Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1191 – 1203
© TÜBİTAK
doi:10.3906/mat-1507-37

Some properties of alternate duals and approximate alternate duals of fusion frames

Ali Akbar AREFIJAMAAL, Fahimeh ARABYANI NEYSHABURI*
Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

Received: 08.07.2015 • **Accepted/Published Online:** 28.11.2016 • **Final Version:** 28.09.2017

Abstract: In this paper we extend the notion of approximate dual to fusion frames and present some approaches to obtain alternate dual and approximate alternate dual fusion frames. We also study the stability of alternate dual and approximate alternate dual fusion frames.

Key words: Fusion frames, alternate dual fusion frames, approximate alternate duals, Riesz fusion bases

1. Introduction and preliminaries

Fusion frame theory is a natural generalization of frame theory in separable Hilbert spaces, introduced by Casazza and Kutyniok in [4]. Fusion frames are applied to signal processing, image processing, sampling theory, filter banks, and a variety of applications that cannot be modeled by discrete frames [11, 14].

Let I be a countable index set and recall that a sequence $\{f_i\}_{i\in I}$ is a *frame* in a separable Hilbert space $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.1}$$

The constants A and B are called the *lower* and *upper frame bounds*, respectively. It is said that $\{f_i\}_{i\in I}$ is a *Bessel sequence* if the right inequality in (1.1) is satisfied. Given a frame $\{f_i\}_{i\in I}$, the *frame operator* is defined by

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \qquad (f \in \mathcal{H}).$$

It is a bounded, invertible, and self-adjoint operator [6]. The family $\{S^{-1}f_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, the so-called *canonical dual* frame. In general, a Bessel sequence $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is called an alternate *dual* or simply a *dual* for the Bessel sequence $\{f_i\}_{i\in I}$ if

$$f = \sum_{i\in I} \langle f, g_i\rangle f_i, \qquad (f \in \mathcal{H}). \tag{1.2}$$

*Correspondence: arabianif@yahoo.com
2010 *AMS Mathematics Subject Classification:* Primary 42C15; Secondary 41A58.

The *synthesis operator* $T : l^2 \to \mathcal{H}$ of a Bessel sequence $\{f_i\}_{i\in I}$ is defined by $T\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i$. By (1.2) two Bessel sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are duals of each other if and only if $T_G T_F^* = I_{\mathcal{H}}$, where T_F and T_G are the synthesis operators $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively. For more details on the frame theory we refer to [3, 6].

Now we review the basic definitions and primary results of fusion frames. Throughout this paper, π_V denotes the orthogonal projection from Hilbert space $\mathcal{H}$ onto a closed subspace V.

Definition 1.1 *Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ a family of weights, i.e. $\omega_i > 0$, $i \in I$. Then $\{(W_i, \omega_i)\}_{i\in I}$ is called a fusion frame for $\mathcal{H}$ if there exist the constants $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_{i\in I} \omega_i^2 \|\pi_{W_i} f\|^2 \le B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.3}$$

The constants A and B are called the *fusion frame bounds.* If we only have the upper bound in (1.3) we call $\{(W_i, \omega_i)\}_{i\in I}$ a *Bessel fusion sequence.* A fusion frame is called A*-tight* if $A = B$, and *Parseval* if $A = B = 1$. If $\omega_i = \omega$ for all $i \in I$, the collection $\{(W_i, \omega_i)\}_{i\in I}$ is called ω*-uniform* and we abbreviate 1- uniform fusion frames as $\{W_i\}_{i\in I}$. A family of closed subspaces $\{W_i\}_{i\in I}$ is called an orthonormal basis for $\mathcal{H}$ when $\oplus_{i\in I} W_i = \mathcal{H}$ and it is a *Riesz decomposition* of $\mathcal{H}$, if for every $f \in \mathcal{H}$ there is a unique choice of $f_i \in W_i$ such that $f = \sum_{i\in I} f_i$. A family of closed subspaces $\{W_i\}_{i\in I}$ is called a Riesz fusion basis whenever it is complete for $\mathcal{H}$ and there exist positive constants A, B such that for every finite subset $J \subset I$ and arbitrary vector $f_i \in W_i$, we have

$$A \sum_{i\in J} \|f_i\|^2 \le \| \sum_{i\in J} f_i\|^2 \le B \sum_{i\in J} \|f_i\|^2.$$

It is clear that every Riesz fusion basis is a 1-uniform fusion frame for $\mathcal{H}$, and also a fusion frame is a Riesz basis if and only if it is a Riesz decomposition for $\mathcal{H}$; see [2, 4].

For every fusion frame a useful local frame is proposed in the following theorem.

Theorem 1.2 *[4] Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ a family of weights. Then $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with bounds A and B, if and only if $\{\omega_i \pi_{W_i} e_j\}_{i\in I, j\in J}$ is a frame for $\mathcal{H}$, with the same bounds, where $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{H}$.*

A connection between local and global properties is given in the next result; see [4].

Theorem 1.3 *For each $i \in I$, let W_i be a closed subspace of $\mathcal{H}$ and $\omega_i > 0$. Also let $\{f_{i,j}\}_{j\in J_i}$ be a frame for W_i with frame bounds α_i and β_i such that*

$$0 < \alpha = inf_{i\in I} \alpha_i \le \beta = sup_{i\in I} \beta_i < \infty. \tag{1.4}$$

Then the following conditions are equivalent:

(i) $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame of $\mathcal{H}$ with bounds C and D.

(ii) $\{\omega_i f_{i,j}\}_{i\in I, j\in J_i}$ is a frame of $\mathcal{H}$ with bounds αC and βD.

