a,\sigma([s,u])]\\&=rm[[a,\sigma([s,u])],a]+r[m,a][a,\sigma([s,u])]+[r,a]m[a,\sigma([s,u])]\\&=r(m[[a,\sigma([s,u])],a]+[m,a][a,\sigma([s,u])])+[r,a]m[a,\sigma([s,u])]\\&=[r,a]m[a,\sigma([s,u])]\end{aligned}$$

is obtained. This implies that

$$[r,a]R[a,\sigma([s,u])]=0,\ \forall r,s\in R,u\in U.$$

Replacing r by r^* and using $a\in S_*(R)$ in the last equation, we have

$$[r,a]^*R[a,\sigma([s,u])]=0,\ \forall r,s\in R,u\in U.$$

It holds that

$$[r,a]R[a,\sigma([s,u])]=[r,a]^*R[a,\sigma([s,u])]=0,\ \forall r,s\in R,u\in U.$$

By the $*$-primeness of R, we get

$$a\in Z(R)\ \text{ or }\ [a,\sigma([s,u])]=0,\ \forall s\in R,u\in U.$$

That is,

$$[\sigma([s,u]),a]=0,\ \forall s\in R,u\in U.$$

Since σ is an automorphism, it implies

$$[[R,\sigma(u)],a]=0,\ \forall u\in U.$$

Thus, we have $[[r,\sigma(u)],a]=0$ for all $r\in R,u\in U$. Using the identity $[[x,y],z]=[[x,z],y]+[x,[y,z]]$ for all $x,y,z\in R$,

$$0=[[r,\sigma(u)],a]=[[r,a],\sigma(u)]+[r,[\sigma(u),a]]$$

is obtained. From Lemma 2.10, we know that $[\sigma(U),a]=0$. It holds that

$$[[r,a],\sigma(u)]=0,\ \forall r\in R,u\in U. \tag{2.8}$$

From (2.8), it implies

$$\left(I_{\sigma(u)}I_a\right)(R) = 0,\ \forall u \in U.$$

Since $a \in S_*(R)$, we know that $I_a * = \pm * I_a$. Hence, according to Lemma 2.7, we have

$$a \in Z(R) \text{ or } \sigma(u) \in Z(R),\ \forall u \in U,$$

which implies that

$$a \in Z(R) \text{ or } U \subset Z(R).$$

Corollary 2.12. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $U \subset S_*(R)$ and $[U,U] = 0$, then $U \subset Z(R)$.*

Theorem 2.13. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $a \in S_*(R)$ and $[U,a]_{\sigma,\tau} = 0$, then $a \in Z(R)$ or $U \subset Z(R)$.*

Proof Since U is a $*$-(σ,τ)-Lie ideal of R, we have $[r\sigma(u), u]_{\sigma,\tau} = [r,u]_{\sigma,\tau}\sigma(u) \in U$ for all $r \in R, u \in U$. From the hypothesis, we have $\left[[r,u]_{\sigma,\tau}\sigma(u), a\right]_{\sigma,\tau} = 0$ for all $r \in R, u \in U$. Expanding this equation by using the hypothesis,

$$\begin{aligned} 0 = \left[[r,u]_{\sigma,\tau}\sigma(u), a\right]_{\sigma,\tau} &= \left[[r,u]_{\sigma,\tau}, a\right]_{\sigma,\tau}\sigma(u) + [r,u]_{\sigma,\tau}\sigma([u,a]) \\ &= [r,u]_{\sigma,\tau}\sigma([u,a]) \end{aligned}$$

is obtained. Thus, we have

$$[r,u]_{\sigma,\tau}\sigma([u,a]) = 0,\ \forall u \in U, r \in R.$$

In the last equality, replacing r by $\tau(s)r$ where $s \in R$, it implies that

$$\tau([s,u])R\sigma([u,a]) = 0,\ \forall u \in U, s \in R. \tag{2.9}$$

Suppose that $u \in U \cap S_*(R)$. In (2.9), replacing s by s^* and using that τ commutes with $*$, we have

$$\tau^*([s,u])R\sigma([u,a]) = 0,\ \forall s \in R, u \in U \cap S_*(R).$$

It follows that

$$\tau([s,u])R\sigma([u,a]) = \tau^*([s,u])R\sigma([u,a]) = 0,\ \forall s \in R, u \in U \cap S_*(R).$$

