Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1184 – 1190
© TÜBİTAK
doi:10.3906/mat-1604-124

New recurrences for Euler's partition function

Mircea MERCA*
Department of Mathematics, University of Craiova, Craiova, Romania

Received: 29.04.2016 • **Accepted/Published Online:** 14.11.2016 • **Final Version:** 28.09.2017

Abstract: In this paper, the author invokes some consequences of the bisectional pentagonal number theorem to derive two linear recurrence relations for Euler's partition function $p(n)$. As a corollary of these results, we obtain an efficient method to compute the parity of Euler's partition function $p(n)$ that requires only the parity of $p(k)$ with $k \leqslant n/4$.

Key words: Partition function, pentagonal number theorem, recurrence relation

1. Introduction

In order to state our results, we recall some basic definitions and results in partition theory. More details and proofs can be found in Andrews's book [1].

An integer partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n, i.e.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

The number of such partitions of n is usually denoted $p(n)$. For example, the partitions of 4 are:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad \text{and} \quad 1+1+1+1.$$

Therefore, $p(4) = 5$.

One way of studying the so-called partition function $p(n)$ is to study its generating function, i.e.

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{for } n > 0 \end{cases}$$

is the q-shifted factorial and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. One of the well-known results in the theory of partitions is the following pentagonal number theorem.

*Correspondence: mircea.merca@profinfo.edu.ro

2010 *AMS Mathematics Subject Classification:* 11P81, 05A17.

Theorem 1.1 (Euler's pentagonal number theorem) *For* $|q|<1$,

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{G_k}=(q;q)_\infty, \tag{1}$$

where the exponents G_k *are called the generalized pentagonal numbers, i.e.*

$$G_k=\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left(3\left\lceil\frac{k}{2}\right\rceil+(-1)^k\right).$$

Euler used this identity to derive the well-known recurrence for the partition function $p(n)$:

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}p\left(n-G_k\right)=\delta_{0,n}, \tag{2}$$

where $\delta_{i,j}$ is Kronecker's delta, and $p(0)=1$ and $p(n)=0$ for any negative integer n.

Recently, Merca [4] considered a bisection of Euler's pentagonal number series (1) and obtained the following result.

Theorem 1.2 (The bisectional pentagonal number theorem) *For* $|q|<1$,

$$\sum_{k=0}^{\infty}\frac{1+(-1)^{G_k}}{2}(-1)^{\lceil k/2\rceil}q^{G_k}=(q^2,q^{12},q^{14},q^{16},q^{18},q^{20},q^{30},q^{32};q^{32})_\infty,$$

where

$$(a_1,a_2\ldots,a_n;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty.$$

For any positive integer n, Theorem 1.2 has the following combinatorial interpretation [4, Corollary 1.1]:

$$\sum_{k=0}^{\infty}\frac{1+(-1)^{G_k}}{2}(-1)^{\lceil k/2\rceil}p\left(n-G_k\right)=L(n),$$

where $L(n)$ is the number of partitions of n into parts not congruent to 0, 2, 12, 14, 16, 18, 20, or 30 mod 32.

In this paper, motivated by these results, we provide two linear recurrence relations for the partition function $p(n)$. As far as we know, these identities are new. As a corollary of these results, we obtain an efficient method to compute the parity of Euler's partition function $p(n)$ that requires only the parity of $p(k)$ with $k\leqslant n/4$.

2. Main results

In terms of the generalized pentagonal numbers, the first result can be written as follows.

Theorem 2.1 *For* $n\geqslant 0$,

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}p\left(n-\frac{G_k}{2}\right)-\sum_{k=0}^{\infty}p\left(\frac{n}{2}-\frac{k(k+1)}{8}\right)=0,$$

with $p(x)=0$ *when* x *is not a nonnegative integer.*

Proof First, we denote by $q_o(n)$ the number of partitions of n into distinct odd parts. It is well known that the partition function $q_o(n)$ has the following generating function:

$$\sum_{n=0}^{\infty} q_o(n)q^n = (-q;q^2)_\infty.$$

The Jacobi triple product identity [2, Theorem 11] can be expressed in terms of the Ramanujan theta function as follows:

$$(-q;qx)_\infty(-x;qx)_\infty(qx;qx)_\infty = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}x^{n(n-1)/2}, \qquad |qx|<1.$$

Replacing x by 1 in this relation, and after some rearranging, we obtain

$$(-q;q)_\infty = \frac{(-q^2;q^2)_\infty}{(q^4;q^4)_\infty}\sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q|<1.$$

Taking into account that

$$(-q;q)_\infty = (-q;q^2)_\infty(-q^2;q^2)_\infty,$$

we deduce the following factorization for the generating function of $q_o(n)$:

$$(-q;q^2)_\infty = \frac{1}{(q^4;q^4)_\infty}\sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q|<1,$$

or

$$\sum_{n=0}^{\infty} q_o(n)q^n = \left(\sum_{n=0}^{\infty} p(n)q^{4n}\right)\left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right), \qquad |q|<1.$$

Applying the Cauchy multiplication in this relation, we obtain the following expression for $q_o(n)$ in terms of $p(n)$, namely

$$q_o(n) = \sum_{k=0}^{\infty} p\left(\frac{n}{4} - \frac{k(k+1)}{8}\right), \tag{3}$$

where $p(x)=0$ if x is not a nonnegative integer.