Recall that for each sequence $\{W_i\}_{i\in I}$ of closed subspaces in $\mathcal{H}$, the space

$$\sum_{i\in I}\oplus W_i=\{\{f_i\}_{i\in I}:f_i\in W_i,\sum_{i\in I}\|f_i\|^2<\infty\},$$

with the inner product

$$\langle\{f_i\}_{i\in I},\{g_i\}_{i\in I}\rangle=\sum_{i\in I}\langle f_i,g_i\rangle,$$

is a Hilbert space. For a Bessel fusion sequence $\{(W_i,\omega_i)\}_{i\in I}$ of $\mathcal{H}$, the *synthesis operator* $T_W:\sum_{i\in I}\oplus W_i\to\mathcal{H}$ is defined by

$$T_W(\{f_i\}_{i\in I})=\sum_{i\in I}\omega_i f_i,\qquad (\{f_i\}_{i\in I}\in\sum_{i\in I}\oplus W_i).$$

Its adjoint operator $T_W^*:\mathcal{H}\to\sum_{i\in I}\oplus W_i$, which is called the *analysis operator*, is given by

$$T_W^*(f)=\{\omega_i\pi_{W_i}(f)\}_{i\in I},\qquad (f\in\mathcal{H}).$$

Let $\{(W_i,\omega_i)\}_{i\in I}$ be a fusion frame. The *fusion frame operator* $S_W:\mathcal{H}\to\mathcal{H}$ is defined by $S_W f=\sum_{i\in I}\omega_i^2\pi_{W_i}f$ is bounded, invertible as well as positive. Hence, we have the following reconstruction formula [4]:

$$f=\sum_{i\in I}\omega_i^2 S_W^{-1}\pi_{W_i}f,\qquad (f\in\mathcal{H}).$$

The family $\{(S_W^{-1}W_i,\omega_i)\}_{i\in I}$, which is also a fusion frame, is called the *canonical dual* of $\{(W_i,\omega_i)\}_{i\in I}$. Also, a Bessel fusion sequence $\{(V_i,\nu_i)\}_{i\in I}$ is called an *alternate dual* of $\{(W_i,\omega_i)\}_{i\in I}$, [8] whenever

$$f=\sum_{i\in I}\omega_i\nu_i\pi_{V_i}S_W^{-1}\pi_{W_i}f,\qquad (f\in\mathcal{H}).\tag{1.5}$$

In [8], it was proved that every alternate dual of a fusion frame is a fusion frame. Also, we can easily see that a Bessel fusion sequence $\{(V_i,\upsilon_i)\}_{i\in I}$ is an alternate dual fusion frame of $\{(W_i,\omega_i)\}_{i\in I}$ if and only if $T_V\phi_{vw}T_W^*=I_{\mathcal{H}}$, where the bounded operator $\phi_{vw}:\sum_{i\in I}\bigoplus W_i\to\sum_{i\in I}\bigoplus V_i$ is given by

$$\phi_{vw}(\{f_i\}_{i\in I})=\{\pi_{V_i}S_W^{-1}f_i\}_{i\in I}.\tag{1.6}$$

Moreover, a Bessel fusion sequence $V=\{(V_i,\omega_i)\}_{i\in I}$ given by $V_i=S_W^{-1}W_i\oplus U_i$ is an alternate dual fusion frame of $\{(W_i,\omega_i)\}_{i\in I}$ in which U_i is a closed subspace of $\mathcal{H}$ for all $i\in I$ [13]. Recently, Heineken et al. introduced the other concept of dual fusion frames [10]. For two fusion frames $\{(W_i,\omega_i)\}_{i\in I}$ and $\{(V_i,\upsilon_i)\}_{i\in I}$, if there exists a mapping $Q\in B(\sum_{i\in I}\bigoplus W_i,\sum_{i\in I}\bigoplus V_i)$, such that $T_VQT_W^*=I_{\mathcal{H}}$, then $\{(V_i,\upsilon_i)\}_{i\in I}$ is called a Q-dual of $\{(W_i,\omega_i)\}_{i\in I}$. Clearly, every alternate dual fusion frame is a ϕ_{vw}-dual. Q-duals are useful tools for establishing the reconstruction formula. For more information on fusion frames, we refer the reader to [2, 4, 5].

2. Alternate approximate duals

Alternate dual fusion frames play a key role in fusion frame theory; however, their explicit computations seem rather intricate. In this section, we introduce the notion of the approximate alternate dual for fusion frames and discuss the existence of alternate dual fusion frames from an approximate alternate dual. Moreover, we present a complete characterization of alternate duals of Riesz fusion bases. The notion of the approximate dual for discrete frames has already been introduced by Christensen and Laugesen in [7] and then for g-frames in [12]; however, many of their results are invalid for fusion frames. Throughout this section we consider a Riesz fusion basis as a 1-uniform fusion frame.

First, we recall the notion of an approximate dual for discrete frames. Let $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ be Bessel sequences for $\mathcal{H}$. Then F and G are called *approximate dual frames* if $\|I_{\mathcal{H}} - T_G T_F^*\| < 1$. In this case, $\{(T_G T_F^*)^{-1} g_i\}$ is a dual of F; see [7].

Now we introduce approximate duality for fusion frames.