Since R is a $*$-prime ring and σ,τ are automorphisms, we get $u \in Z(R)$ or $[u,a] = 0$ for all $u \in U \cap S_*(R)$. Hence, we get

$$[u,a] = 0,\ \forall u \in U \cap S_*(R).$$

Now, suppose that $u \in U$. In this case, we know that $u - u^* \in U \cap S_*(R)$. It follows that $[u - u^*, a] = 0$ from the above equation. Thus, we have

$$[u,a] = [u^*,a],\ \forall u \in U. \tag{2.10}$$

In (2.9), replacing s,u by s^*,u^*, respectively, and using that τ commutes with $*$ and (2.10), we get

$$\tau^*([s,u])R\sigma([u,a]) = 0,\ \forall u \in U, s \in R.$$

Thus,

$$\tau^*([s,u])\,R\sigma([u,a]) = \tau([s,u])\,R\sigma([u,a]) = 0,\ \forall u \in U, s \in R$$

is obtained. Since R is a $*$-prime ring and σ,τ are automorphisms, it implies that

$$u \in Z(R) \text{ or } [u,a] = 0,\ \forall u \in U.$$

That is,

$$[U,a] = 0.$$

Therefore, according to Theorem 2.11,

$$a \in Z(R) \text{ or } U \subset Z(R)$$

is obtained.

Corollary 2.14. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $U \subset S_*(R)$ and $[U,U]_{\sigma,\tau} = 0$, then $U \subset Z(R)$.*

Theorem 2.15. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $a \in S_*(R)$ and $[U,a]_{\sigma,\tau} \subset C_{\sigma,\tau}$, then $a \in Z(R)$ or $U \subset Z(R)$.*

Proof From the hypothesis, we know that $\left[[u,a]_{\sigma,\tau}, r\right]_{\sigma,\tau} = 0$ for all $u \in U, r \in R$. Using the identity $\left[[x,y]_{\sigma,\tau}, z\right]_{\sigma,\tau} = \left[[x,z]_{\sigma,\tau}, y\right]_{\sigma,\tau} + [x,[y,z]]_{\sigma,\tau}$ for all $x,y,z \in R$, for any $r \in R$ and $u \in U$

$$0 = \left[[u,a]_{\sigma,\tau}, r\right]_{\sigma,\tau} = \left[[u,r]_{\sigma,\tau}, a\right]_{\sigma,\tau} + [u,[a,r]]_{\sigma,\tau}$$

is obtained. Since $\left[[u,r]_{\sigma,\tau}, a\right]_{\sigma,\tau} \in C_{\sigma,\tau}$, from the last equation, we have

$$[u,[a,r]]_{\sigma,\tau} \in C_{\sigma,\tau},\ \forall u \in U, r \in R.$$

In this case, we get $\left[[u,[a,r]]_{\sigma,\tau}, s\right]_{\sigma,\tau} = 0$ for all $s \in R$. Replacing r by ra in this equality, we have $\left[[u,[a,ra]]_{\sigma,\tau}, s\right]_{\sigma,\tau} = 0$ for all $r,s \in R$ and $u \in U$. Expanding this equation,

$$\begin{aligned} 0 &= \left[[u,[a,ra]]_{\sigma,\tau}, s\right]_{\sigma,\tau} \\ &= \left[\tau([a,r])\,[u,a]_{\sigma,\tau}, s\right]_{\sigma,\tau} + \left[[u,[a,r]]_{\sigma,\tau}\,\sigma(a), s\right]_{\sigma,\tau} \\ &= \tau([a,r])\left[[u,a]_{\sigma,\tau}, s\right]_{\sigma,\tau} + \tau([[a,r],s])\,[u,a]_{\sigma,\tau} + \\ &\left[[u,[a,r]]_{\sigma,\tau}, s\right]_{\sigma,\tau}\sigma(a) + [u,[a,r]]_{\sigma,\tau}\,\sigma([a,s]) \end{aligned}$$

is obtained. Using that $[u,a]_{\sigma,\tau}$, $[u,[a,r]]_{\sigma,\tau} \in C_{\sigma,\tau}$, it holds that

$$\tau([[a,r],s])\,[u,a]_{\sigma,\tau} + [u,[a,r]]_{\sigma,\tau}\,\sigma([a,s]) = 0,\ \forall u \in U,\ r,s \in R.$$