On the other hand, by [4, Corollaries 4.2], we see that

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil} p\left(n-\frac{G_k}{2}\right) = L_o(n),$$

where $L_0(n)$ is the number of partitions of n into parts not congruent to 0, 1, 6, 7, 8, 9, 10, or $15 \bmod 16$. In addition, by [4, Corollaries 4.3], we have

$$L_o(n) = q_o(2n).$$

Taking into account the last three identities, the proof follows easily. □

Taking into account that the kth generalized pentagonal number is even if and only if k is congruent to $\{0,2,5,7\}$ mod 8, it is an easy exercise to show that the sequence of the even generalized pentagonal numbers divided by 2, i.e.

$$\{a_n\}_{n\geqslant 0} = \{0,1,6,11,13,20,35,46,50,63,88,105,111,130,165,188,196,\ldots\},$$

is the sequence of all integers m such that $48m+1$ is a perfect square. In this context, Theorem 2.1 can be rewritten as follows.

Corollary 2.1 *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}p(n-a_k) - \sum_{k=0}^{\infty} p\left(\left\lfloor \frac{n}{2}\right\rfloor - b_k\right) = 0, \tag{4}$$

where:

1. $a_k = \begin{cases} \dfrac{(6k+1)^2-1}{48}, & \text{for} \quad k \equiv \{0,1\} \pmod 4, \\ \dfrac{(6k+5)^2-1}{48}, & \text{for} \quad k \equiv \{2,3\} \pmod 4. \end{cases}$

2. *For n even,*

$$b_k = \begin{cases} \dfrac{(8k+1)^2-1}{32}, & \text{for } k \text{ even}, \\ \dfrac{(8k+7)^2-1}{32}, & \text{for } k \text{ odd}. \end{cases}$$

3. *For n odd,*

$$b_k = \begin{cases} \dfrac{(8k+7)^2-17}{32}, & \text{for } k \text{ even}, \\ \dfrac{(8k+1)^2-17}{32}, & \text{for } k \text{ odd}. \end{cases}$$

In this corollary, for n even,

$$\{b_k\}_{k\geqslant 0} = \{0,7,9,30,34,69,75,124,132,195,205,282,294,385,\ldots\}$$

is the sequence of all integers m such that $32m+1$ is a perfect square. For n odd,

$$\{b_k\}_{k\geqslant 0} = \{1,2,16,19,47,52,94,101,157,161,236,247,331,344,\ldots\}$$

is the sequence of all integers m such that $32m+17$ is a perfect square.

Example 1 *By Euler's recurrence,*

$$\begin{aligned} p(10) &= p(9)+p(8)-p(5)-p(3) \\ &= 30+22-7-3 \\ &= 42. \end{aligned}$$

By Corollary 2.1,

$$\begin{aligned} p(10) &= p(9)+p(4)+p(5) \\ &= 30+5+7 \\ &= 42. \end{aligned}$$

The second result is similar to Theorem 2.1 but involves the odd generalized pentagonal numbers.

Theorem 2.2 *For $n \geqslant 0$,*

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil} p\left(n-\frac{G_k-1}{2}\right)-\sum_{k=0}^{\infty} p\left(\frac{n}{2}-\frac{(k-1)(k+2)}{8}\right)=0,$$

with $p(x)=0$ when x is not a nonnegative integer.

Proof By [4, Corollary 4.4], we have

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil+1} p\left(n-\frac{G_k-1}{2}\right)=q_o(2n+1).$$

Taking into account (3), the proof follows easily. □

The sequence of the odd generalized pentagonal numbers can be converted into the sequence of all integers m such that $48m+25$ is a perfect square, i.e.

$$\{a'_n\}_{n\geqslant 0}=\{0,2,3,7,17,25,28,38,58,72,77,93,123,143,150,172,212,\ldots\}.$$

Thus, Theorem 2.2 can be rewritten more explicitly in the following way.