Definition 2.1 *Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame. A Bessel fusion sequence $\{(V_i, \upsilon_i)\}_{i\in I}$ is called an approximate alternate dual of $\{(W_i, \omega_i)\}_{i\in I}$ if*

$$\|I_{\mathcal{H}} - T_V \phi_{vw} T_W^*\| < 1.$$

Putting

$$\psi_{vw} = T_V \phi_{vw} T_W^*, \tag{2.1}$$

we have the following reconstruction formula:

$$f = \sum_{i\in I} (\psi_{vw})^{-1} \omega_i \upsilon_i \pi_{V_i} S_W^{-1} \pi_{W_i} f = \sum_{n=0}^{\infty} (I - \psi_{vw})^n \psi_{vw} f, \qquad (f \in \mathcal{H}).$$

Proposition 2.2 *Let $V = \{(V_i, \upsilon_i)\}_{i\in I}$ be an approximate alternate dual of a fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$. Then V is a fusion frame.*

Proof Let B and D be Bessel bounds of W and V, respectively. Then

$$\begin{aligned}
\|\psi_{vw}^* f\|^2 &= \sup_{\|g\|=1} |\langle T_W \phi_{vw}^* T_V^* f, g\rangle|^2 \\
&= \sup_{\|g\|=1} |\langle \sum_{i\in I} \omega_i \upsilon_i \pi_{W_i} S_W^{-1} \pi_{V_i} f, g\rangle|^2 \\
&\leq \sup_{\|g\|=1} \sum_{i\in I} \upsilon_i^2 \|\pi_{V_i} f\|^2 \sum_{i\in I} \omega_i^2 \|S_W^{-1} \pi_{W_i} g\|^2 \\
&\leq \|S_W^{-1}\|^2 B \sum_{i\in I} \upsilon_i^2 \|\pi_{V_i} f\|^2,
\end{aligned}$$

for every $f \in \mathcal{H}$. It follows that

$$\|f\|^2 \frac{\|(\psi_{vw}^{-1})^*\|^{-2}}{\|S_W^{-1}\|^2 B} \leq \sum_{i\in I} \upsilon_i^2 \|\pi_{V_i} f\|^2 \leq D\|f\|^2. \qquad \square$$

The following proposition describes the approximate duality of fusion frames with respect to local frames.

Proposition 2.3 *Let $\{e_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{H}$. Then the Bessel sequence $V=\{(V_i,\upsilon_i)\}_{i\in I}$ is an approximate alternate dual fusion frame of a fusion frame $W=\{(W_i,\omega_i)\}_{i\in I}$ if and only if $\{\upsilon_i\pi_{V_i}e_j\}_{i\in I,j\in J}$ is an approximate dual of $\{\omega_i\pi_{W_i}S_W^{-1}e_j\}_{i\in I,j\in J}$.*

Proof For each $f\in\mathcal{H}$ we have

$$\begin{aligned}\sum_{i\in I,j\in J}|\langle f,\omega_i\pi_{W_i}S_W^{-1}e_j\rangle|^2 &= \sum_{i\in I}\sum_{j\in J}|\langle \omega_i S_W^{-1}\pi_{W_i}f,e_j\rangle|^2\\ &= \sum_{i\in I}\omega_i^2\|S_W^{-1}\pi_{W_i}f\|^2\\ &\leq \|S_W^{-1}\|^2\sum_{i\in I}\omega_i^2\|\pi_{W_i}f\|^2.\end{aligned}$$

This implies that $F=\{\omega_i\pi_{W_i}S_W^{-1}e_j\}_{i\in I,j\in J}$ is a Bessel sequence for $\mathcal{H}$. Similarly, $G=\{\upsilon_i\pi_{V_i}e_j\}_{i\in I,j\in J}$ is also a Bessel sequence for $\mathcal{H}$. Moreover,

$$\begin{aligned}T_GT_F^*f &= \sum_{i\in I,j\in J}\langle f,\omega_i\pi_{W_i}S_W^{-1}e_j\rangle\upsilon_i\pi_{V_i}e_j\\ &= \sum_{i\in I,j\in J}\omega_i\upsilon_i\pi_{V_i}\langle S_W^{-1}\pi_{W_i}f,e_j\rangle e_j\\ &= \sum_{i\in I}\omega_i\upsilon_i\pi_{V_i}S_W^{-1}\pi_{W_i}f\\ &= T_V\phi_{vw}T_W^*f=\psi_{vw}\end{aligned}$$

for all $f\in\mathcal{H}$. This completes the proof. □

Theorem 2.4 *Let $W=\{(W_i,\omega_i)\}_{i\in I}$ be a fusion frame and $V=\{(V_i,\upsilon_i)\}_{i\in I}$ be a Bessel fusion sequence, and also let $\{g_{i,j}\}_{j\in J_i}$ be a frame for V_i with bounds A_i and B_i for every $i\in I$ such that $0<a=\inf_{i\in I}A_i\leq\sup_{i\in I}B_i=b<\infty$. Then V is an approximate alternate dual fusion frame of W if and only if $G=\{\upsilon_i g_{i,j}\}_{i\in I,j\in J_i}$ is an approximate dual of $F=\{\omega_i\pi_{W_i}S_W^{-1}\widetilde{g}_{i,j}\}_{i\in I,j\in J_i}$ where $\{\widetilde{g}_{i,j}\}_{j\in J_i}$ is the canonical dual of $\{g_{i,j}\}_{j\in J_i}$.*

Proof We first show that F is a Bessel sequence for $\mathcal{H}$. Indeed, for each $f\in\mathcal{H}$

$$\begin{aligned}\sum_{i\in I,j\in J_i}|\langle f,\omega_i\pi_{W_i}S_W^{-1}\widetilde{g}_{i,j}\rangle|^2 &= \sum_{i\in I}\omega_i^2\sum_{j\in J_i}|\langle \pi_{V_i}S_W^{-1}\pi_{W_i}f,\widetilde{g}_{i,j}\rangle|^2\\ &\leq \sum_{i\in I}\frac{\omega_i^2}{A_i}\|\pi_{V_i}S_W^{-1}\pi_{W_i}f\|^2\\ &\leq \frac{\|S_W^{-1}\|^2}{a}\sum_{i\in I}\omega_i^2\|\pi_{W_i}f\|^2.\end{aligned}$$

Moreover, by Theorem 1.3, G is a Bessel sequence for $\mathcal{H}$. On the other hand,

$$\begin{aligned} T_V\phi_{vw}T_W^*f &= \sum_{i\in I}\omega_i\upsilon_i\pi_{V_i}S_W^{-1}\pi_{W_i}f \\ &= \sum_{i\in I}\omega_i\upsilon_i\sum_{j\in J_i}\langle\pi_{V_i}S_W^{-1}\pi_{W_i}f,\widetilde{g}_{i,j}\rangle g_{i,j} \\ &= \sum_{i\in I,j\in J_i}\langle f,\omega_i\pi_{W_i}S_W^{-1}\widetilde{g}_{i,j}\rangle\upsilon_i g_{i,j} = T_GT_F^*f. \end{aligned}$$

This completes the proof. □

The following theorem gives the idea that will lead to one of the main results of this section.

