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Subspace condition for Bernstein's lethargy theorem

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Abstract: In this paper, we consider a condition on subspaces in order to improve bounds given in Bernstein's lethargy theorem for Banach spaces. Let $d_1 \geq d_2 \geq \dots d_n \geq \dots > 0$ be an infinite sequence of numbers converging to 0, and let $Y_1 \subset Y_2 \subset \dots \subset Y_n \subset \dots \subset X$ be a sequence of closed nested subspaces in a Banach space X with the property that $\overline{Y}_n \subset Y_{n+1}$ for all $n \geq 1$. We prove that for any $c \in (0,1]$ there exists an element $x_c \in X$ such that

$$cd_n \leq \rho(x_c, Y_n) \leq \min(4, \tilde{a}) c\, d_n.$$

Here, $\rho(x, Y_n) = \inf\{||x-y||: \ y \in Y_n\}$,

$$\tilde{a} = \sup_{i \geq 1} \sup_{\{q_i\}} \left\{ a_{n_{i+1}-1}^{-3} \right\}$$

where the sequence $\{a_n\}$ is defined as: for all $n \geq 1$,

$$a_n = \inf_{l \geq n} \inf_{q \in \langle q_l, q_{l+1}, \dots \rangle} \frac{\rho(q, Y_l)}{||q||}$$

in which each point q_n is taken from $Y_{n+1} \setminus Y_n$, and satisfies $\inf\limits_{n \geq 1} a_n > 0$. The sequence $\{n_i\}_{i \geq 1}$ is given by

$$n_1 = 1; \ n_{i+1} = \min\left\{ n \geq 1 : \frac{d_n}{a_n^2} \leq d_{n_i} \right\}, \ i \geq 1.$$

Key words: Best approximation, Bernstein's lethargy theorem, Banach spaces

1. Introduction

Bernstein's lethargy theorem (BLT) [7] involves finding approximations of an element in a space X when those approximations are limited to some sequence of subspaces. Before we can compare approximations, we need a function to determine how close an approximation is to the desired target. In the following we define a distance function, which we call the ρ-function:

Definition 1 *Let $(X, \|\cdot\|)$ be a Banach space and let S be a subspace of X. Then, for any point $x \in X$, we can define the distance from x to S as*

$$dist(x, S) = \rho(x, S) = \inf_{y \in S} ||x - y||.$$

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If $Y_1 \subset Y_2 \subset \ldots$ is a sequence of strictly embedded linear subspaces of X, then for each $x \in X$ there exists a nonincreasing sequence of best approximation errors

$$\rho(x, Y_1) \geq \rho(x, Y_2) \geq \ldots.$$

The general objective is to characterize these sequences of best approximation errors. For example, one can ask if it is true that for any nonincreasing sequence $\{d_n\}$ with $\lim_{n\to\infty} d_n = 0$ there exists an element $x \in X$ such that

$$\rho(x, Y_n) = d_n \quad \text{for all} \quad n = 1, 2, \ldots.$$

Bernstein [7] proved that in the case $X = C[a, b]$ and $Y_n = P_n$, the space of polynomials of degree at most n, **any** sequence converging to zero is a sequence of best approximations. This theorem is sometimes referred to as Bernstein's Lethargy Theorem or in short BLT and it has been applied to the theory of quasi analytic functions in several complex variables [13] and used in the constructive theory of functions [16]. Note that the density of polynomials in $C[0, 1]$ (Weierstrass approximation theorem [9]) implies that $\lim_{n\to\infty} \rho(f, P_n) = 0$. However, the Weierstrass approximation theorem gives no information about the speed of convergence for $\rho(f, Y_n)$. Following the proof given by Bernstein, Timan [17] extended his result to an arbitrary system of strictly embedded *finite-dimensional* subspaces Y_n. Later Shapiro [15], replacing $C[0, 1]$ with an arbitrary infinite-dimensional Banach space $(X, \|\cdot\|)$ and the sequence of n-dimensional subspaces of polynomials of degree $\leq n$ by a sequence $\{Y_n\}$ where $Y_1 \subset Y_2 \subset \cdots$ are strictly embedded *closed subspaces* of X, showed that in this setting, for each null sequence $\{d_n\}$ of nonnegative numbers, there is a vector $x \in X$ such that

$$\rho(x, Y_n) \neq O(d_n), \text{ as } n \to \infty.$$

Thus, there is no $M > 0$ such that $\rho(x, Y_n) \leq M d_n$ for all n. In other words, $\rho(x, Y_n)$ can decay arbitrarily slowly. This result was strengthened by Tyuriemskih [18], who established that the sequence of best approximations may converge to zero at an arbitrary slow rate; for any expanding sequence $\{Y_n\}$ of subspaces and for any sequence $\{d_n\}$ of positive numbers converging to zero, he constructed an element $x \in X$ such that $\lim_{n\to\infty} \rho(x, Y_n) = 0$ and $\rho(x, Y_n) \geq d_n$ for all n. For a generalization of Shapiro's theorem we refer the reader to [6]. For an application of Tyuriemskih's theorem to convergence of a sequence of bounded linear operators, consult [10]. For other versions of BLT, see [1–3, 5, 12, 14].