In the last equation, taking a instead of s, we have

$$\tau\left(\left[\left[a,r\right],a\right]\right)\left[u,a\right]_{\sigma,\tau}=0,\ \forall u\in U,\ r\in R.$$

Since $\left[u,a\right]_{\sigma,\tau}\in C_{\sigma,\tau}$, it implies

$$\tau\left(\left[\left[a,r\right],a\right]\right)R\left[u,a\right]_{\sigma,\tau}=0,\ \forall u\in U,\ r\in R.$$

In the above equality, replacing r by r^* and using that $a\in S_*\left(R\right)$ and τ commutes with $*$, we get

$$\left(\tau\left(\left[\left[a,r\right],a\right]\right)\right)^*R\left[u,a\right]_{\sigma,\tau}=0,\ \forall u\in U,\ r\in R.$$

Thus,

$$\tau\left(\left[\left[a,r\right],a\right]\right)R\left[u,a\right]_{\sigma,\tau}=\left(\tau\left(\left[\left[a,r\right],a\right]\right)\right)^*R\left[u,a\right]_{\sigma,\tau}=0,\ \forall u\in U,\ r\in R.$$

Since R is a $*$-prime ring and τ is an automorphism,

$$\left[\left[r,a\right],a\right]=0 \text{ or } \left[u,a\right]_{\sigma,\tau}=0,\ \forall u\in U,\ r\in R \tag{2.11}$$

is obtained. From (2.11), it follows that

$$I_a^2\left(R\right)=0 \text{ or } \left[U,a\right]_{\sigma,\tau}=0.$$

Since $a\in S_*\left(R\right)$, we know that $I_a*=\pm*I_a$. According to Lemma 2.3 and Theorem 2.13,

$$a\in Z\left(R\right) \text{ or } U\subset Z\left(R\right)$$

is obtained.

Corollary 2.16. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $U\subset S_*\left(R\right)$ and $\left[U,U\right]_{\sigma,\tau}\subset C_{\sigma,\tau}$, then $U\subset Z\left(R\right)$.*

Theorem 2.17. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$. If $U\not\subset Z\left(R\right)$ and $U\not\subset C_{\sigma,\tau}$, then there exists a nonzero $*$-ideal M of R such that $\left[R,M\right]_{\sigma,\tau}\subset U$ but $\left[R,M\right]_{\sigma,\tau}\not\subset C_{\sigma,\tau}$.*

Proof It follows from Lemma 2.1 that $T\left(U\right)=\left\{c\in R\mid\left[R,c\right]_{\sigma,\tau}\subset U\right\}$ is a subring and Lie ideal of R. Moreover, $U\subset T\left(U\right)$. On the other hand, it follows from Lemma 2.2 that $\left[T\left(U\right),\tau\left(T\left(U\right)\right)\right]R\subset T\left(U\right)$. Since $U\subset T\left(U\right)$, it implies that

$$\left[U,\tau\left(U\right)\right]R\subset\left[T\left(U\right),\tau\left(T\left(U\right)\right)\right]R\subset T\left(U\right).$$

That is,

$$\left[U,\tau\left(U\right)\right]R\subset T\left(U\right).$$

Since $T\left(U\right)$ is a Lie ideal of R, it holds that $\left[R,\left[U,\tau\left(U\right)\right]R\right]\subset T\left(U\right)$. Thus, for any $s,r\in R,u,v\in U$

$$\left[s,\left[u,\tau\left(v\right)\right]r\right]=\left[u,\tau\left(v\right)\right]\left[s,r\right]+\left[s,\left[u,\tau\left(v\right)\right]\right]r\in T\left(U\right).$$