Theorem 2.5 *Let $W=\{(W_i,\omega_i)\}_{i\in I}$ be a Riesz fusion basis. For an approximate alternate dual fusion frame $\{(V_i,\omega_i)\}_{i\in I}$ of W, the sequence $\{(\psi_{vw}^{-1}V_i,\omega_i)\}_{i\in I}$ is an alternate dual fusion frame of W.*

Proof Suppose that $\{e_{i,j}\}_{j\in J_i}$ is an orthonormal basis of W_i, for each $i\in I$. Then $F:=\{\omega_ie_{i,j}\}_{i\in I,j\in J_i}$ is a frame for $\mathcal{H}$ by Theorem 1.3. Now, for each $f\in\mathcal{H}$, we obtain

$$\begin{aligned} \sum_{i\in I,j\in J_i}|\langle f,\omega_i\pi_{V_i}S_W^{-1}e_{i,j}\rangle|^2 &= \sum_{i\in I}\sum_{j\in J_i}|\langle\omega_iS_W^{-1}\pi_{V_i}f,e_{i,j}\rangle|^2 \\ &\leq \sum_{i\in I}\omega_i^2\|S_W^{-1}\pi_{V_i}f\|^2 \\ &\leq \|S_W^{-1}\|^2\sum_{i\in I}\omega_i^2\|\pi_{V_i}f\|^2. \end{aligned}$$

Thus, $G:=\{\omega_i\pi_{V_i}S_W^{-1}e_{i,j}\}_{i\in I,j\in J_i}$ is a Bessel sequence for $\mathcal{H}$. Moreover,

$$\begin{aligned} \psi_{vw}f=\sum_{i\in I}\omega_i^2\pi_{V_i}S_W^{-1}\pi_{W_i}f &= \sum_{i\in I,j\in J_i}\omega_i^2\pi_{V_i}S_W^{-1}\langle f,e_{i,j}\rangle e_{i,j} \\ &= \sum_{i\in I,j\in J_i}\langle f,\omega_ie_{i,j}\rangle\omega_i\pi_{V_i}S_W^{-1}e_{i,j} = T_GT_F^*f. \end{aligned}$$

Hence, by the assumption, G is an approximate dual of F. This implies that the sequence $\{(T_GT_F^*)^{-1}\omega_i\pi_{V_i}S_W^{-1}e_{i,j}\}_{i\in I,j\in J_i}$ is a dual for $\{\omega_ie_{i,j}\}_{i\in I,j\in J_i}$. On the other hand, the sequence $\{\omega_ie_{i,j}\}_{i\in I,j\in J_i}$ is a Riesz basis for $\mathcal{H}$ by Theorem 3.6 of [2]. Using the fact that discrete Riesz bases have only one dual, we obtain

$$(T_GT_F^*)^{-1}\omega_i\pi_{V_i}S_W^{-1}e_{i,j}=S_F^{-1}\omega_ie_{i,j}\quad(i\in I,j\in J_i). \tag{2.2}$$

Furthermore, it is not difficult to see that $S_F=S_W$. Indeed, for all $f\in\mathcal{H}$ we have

$$\begin{aligned} S_Ff &= \sum_{i\in I,j\in J_i}\langle f,\omega_ie_{i,j}\rangle\omega_ie_{i,j} \\ &= \sum_{i\in I}\omega_i^2\sum_{j\in J_i}\langle\pi_{W_i}f,e_{i,j}\rangle e_{i,j} \\ &= \sum_{i\in I}\omega_i^2\pi_{W_i}f=S_Wf. \end{aligned}$$

Now, since $T_G T_F^* = \psi_{vw}$, by substituting ψ_{vw} and S_W in (2.2), we finally conclude that

$$\psi_{vw}^{-1}\pi_{V_i} S_W^{-1} e_{i,j} = S_W^{-1} e_{i,j}, \quad (i \in I, j \in J_i).$$

In particular,

$$\psi_{vw}^{-1} V_i \supseteq S_W^{-1} W_i, \quad (i \in I). \tag{2.3}$$

It immediately follows that $\{(\psi_{vw}^{-1} V_i, \omega_i)\}_{i\in I}$ is an alternate dual fusion frame of $\{(W_i, \omega_i)\}_{i\in I}$. □

By the above theorem we obtain the following characterization of alternate duals of Riesz fusion bases.

Corollary 2.6 *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ be a Riesz fusion basis. A Bessel sequence $V = \{(V_i, \omega_i)\}_{i\in I}$ is an alternate dual fusion frame of W if and only if*

$$V_i \supseteq S_W^{-1} W_i, \quad (i \in I). \tag{2.4}$$

Proof If V satisfies (2.4), clearly V is an alternate dual of W. On the other hand, since every alternate dual fusion frame is an approximate alternate dual with $\psi_{vw} = I_{\mathcal{H}}$, by (2.3) the result follows. □

Corollary 2.6 also shows that, unlike discrete frames, Riesz fusion bases may have more than one dual. Moreover, in the next proposition, we show that every fusion frame has at least an alternate dual.