We now consider the following well-known BLT [7], stated for the case of finite-dimensional subspaces of a Banach space X.

Theorem 2 (Lethargy) *Given a Banach space X and a series of nested finite-dimensional subspaces $Y_1 \subset Y_2 \subset \cdots \subset X$. If $\{d_k\}_{k\geq 1}$ is a monotone decreasing sequence converging to 0, then there exists a point $x \in X$ such that $\rho(x, Y_k) = d_k$ for all $k \geq 1$.*

The above theorem can be extended to infinite-dimensional subspaces, by considering some extra conditions. Borodin [8] has provided two sets of conditions. One condition is on the sequence $\{d_n\}$ and the other on both the subspaces $\{Y_n\}$ and the sequence $\{d_n\}$. In both cases, he proves the existence of an element $x \in X$ with $\rho(x, Y_k) = d_k,\ k \geq 1$. These two results are explicitly presented as follows.

Theorem 3 *(see [8]) Let X be an arbitrary infinite-dimensional Banach space, $Y_1 \subset Y_2 \subset \dots$ be an arbitrary system of strictly embedded subspaces in X, and the number sequence $\{d_n\}$ be such that*

$$d_n > \sum_{k=n+1}^{\infty} d_k \tag{1}$$

for every positive integer $n \geq n_0$ for which $d_n > 0$. Then there exists an element $x \in X$ such that $\rho(x, Y_n) = d_n$ for $n \geq 1$.

Theorem 4 *(see [8]) Let $d_0 \geq d_1 \geq d_2 \geq \cdots > 0$ be a nonincreasing sequence converging to 0 and $Y_1 \subset Y_2 \subset \cdots \subset X$ be a system of strictly nested subspaces of an infinite-dimensional Banach space X that meets the following property: there exists a series of nonzero elements q_n such that $q_n \in Y_{n+1} \setminus Y_n$, and the following inequality*

$$\|q\| \leq \frac{d_{k-1}}{d_k} \rho(q, Y_k) \tag{2}$$

holds for all $k \in \mathbb{N}$ and any nonzero element q in the linear span $\langle q_k, q_{k+1}, \dots \rangle$. Then there is some element x in the closed linear span $\overline{\langle q_1, q_2, \dots \rangle}$ satisfying

$$\rho(x, Y_n) = d_n \quad for\ all\ n \geq 1.$$

Recently Konyagin [11] showed that under the same assumptions in Theorem 3, except that the sequence $\{d_n\}$ can go to 0 with arbitrary rate, there is $x \in X$ such that

$$d_n \leq \rho(x, Y_n) \leq 8d_n, \text{ for } n \geq 1. \tag{3}$$

The proof is based on Theorem 3. Note that the statements in Theorem 4 are similar to that of Theorem 3. We can now adapt the idea of the proof of Konyagin's [11] with Borodin's theorem [8] to improve the bounds of $\rho(x, Y_n)$ in (3).

In Konyagin's paper [11], it is assumed that Y_n are closed and strictly increasing. In Borodin's paper, this is not specified, but from the proof of his theorem it is clear that his proof works only under the assumption that $\overline{Y_n}$ is strictly included in Y_{n+1}. The necessity of this assumption on subspaces is illustrated by the following:

Example 5 *Let $X = L^\infty[0,1]$ and consider $C[0,1] \subset L^\infty[0,1]$. Define the subspaces of X as follows:*

- $Y_1 =$ *all polynomials*
- $Y_2 =$ *span*$[Y_1 \cup f_1]$, *where* $f_1 \in C[0,1] \setminus Y_1$,
- $Y_{n+1} =$ *span*$[Y_n \cup f_n]$, *where* $f_n \in C[0,1] \setminus Y_n$.