Using that $[U,\tau(U)]R \subset T(U)$, we have $[s,[u,\tau(v)]]r \in T(U)$ for all $u,\ v \in U,\ r,s \in R$. Expanding this equation,

$$[s,[u,\tau(v)]]r = s[u,\tau(v)]r - [u,\tau(v)]sr \in T(U)$$

and using that $[U,\tau(U)]R \subset T(U)$,

$$s[u,\tau(v)]r \in T(U),\ \forall u,v \in U, r,s \in R$$

is obtained. It holds that

$$R[U,\tau(U)]R \subset T(U). \tag{2.12}$$

Set $M = R[U,\tau(U)]R$. Thus, M is an ideal of R. Now, assume that $M = 0$. It follows that $R[U,\tau(U)]R = 0$. Since R is a $*$-ideal of R, according to Lemma 2.5, it holds that

$$[U,\tau(U)] = 0.$$

On the other hand, we know that $\tau(U) \cap S_*(R) \subset S_*(R)$ and $\tau(U) \cap S_*(R) \subset \tau(U)$. It follows that $[U,\tau(U) \cap S_*(R)] \subset [U,\tau(U)] = 0$. Therefore, it implies

$$[U,\tau(U) \cap S_*(R)] = 0.$$

That is, we have $\tau(U) \cap S_*(R) \subset S_*(R)$ and $[U,\tau(U) \cap S_*(R)] = 0$. Since $U \not\subset Z(R)$, it holds from Theorem 2.11 that

$$\tau(U) \cap S_*(R) \subset Z(R).$$

Now, for any $u \in U$, we get $\tau(u-u^*), \tau(u+u^*) \in \tau(U) \cap S_*(R) \subset Z(R)$. It yields $2\tau(u) \in Z(R)$. Since the characteristic is not 2, we obtain $u \in Z(R)$ for all $u \in U$. This is a contradiction, which implies $M \neq 0$. Moreover, we have

$$M^* = (R[U,\tau(U)]R)^* = R[U,\tau(U)]R = M.$$

Thus, M is a nonzero $*$-ideal of R. It also follows from (2.12) that $0 \neq M \subset T(U)$. From the definition of $T(U)$, we get

$$[R,M]_{\sigma,\tau} \subset [R,T(U)]_{\sigma,\tau} \subset U,$$

which implies that

$$[R,M]_{\sigma,\tau} \subset U.$$

Suppose that $[R,M]_{\sigma,\tau} \subset C_{\sigma,\tau}$. From the assumption, $d_r(M) \subset C_{\sigma,\tau}$ for all $r \in R$. According to Lemma 2.6, R is commutative. This is a contradiction, so we have

$$[R,M]_{\sigma,\tau} \not\subset C_{\sigma,\tau}.$$

Theorem 2.18. *Let R be a $*$-prime ring with characteristic not 2, and let U be a nonzero $*$-(σ,τ)-Lie ideal of R such that τ commutes with $*$ and $a,b \in R$. If $U \not\subset Z(R)$ and $U \not\subset C_{\sigma,\tau}$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Proof Since $U \not\subset Z(R)$ and $U \not\subset C_{\sigma,\tau}$, by Theorem 2.17, there exists a nonzero $*$-ideal M of R such that $[R,M]_{\sigma,\tau} \subset U$ but $[R,M]_{\sigma,\tau} \not\subset C_{\sigma,\tau}$. Since $[R,M]_{\sigma,\tau} \subset U$, using the hypothesis, we have $a[R,M]_{\sigma,\tau} b = a^*[R,M]_{\sigma,\tau} b = 0$. For any $m \in M$ and $u \in U$, it implies $a[a^*u,m]_{\sigma,\tau} b = 0$. Expanding this equation,

$$0 = a[a^*u,m]_{\sigma,\tau} b = aa^*[u,m]_{\sigma,\tau} b + a[a^*,\tau(m)]ub,$$

and using the hypothesis,

$$a[a^*,\tau(m)]ub = 0$$

is obtained. Expanding this by using the hypothesis, it yields

$$\begin{aligned} 0 &= aa^*\tau(m)ub - a\tau(m)a^*ub \\ &= aa^*\tau(m)ub. \end{aligned}$$