Proposition 2.7 *Every fusion frame has an alternate dual fusion frame different from the canonical dual fusion frame.*

Proof Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame with frame operator S_W. First suppose that there exists $i_0 \in I$ such that $W_{i_0} \neq \mathcal{H}$. Take $V_i = S_W^{-1} W_i$ for $i \neq i_0$ and $V_{i_0} = S_W^{-1} W_{i_0} \oplus U_{i_0}$ where $U_{i_0} \subseteq (S_W^{-1} W_{i_0})^{\perp}$ is an arbitrary closed subspace. Obviously, $\{(V_i, \omega_i)\}_{i\in I}$ is an alternate dual fusion frame of $\{(W_i, \omega_i)\}_{i\in I}$. Indeed,

$$\begin{aligned}
T_V \phi_{vw} T_W^* f &= \sum_{i\in I, i\neq i_0} \omega_i^2 \pi_{V_i} S_W^{-1} \pi_{W_i} f + \omega_{i_0}^2 \pi_{V_{i_0}} S_W^{-1} \pi_{W_{i_0}} f \\
&= \sum_{i\in I, i\neq i_0} \omega_i^2 \pi_{S_W^{-1} W_i} S_W^{-1} \pi_{W_i} f + \omega_{i_0}^2 \pi_{S_W^{-1} W_{i_0} \oplus U_{i_0}} S_W^{-1} \pi_{W_{i_0}} f \\
&= \sum_{i\in I} \omega_i^2 \pi_{S_W^{-1} W_i} S_W^{-1} \pi_{W_i} f = f,
\end{aligned}$$

for every $f \in \mathcal{H}$. On the other hand, assume that $W_i = \mathcal{H}$ for all $i \in I$. It immediately follows that $\{\omega_i\}_{i\in I} \in l^2$. Take $V_1 = \mathcal{H}$ and $V_i = \{0\}$ for $i > 1$, and assume that $\nu_1 = \frac{\sum_{i\in I}\omega_i^2}{\omega_1}$ and $\nu_i = \omega_i$ for $i > 1$. Then $S_W f = \left(\sum_{i\in I} \omega_i^2\right) f$ and for every $f \in \mathcal{H}$ we have

$$\begin{aligned}
\sum_{i\in I} \omega_i \nu_i \pi_{V_i} S_W^{-1} \pi_{W_i} f &= \sum_{i\in I} \omega_i \nu_i \pi_{V_i} S_W^{-1} f = \omega_1 \nu_1 \pi_{V_1} S_W^{-1} f \\
&= \left(\sum_{i\in I} \omega_i^2\right) S_W^{-1} f = f.
\end{aligned}$$

This shows that $\{(V_i, \nu_i)\}_{i\in I}$ is an alternate dual fusion frame of $\{(W_i, \omega_i)\}_{i\in I}$. □

Example 2.8 *Let $W=\{W_i\}_{i\in I}$ be a Riesz fusion basis. Considering*

$$V_i = (\overline{span}_{j\neq i}\{W_j\})^{\perp}, \quad (i\in I),$$

we claim that $V=\{(V_i,\upsilon_i)\}_{i\in I}$ is an alternate dual of $\{W_i\}_{i\in I}$ for all $\{\upsilon_i\}_{i\in I}\in l^2$. Take $f_i\in W_i$; since $\{S_W^{-1/2}W_i\}_{i\in I}$ is an orthogonal family of subspaces in $\mathcal{H}$ so $S_W^{-1}f_i\in V_i$. Hence, $V_i\supseteq S_W^{-1}W_i$ for every $i\in I$ and so V is a dual of W by Corollary 2.6. In fact, this dual is the unique maximal biorthogonal sequence for $\{W_i\}_{i\in I}$; see also Proposition 4.3 in [4].

Suppose that $\{W_i\}_{i\in I}$ is a Riesz fusion basis. By Theorem 3.9 in [2], there exists an orthonormal fusion basis $\{U_i\}_{i\in I}$ and a bounded bijective linear operator $T:\mathcal{H}\to\mathcal{H}$ for which $TU_i=W_i$ for all $i\in I$. Therefore, the canonical dual of a Riesz fusion basis is also a Riesz fusion basis. The following theorem shows that other alternate duals of $\{W_i\}_{i\in I}$ are not Riesz fusion basis. This result is the infinite dimensional version for alternate dual frames of Proposition 3.7 (2) in [9].

Theorem 2.9 *Let $W=\{W_i\}_{i\in I}$ be a Riesz fusion basis. The only dual $\{V_i\}_{i\in I}$ of W that is Riesz basis is the canonical dual.*

Proof Suppose that the Riesz basis $\{V_i\}_{i\in I}$ is an alternate dual fusion frame of W. By Corollary 2.6, $S_W^{-1}W_i\subseteq V_i$ for all $i\in I$. Assume that there exists $j\in I$ such that $S_W^{-1}W_j\subset V_j$, and pick a nonzero $0\neq f\in V_j\cap(S_W^{-1}W_j)^{\perp}$. Since $\{S_W^{-1}W_i\}_{i\in I}$ is a Riesz fusion basis we can choose a unique sequence $\{g_i\}_{i\in I}$ such that $f=\sum_{i\in I}g_i$ where $g_i\in S_W^{-1}W_i$ for all $i\in I$. Therefore, the vector f has two representations of the elements in the Riesz fusion basis $\{V_i\}_{i\in I}$, which is a contradiction. Hence, $V_i=S_W^{-1}W_i$ for every $i\in I$. □

Suppose that $L\in B(\mathcal{H})$ is invertible and $\{(V_i,\upsilon_i)\}_{i\in I}$ is an alternate dual (approximate alternate dual) fusion frame of $W=\{(W_i,\omega_i)\}_{i\in I}$. It is natural to ask whether $\{(LV_i,\upsilon_i)\}_{i\in I}$ is an alternate dual (approximate alternate dual) fusion frame of $\{(LW_i,\omega_i)\}_{i\in I}$.