Observe that by Weierstrass theorem, $\overline{Y_n} = C[0,1]$ for any $n \geq 1$. Take any $f \in L^\infty[0,1]$ and consider the following cases:

1. *If $f \in C[0,1]$, then*

$$dist(f, Y_n) = dist(f, C[0,1]) = 0 \ for\ any\ n \geq 1.$$

2. If $f \in L^\infty[0,1] \setminus C[0,1]$, then

$$dist(f, Y_n) = dist(f, C[0,1]) = d > 0 \quad \textit{(independent of } n\textit{).}$$

Hence in this case BLT does not hold. (Note that in the above, we used the fact that $dist(f,Y) = dist(f,\overline{Y})$.)

Note that Borodin's condition on sequence $\{d_n\}$, namely $d_n > \sum_{k=n+1}^{\infty} d_k$ is not satisfied when $d_n = \frac{1}{2^n}$; however, it is satisfied when $d_n = \frac{1}{(2+\epsilon)^n}$ for $\epsilon > 0$. Thus, it is natural to ask whether the condition (1) is necessary for the results in Theorem 3 to hold?

In [4] , it is shown that weakening the condition (1) in Theorem 3 above yields an improvement in the bounds in the inequality (3) in Konyagin's theorem. In this paper, we take a different approach. We concentrate on Borodin's second condition on subspaces, namely on the inequality (2) of Theorem 4 above and obtain better bounds for the inequalities in (3). The statements in Theorem 4 are similar to that of Theorem 3; thus, we can now adapt the idea of the proof of Konyagin's [11] with Borodin's theorem [8] to improve the bounds of $\rho(x, Y_n)$ in (3).

2. Main result

Let X be an arbitrary infinite-dimensional Banach space. Given $Y_1 \subset Y_2 \subset \cdots \subset X$, an arbitrary system of strictly embedded closed subspaces and $d_1 \geq d_2 \geq \cdots \geq 0$, a nonincreasing sequence converging to 0. The goal of this paper is to improve Konyagin's result (3) under conditions on subspaces $\{Y_n\}$ and $\{d_n\}$. It is worth noting that if $\{Y_n\}$ and $\{d_n\}$ are finite sequences, we have the best approximations of the sequence d_n in terms of the distances $\rho(x, Y_n)$, i.e. the following lemma holds:

Lemma 6 *Let $d_1 > d_2 > \cdots > d_n > 0$ be a finite decreasing sequence and $Y_1 \subset Y_2 \subset \cdots \subset Y_n$ be a system of strictly nested closed subspaces of Banach space X. Then for any $c \in (0,1]$, there exists an element $x_c \in X$ such that $\rho(x_c, Y_k) = cd_k$, for $k = 1, \ldots n$.*

Proof First, from [8] and [17], we see Lemma 6 is true for $c = 1$. Next for any $c \in (0,1]$, let $\tilde{d}_k = cd_k$. It is easy to see the sequence of numbers $\{\tilde{d}_k\}$ satisfies Lemma 6; therefore there exists an element $x_c \in X$ such that $\rho(x, Y_k) = \tilde{d}_k$, for $k = 1, \ldots, n$. □

Now we consider the case when $\{Y_n\}$ and $\{d_n\}$ are infinite sequences and state our main result.

Theorem 7 *Let X be an arbitrary infinite-dimensional Banach space, and let $Y_1 \subset Y_2 \subset \cdots \subset X$ be an arbitrary system of strictly embedded closed linear subspaces. Let $\{d_n\}$ be a nonincreasing sequence of real numbers converging to 0. Assume that, for any sequence of elements q_i such that $q_n \in Y_{n+1} \setminus Y_n$ for all n, we have that*

$$\inf_{n \geq 1} a_n > 0, \tag{4}$$

where for each $n \geq 1$, a_n is defined by

$$a_n = \inf_{l \geq n} \inf_{q \in \langle q_l, q_{l+1}, \ldots \rangle} \frac{\rho(q, Y_l)}{||q||}$$

for these elements q_i. Then for any constant $c \in (0,1]$ there exists an element $x_c \in X$ such that

$$cd_n \le \rho(x_c, Y_n) \le \min(4, \tilde{a}) cd_n, \tag{5}$$

where

$$\tilde{a} = \sup_{i\ge 1} \sup_{\{q_i\}} \left\{ a_{n_{i+1}-1}^{-3} \right\}$$

and n_i satisfies

$$\begin{aligned} n_1 &= 1; \\ n_{i+1} &= \min\left\{ n \ge 1 : \frac{d_n}{a_n^2} \le d_{n_i} \right\}, \ i \ge 1. \end{aligned} \tag{6}$$