Corollary 2.2 *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil} p(n-a'_k)-\sum_{k=0}^{\infty} p\left(\left\lfloor\frac{n}{2}\right\rfloor-b'_k\right)=0, \tag{5}$$

where:

1. $$a'_k=\begin{cases}\dfrac{(6k+5)^2-25}{48}, & \text{for} \quad k\equiv\{0,1\} \pmod 4,\\[2ex] \dfrac{(6k+1)^2-25}{48}, & \text{for} \quad k\equiv\{2,3\} \pmod 4.\end{cases}$$

2. *For n even,*

$$b'_k=\begin{cases}\dfrac{(8k+3)^2-9}{32}, & \text{for } k \text{ even},\\[2ex] \dfrac{(8k+5)^2-9}{32}, & \text{for } k \text{ odd}.\end{cases}$$

3. For n odd,

$$b'_k = \begin{cases} \dfrac{(8k+5)^2-25}{32}, & \text{for } k \text{ even,} \\ \dfrac{(8k+3)^2-25}{32}, & \text{for } k \text{ odd.} \end{cases}$$

For n even, we remark that

$$\{b'_k\}_{k\geqslant 0} = \{0, 5, 11, 26, 38, 63, 81, 116, 140, 185, 215, 270, 306, 371, \ldots\}$$

is the sequence of all integers m such that $32m+9$ is a perfect square. For n odd,

$$\{b'_k\}_{k\geqslant 0} = \{0, 3, 13, 22, 42, 57, 87, 108, 148, 175, 225, 258, 318, 357, \ldots\}$$

is the sequence of all integers m such that $32m+25$ is a perfect square.

Example 2 *By Euler's recurrence,*

$$\begin{aligned} p(15) &= p(14)+p(13)-p(10)-p(8)+p(3)+p(0) \\ &= 135+101-42-22+3+1 \\ &= 176. \end{aligned}$$

By Corollary 2.2,

$$\begin{aligned} p(15) &= p(13)+p(12)-p(8)+p(7)+p(4) \\ &= 101+77-22+15+5 \\ &= 176. \end{aligned}$$

3. Concluding remarks

New recurrence relations for the partition function $p(n)$ are derived in this paper as consequences of the bisectional pentagonal number theorem. We remark that the formula (3) can easily be derived from Watson's paper [6, p. 551]. It seems that Watson [6, p. 552] was the first to note that the partition functions $q_0(n)$ and $p(n)$ are of the same parity. According to Corollaries 2.1 and 2.2, we derive the following results related to the parity of the partition function $p(n)$.

Corollary 3.1 *Let n be a nonnegative integer. For $r \in \{0,1\}$,*

$$p(2n+r) \equiv 0 \pmod 2$$

if and only if

$$\sum_{24(2k+r)+1 \text{ square}} p(n-k) \equiv 0 \pmod 2.$$

Corollary 3.2 *Let n be a nonnegative integer. For $r \in \{0,1,2,3\}$,*

$$p(4n+r) \equiv 0 \pmod 2$$

if and only if

$$\sum_{8(4k+r)+1 \ square} p(n-k) \equiv 0 \pmod 2.$$

The last corollary provides an efficient method to compute the parity of Euler's partition function $p(n)$ that requires only the parity of $p(k)$ with $k \leqslant n/4$.

Related to Corollaries 3.1 and 3.2, we remark that the pairs $(a,b) \in \mathbb{N}^2$ for which

$$\sum_{ak+1 \text{ square}} p(n-k) \equiv 1 \pmod 2 \qquad \text{if and only if} \qquad bn+1 \text{ is a square}$$

were investigated very recently by Ballantine and Merca [3].

On the other hand, in a recent paper [5], Merca provided a method to compute the values of Euler's partition function $p(n)$ that requires only the values of $p(k)$ with $k \leqslant n/2$. There is a natural question arising from our remarks: Is it possible to compute the values of $p(n)$ using only the values of $p(k)$ with $k \leqslant n/4$?

Acknowledgments

The author thanks the referees for their helpful comments. Special thanks go to Dr Oana Merca for the careful reading of the manuscript and helpful remarks.

References

[1] Andrews GE. The Theory of Partitions. New York, NY, USA: Addison-Wesley Publishing, 1976.

[2] Andrews GE, Eriksson K. Integer Partitions. Cambridge, UK: Cambridge University Press, 2004.

[3] Ballantine C, Merca M. Parity of sums of partition numbers and squares in arithmetic progressions. Ramanujan J (in press).

[4] Merca M. The bisectional pentagonal number theorem. J Number Theory 2015; 157: 223-232.

[5] Merca M. Fast computation of the partition function. J Number Theory 2016; 164: 405-416.

[6] Watson GN. Two tables of partitions. P Lond Math Soc 1937; 42: 550-556.