Theorem 2.10 *Let $W=\{(W_i,\omega_i)\}_{i\in I}$ be a fusion frame and $L\in B(\mathcal{H})$ be invertible such that $L^*LW_i\subseteq W_i$ for every $i\in I$. The following statements hold:*

(i) If $V=\{(V_i,\upsilon_i)\}_{i\in I}$ is an alternate dual fusion frame of W, then the sequence $LV=\{(LV_i,\upsilon_i)\}_{i\in I}$ is an alternate dual fusion frame of $LW=\{(LW_i,\omega_i)\}_{i\in I}$.

(ii) If V is an approximate alternate dual fusion frame of W such that

$$\|I_{\mathcal{H}}-\psi_{vw}\| < \|L\|^{-1}\|L^{-1}\|^{-1},$$

then LV is an approximate alternate dual fusion frame of LW.

Proof The sequence $\{(LW_i,\omega_i)\}_{i\in I}$ is a fusion frame with the frame operator LS_WL^{-1} and $\pi_{LW_i}=L\pi_{W_i}L^{-1}$; see [5]. Therefore, for each $f\in\mathcal{H}$, we obtain

$$\begin{aligned}\sum_{i\in I}\omega_i\upsilon_i\pi_{LV_i}S_{LW}^{-1}\pi_{LW_i}f &= \sum_{i\in I}\omega_i\upsilon_i L\pi_{V_i}S_W^{-1}\pi_{W_i}L^{-1}f\\ &= LL^{-1}f=f.\end{aligned}$$

This proves (i). To show (ii) first note that $\psi_{Lv,Lw} = L\psi_{vw}L^{-1}$ and hence

$$\begin{aligned}\|(I_{\mathcal{H}} - \psi_{Lv,Lw})f\| &= \|(I_{\mathcal{H}} - L\psi_{v,w}L^{-1})f\| \\ &= \|L(I_{\mathcal{H}} - \psi_{v,w})L^{-1}f\| \\ &< \|f\|\end{aligned}$$

for all $f \in \mathcal{H}$. This follows the result. □

3. Stability of approximate alternate duals

In frame theory, every $f \in \mathcal{H}$ is represented by the collection of coefficients $\{\langle f, f_i\rangle\}_{i\in I}$. From these coefficients, f can be recovered using a reconstruction formula by dual frames. In real applications, in these transmissions usually a part of the data vectors changes or reshapes; in other words, disturbances affect the information. In this respect, the stability of frames and dual frames under perturbations has a key role in practice. The stability of approximate duals of discrete frames and g-frames can be found in [7, 12]. In the following, we discuss the stability of approximate alternate dual fusion frames under some perturbations. First, we fix the definition of perturbation.

Definition 3.1 *Let $\{W_i\}_{i\in I}$ and $\{\widetilde{W}_i\}_{i\in I}$ be closed subspaces in $\mathcal{H}$. Also let $\{\omega_i\}_{i\in I}$ be positive numbers and $\epsilon > 0$. We call $\{(\widetilde{W}_i, \omega_i)\}_{i\in I}$ an ϵ-perturbation of $\{(W_i, \omega_i)\}_{i\in I}$ whenever, for every $f \in \mathcal{H}$,*

$$\sum_{i\in I} \omega_i^2 \|(\pi_{\widetilde{W}_i} - \pi_{W_i})f\|^2 < \epsilon \|f\|^2.$$

Theorem 3.2 *Let $V = \{(V_i, \upsilon_i)\}_{i\in I}$ be an approximate alternate dual fusion frame of a fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$. Also let $\{(U_i, \upsilon_i)\}_{i\in I}$ be an ϵ-perturbation of V, such that*

$$\epsilon < \left(\frac{1 - \|I_{\mathcal{H}} - \psi_{vw}\|}{\sqrt{B}\|S_W^{-1}\|}\right)^2, \tag{3.1}$$

where B is the upper bound of W. Then $\{(U_i, \upsilon_i)\}_{i\in I}$ is also an approximate alternate dual fusion frame of W. In particular, if W is a Parseval fusion frame and we choose $V = W$, then the result holds for $\epsilon < 1$.

Proof Notice that $\{(U_i, \upsilon_i)\}_{i\in I}$ is a Bessel fusion sequence; in fact,

$$\begin{aligned}\sum_{i\in I} \upsilon_i^2 \|\pi_{U_i} f\|^2 &= \sum_{i\in I} \upsilon_i^2 \|\pi_{V_i} f + (\pi_{U_i} - \pi_{V_i})f\|^2 \\ &\leq \left(\left(\sum_{i\in I} \upsilon_i^2 \|\pi_{V_i} f\|^2\right)^{1/2} + \left(\sum_{i\in I} \upsilon_i^2 \|(\pi_{U_i} - \pi_{V_i})f\|^2\right)^{1/2}\right)^2 \\ &\leq (\sqrt{D} + \sqrt{\epsilon})^2 \|f\|^2,\end{aligned}$$

where D is the upper bound of V. On the other hand,

$$\begin{aligned}
\|(I_{\mathcal{H}}-\psi_{uw})f\| &\leq \|(I_{\mathcal{H}}-\psi_{vw})f\|+\|(\psi_{vw}-\psi_{uw})f\| \\
&\leq \|(I_{\mathcal{H}}-\psi_{vw})f\|+\sup_{\|g\|=1}\left|\left\langle\sum_{i\in I}\omega_i\upsilon_i(\pi_{V_i}-\pi_{U_i})S_W^{-1}\pi_{W_i}f,g\right\rangle\right| \\
&= \|(I_{\mathcal{H}}-\psi_{vw})f\|+\sup_{\|g\|=1}\left|\sum_{i\in I}\left\langle\omega_i S_W^{-1}\pi_{W_i}f,\upsilon_i(\pi_{V_i}-\pi_{U_i})g\right\rangle\right| \\
&\leq \|(I_{\mathcal{H}}-\psi_{vw})f\|+\sqrt{\epsilon}\left(\sum_{i\in I}\omega_i^2\|S_W^{-1}\pi_{W_i}f\|^2\right)^{1/2} \\
&\leq \|(I_{\mathcal{H}}-\psi_{vw})f\|+\sqrt{\epsilon B}\|S_W^{-1}\|\|f\|<\|f\|,
\end{aligned}$$

where the last inequality is implied from (3.1). The rest follows by the fact that each Parseval fusion frame is a dual of itself. $\square$