Proof If $d_n = 0$ for some n, then Theorem 7 holds by using Lemma 1 in [8] and Lemma 6. Thus, we will assume that $d_n > 0$ for all $n \ge 1$. Take the sequence $\{n_i\}$ defined in (6). Define a sequence of positive integers $\{j_i\}$ such that

$$j_1 = 1, \ j_{i+1} = \begin{cases} j_i + 1 & \text{if } n_{i+1} = n_i + 1; \\ j_i + 2 & \text{if } n_{i+1} > n_i + 1, \end{cases} \quad \text{for } i \ge 1.$$

Let

$$m_j = \begin{cases} n_i & \text{if } j = j_i; \\ n_{i+1} - 1 & \text{if } j_i < j < j_{i+1}. \end{cases}$$

Clearly the sequence $\{m_j\}_{j\ge1}$ is strictly increasing. Now we define the sequences of subspaces $\{Z_j\}_{j\ge1}$ and numbers $\{e_j\}_{j\ge1}$ to be

$$\begin{aligned} Z_j &= Y_{m_j}, \\ e_j &= \begin{cases} \frac{c}{a_{m_{j+1}}} d_{n_i} & \text{if } j = j_i & \text{for some } i; \\ cd_{n_i} & \text{if } j_i < j < j_{i+1} & \text{for some } i. \end{cases} \end{aligned}$$

Hence, for any $j \ge 1$, 3 cases follow:

Case 1:
if $j_i < j < j_{i+1}$ for some i, then $j+1 = j_{i+1}$. By the definition of e_j, the facts that $n_{i+1} = m_{j+1}$ and $\{a_n\}_n$ is increasing, we obtain

$$e_{j+1} = \frac{c}{a_{m_{j+2}}} d_{n_{i+1}} \le \frac{c}{a_{m_{j+2}}} a_{n_{i+1}}^2 d_{n_i} = \frac{c}{a_{m_{j+2}}} a_{m_{j+1}}^2 d_{n_i} \le a_{m_{j+1}} e_j.$$

Case 2:
if $j = j_i$ for some i and $j+1 < j_{i+1}$, then

$$e_{j+1} = cd_{n_i} = a_{m_{j+1}} e_j.$$

Case 3:
if $j = j_i$ for some i and $j+1 = j_{i+1}$, then

$$e_{j+1} = \frac{c}{a_{m_{j+2}}} d_{n_{i+1}} \le \frac{c}{a_{m_{j+2}}} a_{n_{i+1}}^2 d_{n_i} = \frac{a_{m_{j+1}} a_{n_{i+1}}^2}{a_{m_{j+2}}} e_j \le a_{m_{j+1}} e_j.$$

Thus, we conclude that

$$e_{j+1} \leq a_{m_{j+1}} e_j, \text{ for all } j \geq 1.$$

Note that for all $q \in \langle q_{m_{j+1}}, q_{m_{j+1}+1}, \ldots \rangle$,

$$\frac{e_{j+1}}{e_j} \leq a_{m_{j+1}} \leq \frac{\rho(q, Y_{m_{j+1}})}{||q||} = \frac{\rho(q, Z_{j+1})}{||q||}.$$

Therefore, we can apply Theorem 4 to the sequence $\{Z_j\}_{j\geq 1}$ of subspaces and the sequence of numbers $\{e_j\}_{j\geq 1}$, to obtain the existence of an element $x'_c \in \overline{< q_1, q_2, \ldots >}$ such that

$$\rho(x'_c, Z_j) = e_j, \text{ for } j \geq 1.$$

If $n = n_i$ for some i, then for $j = j_i$ we have $n = m_j$, $Y_n = Z_j$ and

$$\rho(x'_c, Y_n) = \rho(x'_c, Z_j) = e_j = \frac{c}{a_{m_{j+1}}} d_n.$$

Now let $n_i < n < n_{i+1}$ for some i and $j = j_i$. Then

$$m_j = n_i < n \leq n_{i+1} - 1 = m_{j+1}. \tag{7}$$

It leads to the lower bound of $\rho(x'_c, Y_n)$ in terms of d_n:

$$\rho(x'_c, Y_n) \geq \rho(x'_c, Z_{j+1}) = e_{j+1} = cd_{n_i} \geq cd_n. \tag{8}$$

To obtain the upper bound of $\rho(x'_c, Y_n)$ we observe from (7) that

$$\rho(x'_c, Y_n) \leq \rho(x'_c, Y_{n_i}) = \rho(x'_c, Z_j) = e_j = \tfrac{c}{a_{m_{j+1}}} d_{n_i}.$$