That is,

$$aa^*\tau(M)ub = 0, \ \forall u \in U.$$

Since $(aa^*)^* = aa^*$, it also holds that

$$aa^*\tau(M)ub = (aa^*)^*\tau(M)ub = 0, \ \forall u \in U.$$

Since M is a nonzero $*$-ideal of R and τ commutes with $*$, $\tau(M)$ is a nonzero $*$-ideal of R. By Lemma 2.4 and Lemma 2.9, it follows that

$$aa^* = 0 \text{ or } b = 0.$$

Assume that $b \neq 0$. In this case $aa^* = 0$. Since $[R,M]_{\sigma,\tau} \subset U$, we get $a[a^*n,m]_{\sigma,\tau} b = 0$ for all $n,m \in M$. Expanding this equation by using the assumption,

$$\begin{aligned} 0 &= a[a^*n,m]_{\sigma,\tau} b = aa^*n\sigma(m)b - a\tau(m)a^*nb \\ &= a\tau(m)a^*nb \end{aligned}$$

is obtained, so it holds that

$$a\tau(M)a^*Mb = 0.$$

Since $(a\tau(M)a^*)^* = a\tau(M)a^*$, we have

$$a\tau(M)a^*Mb = (a\tau(M)a^*)^*Mb = 0.$$

According to Lemma 2.4, since $b \neq 0$, it follows that

$$a\tau(M)a^* = 0. \tag{2.13}$$

On the other hand, since $[R,M]_{\sigma,\tau} \subset U$, we have $a[au,m]_{\sigma,\tau} b = 0$ for all $u \in U$, $m \in M$. Expanding this equation by using hypothesis, we have

$$\begin{aligned} 0 &= a[au,m]_{\sigma,\tau} b = aa[u,m]_{\sigma,\tau} b + a[a,\tau(m)]ub \\ &= a[a,\tau(m)]ub \\ &= a^2\tau(m)ub - a\tau(m)aub \\ &= a^2\tau(m)ub. \end{aligned}$$

That is,

$$a^2\tau(M)\,ub = 0,\ \forall u \in U.$$

Similarly, $a^*[a^*u,m]_{\sigma,\tau}\,b = 0$ for all $u \in U, m \in M$. Expanding this equation by using the hypothesis, we get

$$\left(a^2\right)^*\tau(M)\,ub = 0,\ \forall u \in U.$$

This implies that

$$a^2\tau(M)\,ub = \left(a^2\right)^*\tau(M)\,ub = 0,\ \forall u \in U.$$

Since $b \neq 0$, by Lemma 2.4 and Lemma 2.9, we get

$$a^2 = 0.$$

Moreover, for any $n,m \in M$, we have $a\,[an,m]_{\sigma,\tau}\,b = 0$ from the hypothesis. Expanding this equation by using that $a^2 = 0$,

$$\begin{aligned} 0 &= a\,[an,m]_{\sigma,\tau}\,b = a^2n\sigma(m)\,b - a\tau(m)\,anb \\ &= a\tau(m)\,anb \end{aligned}$$

is obtained. Thus, we have

$$a\tau(M)\,aMb = 0.$$

Similarly, we know that $a^*[a^*n,m]_{\sigma,\tau}\,b = 0$ for all $n,m \in M$. Using that $a^2 = 0$, we get

$$a^*\tau(M)\,a^*Mb = 0.$$

Since $(a\tau(M)\,a)^* = a^*\tau(M)\,a^*$, it holds that

$$(a\tau(M)\,a)^*\,Mb = 0.$$

Thus,

$$a\tau(M)\,aMb = (a\tau(M)\,a)^*\,Mb = 0$$

is obtained. By Lemma 2.4, we have

$$a\tau(M)\,a = 0. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$a\tau(M)\,a^* = a\tau(M)\,a = 0.$$

According to Lemma 2.4, it is implied that $a = 0$.

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