Example 3.3 *Consider*

$$W_1=\mathbb{R}^2\times\{0\},\quad W_2=\{0\}\times\mathbb{R}^2,\quad W_3=span\{(1,0,0)\},$$

$$V_1=span\{(0,1,0)\},\quad V_2=\{0\}\times\mathbb{R}^2,\quad V_3=span\{(1,0,0)\}.$$

Then $W=\{W_i\}_{i=1}^3$ is a fusion frame and $\|S_W^{-1}\|=1$. Also, we have $\|I_{\mathcal{H}}-\psi_{vw}\|=\frac{1}{2}$, and so the Bessel sequence $V=\{V_i\}_{i=1}^3$ is an approximate alternate dual fusion frame of W. Now, if we take

$$U_1=V_1,\quad U_2=V_2,\quad U_3=span\{(\alpha,\beta,0)\},$$

where $\frac{1}{2}\leq\alpha<1$ and $0\leq\beta\leq\frac{1}{100}$, then $U=\{U_i\}_{i\in I}$ is an ϵ-perturbation of V with $\epsilon<\frac{1}{8}$. Hence, by Theorem 3.2, U is also an approximate alternate dual fusion frame of W.

The next result is obtained immediately from Theorem 3.2.

Corollary 3.4 *Let $\{(V_i,\upsilon_i)\}_{i\in I}$ be an alternate dual fusion frame of a fusion frame $W=\{(W_i,\omega_i)\}_{i\in I}$. Also, let $\{(U_i,\upsilon_i))\}_{i\in I}$ be an ϵ-perturbation of V, and*

$$\sqrt{\epsilon B}\leq\frac{1}{\|S_W^{-1}\|}, \tag{3.2}$$

where B is the upper bound of W. Then $\{(U_i,\upsilon_i)\}_{i\in I}$ is an approximate alternate dual fusion frame of W.

Theorem 3.5 *Let $V=\{(V_i,\upsilon_i)\}_{i\in I}$ be an approximate alternate dual fusion frame of a fusion frame $W=\{(W_i,\omega_i)\}_{i\in I}$. Also, let $\{U_i\}_{i\in I}$ be an ϵ-perturbation of W with*

$$\sqrt{\epsilon}<\frac{1-(\sqrt{BD}\|S_W^{-1}-S_U^{-1}\|+\|I_{\mathcal{H}}-\psi_{vw}\|)}{\sqrt{D}\|S_U^{-1}\|}, \tag{3.3}$$

where B and D are the upper bounds of W and V, respectively. Then $\{(V_i, \upsilon_i)\}_{i\in I}$ is also an approximate alternate dual fusion frame of $U = \{(U_i, \omega_i)\}_{i\in I}$.

Proof Applying the Cauchy–Schwarz inequality for every $f \in \mathcal{H}$, we have

$$\begin{aligned}
\|(I_{\mathcal{H}} - \psi_{vu})f\| &\leq \|(I_{\mathcal{H}} - \psi_{vw})f\| + \|(\psi_{vw} - \psi_{vu})f\| \\
&\leq \|(I_{\mathcal{H}} - \psi_{vw})f\| + \sup_{\|g\|=1} \left| \left\langle \sum_{i\in I} \omega_i \upsilon_i \pi_{V_i} (S_W^{-1}\pi_{W_i} - S_U^{-1}\pi_{U_i}) f, g \right\rangle \right| \\
&\leq \|(I_{\mathcal{H}} - \psi_{vw})f\| + \sup_{\|g\|=1} \left| \left\langle \sum_{i\in I} \omega_i \upsilon_i (S_W^{-1} - S_U^{-1}) \pi_{W_i} f, \pi_{V_i} g \right\rangle \right| \\
&+ \sup_{\|g\|=1} \left| \left\langle \sum_{i\in I} \omega_i \upsilon_i S_U^{-1} (\pi_{W_i} - \pi_{U_i}) f, \pi_{V_i} g \right\rangle \right| \\
&\leq \|(I_{\mathcal{H}} - \psi_{vw})f\| + \sqrt{D} \left(\|S_W^{-1} - S_U^{-1}\| \sqrt{B} + \sqrt{\epsilon} \|S_U^{-1}\| \right) \|f\| \\
&< \|f\|,
\end{aligned}$$

where the last inequality is obtained by the assumption.

□

Example 3.6 *Consider*

$$V_1 = \mathbb{R}^3, \quad V_2 = \{0\} \times \mathbb{R}^2, \quad V_3 = span\{(1, 0, 0)\}.$$

Then $V = \{V_i\}_{i=1}^3$ is an alternate dual of Parseval fusion frame $W = \{W_i\}_{i=1}^3$, in which

$$W_1 = span\{(0, 0, 1)\}, \quad W_2 = span\{(0, 1, 0)\}, \quad W_3 = span\{(1, 0, 0)\}.$$

On the other hand, letting

$$U_1 = W_1, \quad U_2 = W_2, \quad U_3 = span\{(1, \frac{1}{50}, 0)\},$$

then $\{U_i\}_{i\in I}$ is an ϵ-perturbation of W with $\epsilon < 0.02$. Using the fact that

$$0.02 < \frac{1 - \sqrt{2}\|I_{\mathcal{H}} - S_U^{-1}\|}{\sqrt{2}\|S_U^{-1}\|},$$

we obtain that V is an approximate alternate dual fusion frame of $\{U_i\}_{i\in I}$ by Theorem 3.5.