Since $n_i < n_{i+1} - 1 < n_{i+1}$, then we have

$$a^2_{n_{i+1}-1} d_{n_i} \leq d_{n_{i+1}-1} \leq d_n.$$

Consequently,

$$\rho(x'_c, Y_n) \leq \frac{c}{a_{m_{j+1}}} d_{n_i} \leq \frac{c}{a^2_{n_{i+1}-1} a_{m_{j+1}}} d_n \leq \frac{c}{a^3_{n_{i+1}-1}} d_n. \tag{9}$$

It follows from (8) and (9) that

$$cd_n \leq cd_{n_i} \leq \rho(x'_c, Y_n) \leq \frac{c}{a^3_{n_{i+1}-1}} d_n.$$

Notice that $a_{n_{i+1}-1}$ only depends on the sequences $\{n_i\}$ and $\{q_i\}$. Therefore, by taking supremum over $\{q_i\}$ and $\{n_i\}$ we proved

$$cd_n \leq \rho(x'_c, Y_n) \leq \tilde{a} c d_n. \tag{10}$$

Also note that in [4] it is shown that, for the same sequences $\{d_n\}$ and $\{Y_n\}$ as in Theorem 7, there is another element $x''_c \in X$ such that

$$cd_n \leq \rho(x''_c, Y_n) \leq 4cd_n. \tag{11}$$

Therefore if $\tilde{a} \leq 4$,

$$cd_n \leq \rho(x'_c, Y_n) \leq \tilde{a}cd_n = \min(4, \tilde{a})cd_n;$$

if $\tilde{a} > 4$,

$$cd_n \leq \rho(x''_c, Y_n) \leq 4cd_n = \min(4, \tilde{a})cd_n.$$

Thus by taking

$$x_c = \begin{cases} x'_c & \text{if } \tilde{a} \leq 4; \\ x''_c & \text{if } \tilde{a} > 4, \end{cases}$$

we have proven Theorem 7. □

Remark 8 *One can observe that the inequalities in (5) are stronger than the inequalities given by Konyagin in [11].*

References

[1] Aksoy AG, Almira JM. On Shapiro's lethargy theorem and some applications. Jaen J Approx 2014; 6: 87-116.

[2] Aksoy AG, Lewicki G. Bernstein lethargy theorem in Fréchet spaces. J Approx Theory 2016; 209: 58-77.

[3] Aksoy AG, Lewicki G. Diagonal operators, s-numbers and Bernstein pairs. Note Mat 1999; 17: 209-216.

[4] Aksoy AG, Peng Q. On a theorem of S. N. Bernstein for Banach spaces. ArXiv: 1605.04592.

[5] Almira JM, del Toro N. Some remarks on negative results in approximation theory. Proceedings of the Fourth International Conference on Functional Analysis and Approximation Theory, Vol. I (Potenza 2000) Rend. Circ. Mat. Palermo (2) Suppl., 2002, no. 68, part I, pp. 245-256.

[6] Almira JM, Oikhberg T. Approximation schemes satisfying Shapiro's theorem. J Approx Theory 2012; 5: 534-571.

[7] Bernstein SN. On the inverse problem in the theory of best approximation of continuous functions. Collected works (in Russian), Izd Akad Nauk, USSR 1954; vol II: 292-294.

[8] Borodin PA. On the existence of an element with given deviations from an expanding system of subspaces. Math Notes 2006; 80: 621-630 (Translated from Mathematicheskie Zametki).

[9] Cheney EW. Introduction to Approximation Theory. Providence, RI, USA: AMS Chelsea Publishing, 1998.

[10] Deutsch F, Hundal H. A generalization of Tyuriemskih's lethargy theorem and some applications. Numer Func Anal Opt 2013; 34: 1033-1040.

[11] Konyagin SV. Deviation of elements of a Banach space from a system of subspaces. Proceedings of the Steklov Institute of Mathematics; 2014; 284 (1), pp. 204-207.

[12] Lewicki G. A theorem of Bernstein's type for linear projections. Univ Iagellon Acta Math 1988; 27: 23-27.

[13] Pleśniak W. Quasianalytic functions in the sense of Bernstein. Dissert Math 1977; 147: 1-70.

[14] Plichko A. Rate of decay of the Bernstein numbers. Zh Mat Fiz Anal Geom 2013; 9: 59-72.

[15] Shapiro HS. Some negative theorems of approximation theory. Michigan Math J 1964; 11: 211-217.

[16] Singer I. Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces. Berlin, Germany: Springer-Verlag, 1970.

[17] Timan AF. The Theory of Approximating Functions of Real Variables. Oxford, UK: Macmillan, 1994.

[18] Tyuriemskih IS. On one problem of S. N. Bernstein. Scientific Proceedings of Kaliningrad State Pedagog. Inst.; 1967; 52: 123-129.