We know that many concepts of the classical frame theory have not been generalized to the fusion frames. For example, in the duality discussion, if $V = \{(V_i, \upsilon_i)\}_{i\in I}$ is an alternate dual of fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$, then W is not an alternate dual fusion frame of V. Indeed, take

$$W_1 = span\{(1, 0, 0)\}, \quad W_2 = span\{(1, 1, 0)\},$$
$$W_3 = span\{(0, 1, 0)\}, \quad W_4 = span\{(0, 0, 1)\},$$

and $\omega_1 = \omega_3 = \omega_4 = 1$, $\omega_2 = \sqrt{2}$. Then $W = \{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathbb{R}^3$ with an alternate dual as $V = \{(V_i, \upsilon_i)\}_{i\in I}$ where

$$V_1 = span\{(0,1,0)\}, \quad V_2 = \mathbb{R}^3, \quad V_3 = span\{(1,0,0)\}, \quad V_4 = span\{(0,0,1)\},$$

and $\upsilon_1 = \upsilon_3 = 3$, $\upsilon_2 = 3\sqrt{2}$, $\upsilon_4 = 1$; see Example 3.1 of [1]. A straightforward calculation shows that W is not an alternate dual fusion frame of V. Moreover, for an alternate dual fusion frame V of W, the fusion frame W is not an approximate alternate dual fusion frame of V in general. The next theorem gives a sufficient condition for a fusion frame being an approximate alternate dual of its dual.

Theorem 3.7 *Let $\{(V_i, \upsilon_i)\}_{i\in I}$ be an alternate dual of fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ such that*

$$\|S_W^{-1} - S_V^{-1}\| < \|S_W\|^{-1/2}\|S_V\|^{-1/2}.$$

Then $\{(W_i, \omega_i)\}_{i\in I}$ is an approximate alternate dual fusion frame of $\{(V_i, \upsilon_i)\}_{i\in I}$.

Proof By the assumption $T_V\phi_{vw}T_W^* = I_{\mathcal{H}}$, where ϕ_{vw} is given by (1.6). Also, it is not difficult to see that $\phi_{vw}^*\{f_i\} = \{\pi_{W_i}S_W^{-1}f_i\}$ for all $\{f_i\} \in \sum_{i\in I}\oplus V_i$. Hence,

$$\begin{aligned} \|I_{\mathcal{H}} - T_W\phi_{wv}T_V^*\| &= \|T_W\phi_{wv}T_V^* - T_W\phi_{vw}^*T_V^*\| \\ &\leq \|T_W\|\|T_V\|\|\phi_{wv} - \phi_{vw}^*\| \\ &\leq \|T_W\|\|T_V\|\|S_W^{-1} - S_V^{-1}\| < 1. \end{aligned}$$

□

The fusion frame W in Example 3.6 is not an alternate dual of V; however, a straightforward calculation shows that

$$\|S_V^{-1} - S_W^{-1}\| = \frac{1}{2}, \quad \|S_V\| = 2.$$

Hence, W is an approximate alternate dual of V by Theorem 3.7. It is worth noticing that, unlike discrete frames, $\{\psi_{wv}^{-1}W_i\}_{i=1}^3$ is not dual of $\{V_i\}_{i=1}^3$. Indeed, $\psi_{wv}^{-1} = 2I_{\mathcal{H}}$ and so

$$\sum_{i\in I}\pi_{\psi_{wv}^{-1}W_i}S_V^{-1}\pi_{V_i} = \frac{1}{2}I_{\mathcal{H}}.$$

References

[1] Amiri Z, Dehghan MA, Rahimi E. Subfusion frames. Abstr Appl Anal 2012; 2012: 1-12.

[2] Asgari MS. On the Riesz fusion bases in Hilbert spaces. Egypt Math Soc 2013; 21: 79-86.

[3] Casazza PG, Kutyniok G, Lammers MC. Duality principles in frame theory. J Fourier Anal Appl 2004; 10: 383-408.

[4] Casazza PG, Kutyniok G. Frames of subspaces. Contemp Math 2004; 345: 87-114.

[5] Casazza PG, Kutyniok G, Li S. Fusion frames and distributed processing. Appl Comput Harmon Anal 2008; 25: 114-132.

[6] Christensen O. Frames and Bases: An Introductory Course. Boston, MA, USA: Birkhäuser, 2008.

[7] Christensen O, Laugesen RS. Approximately dual frames in Hilbert spaces and applications to Gabor frames. Sampling Theory in Signal and Image Processing 2010; 9: 77-89.

[8] Găvruţa P. On the duality of fusion frames. J Math Anal Appl 2007; 333: 871-879.

[9] Heineken SB, Morillas PM. Properties of finite dual fusion frames. Linear Algebra Appl 2014; 453: 1-27.

[10] Heineken SB, Morillas PM, Benavente AM, Zakowicz MI. Dual fusion frames. Arch Math 2014; 103: 355-365.

[11] Iyengar SS, Brooks RR, editors. Distributed Sensor Networks. Baton Rouge, LA, USA: Chapman, 2005.

[12] Khosravi A, Mirzaee Azandaryani M. Approximate duality of g-frames in Hilbert spaces. Acta Math Sci Ser B Engl Ed 2014; 34: 639-652.

[13] Leng J, Gue Q, Huang T. The duals of fusion frames for experimental data transmission coding of energy physics. Advances in High Energy Physics 2013; 2013: 837129.

[14] Rozell CJ, Jahnson DH. Analyzing the robustness of redundant population codes in sensory and feature extraction systems. Neurocomputing 2006; 69: 1215-1218.