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Character analogue of the Boole summation formula with applications

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Abstract: In this paper, we present the character analogue of the Boole summation formula. Using this formula, an integral representation is derived for the alternating Dirichlet L-function and its derivative is evaluated at $s=0$. Some applications of the character analogue of the Boole summation formula and the integral representation are given about the alternating Dirichlet L-function. Moreover, the reciprocity formulas for two new arithmetic sums, arising from the summation formulas, and for Hardy–Berndt sum $S_p(b,c:\chi)$ are proved.

Key words: Boole summation formula, Dirichlet L-function, Hardy–Berndt sum, Bernoulli and Euler polynomials

1. Introduction

The Euler–MacLaurin summation formula is a well-known formula from classical analysis giving a relation between the finite sum of values of a function and its integral. One of the generalizations of the Euler–MacLaurin summation formula is the character analogue due to Berndt [4], which is presented here in the following form:

Theorem 1.1 *([4, Theorem 4.1]) Let χ be a primitive character of modulus k with $k>1$. For $f \in C^{(l+1)}[\alpha,\beta]$, $-\infty<\alpha<\beta<\infty$*

$$\sum_{\alpha\le n\le\beta}{}' \chi(n) f(n) = \chi(-1)\sum_{j=0}^{l}\frac{(-1)^{j+1}}{(j+1)!}\left(\overline{B}_{j+1,\overline{\chi}}(\beta) f^{(j)}(\beta) - \overline{B}_{j+1,\overline{\chi}}(\alpha) f^{(j)}(\alpha)\right)$$

$$+\chi(-1)\frac{(-1)^{l}}{(l+1)!}\int_{\alpha}^{\beta}\overline{B}_{l+1,\overline{\chi}}(u) f^{(l+1)}(u)du,$$

where the dash indicates that if $n=\alpha$ or $n=\beta$, then only $\frac{1}{2}\chi(\alpha)f(\alpha)$ or $\frac{1}{2}\chi(\beta)f(\beta)$ is counted, respectively. Also, $\overline{B}_{p,\chi}(x)$ denotes the generalized Bernoulli function defined by (2.3).

The alternating version of the Euler–MacLaurin summation formula is known as the Boole summation formula ([24, 24.17.1–2]), as pointed out by Nörlund [25], which was also formulated by Euler.

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Theorem 1.2 *(Boole summation formula) For integers α, β, and l such that $\alpha < \beta$ and $l > 0$,*

$$2\sum_{n=\alpha}^{\beta-1}(-1)^n f(n) = \sum_{j=0}^{l-1}\frac{E_j(0)}{j!}\left((-1)^{\beta-1} f^{(j)}(\beta) + (-1)^{\alpha} f^{(j)}(\alpha)\right) + \frac{1}{(l-1)!}\int_{\alpha}^{\beta} f^{(l)}(x)\overline{E}_{l-1}(-x)\,dx,$$

where $f^{(l)}(x)$ is absolutely integrable over $[\alpha,\beta]$, and $\overline{E}_p(x)$ is the Euler function defined by (2.1).

To the authors' knowledge, the character generalization of the Boole summation formula is not available. In this paper, we first present the character analogue of the Boole summation formula as follows:

Theorem 1.3 *Let χ be a primitive character of modulus k with $k > 1$ odd. If $f \in C^{(l+1)}[\alpha,\beta]$, $-\infty < \alpha < \beta < \infty$, then*

$$2\sum_{\alpha<n<\beta}(-1)^n \chi(n) f(n) = \chi(-1)\sum_{j=0}^{l}\frac{(-1)^j}{j!}\left(\overline{E}_{j,\overline{\chi}}(\beta) f^{(j)}(\beta) - \overline{E}_{j,\overline{\chi}}(\alpha) f^{(j)}(\alpha)\right) - \chi(-1)\frac{(-1)^l}{l!}\int_{\alpha}^{\beta}\overline{E}_{l,\overline{\chi}}(t) f^{(l+1)}(t)dt,$$

where $\overline{E}_{l,\chi}(t)$ is the generalized Euler function defined by (2.4).

Later, we give applications of this formula for two subjects. The first one is about the alternating Dirichlet L-function. For $a \neq -1,-2,-3,\ldots$, let $\ell(s,a,\chi)$ denote the alternating Dirichlet L-function

$$\ell(s,a,\chi) = \sum_{n=1}^{\infty}(-1)^n\frac{\chi(n)}{(n+a)^s}, \quad \mathrm{Re}(s) > 0,$$

which can be written in terms of Hurwitz zeta function $\zeta(s,a)$ as

$$\ell(s,a,\chi) = (2k)^{-s}\sum_{j=1}^{2k-1}(-1)^j\chi(j)\zeta\left(s,\frac{a+j}{2k}\right)$$

for $\mathrm{Re}(s) > 1$. Also let

$$\ell_s(x,a,\chi) = \sum_{1\le n\le x}(-1)^n\chi(n)(n+a)^s, \quad x \ge 0.$$

Then the integral representations for $\ell(s,a,\chi)$ and $\ell_s(x,a,\chi)$ are derived as in the following.

Theorem 1.4 *Let χ be a primitive character of modulus k with $k>1$ odd. For $l\geq 0$ with $l>\operatorname{Re}(s)$ and for any $x\geq 0$, we have the integral representation*

$$2\ell_s(x,a,\chi)=\chi(-1)\sum_{j=0}^{l}(-1)^j\frac{(s)_j}{j!}\overline{E}_{j,\overline{\chi}}(x)(x+a)^{s-j}+2\ell(-s,a,\chi)$$
$$-\frac{(s)_{l+1}}{l!}\int_x^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt,$$

where $(s)_j=s(s-1)\cdots(s-j+1)$ with $(s)_0=1$.

Moreover, for $x=0$ we have

$$2\ell(-s,a,\chi)=\sum_{j=0}^{l}\frac{(s)_j}{j!}E_{j,\overline{\chi}}(0)a^{s-j}+\frac{(s)_{l+1}}{l!}\int_0^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt.$$

Furthermore, some formulas, such as the character analogue of Lerch's formula for the Hurwitz zeta function (see (4.9)) and the character analogues of Stirling's formula for $\log\Gamma(a)$ and of the Weierstrass product for $\Gamma(a)$ (see Propositions 4.4 and 4.7 below, respectively) are deduced via Theorems 1.3 and 1.4.

The second is about the Hardy–Berndt sums. Let us mention that, utilizing the summation formulas, alternative proofs of the reciprocity formulas of certain Dedekind sums and their analogues were offered in [9, 10, 14, 15, 18]. Here we reveal that new arithmetic sums obeying the reciprocity law may be defined by the summation formulas mentioned above. We describe two such sums as

$$S_p^{(1)}(b,c:\chi)=\sum_{n=0}^{ck-1}(-1)^n\,\overline{E}_{p,\overline{\chi}}\left(\frac{bn}{c}\right),$$
$$S_p^{(2)}(b,c:\chi)=\sum_{n=1}^{ck}(-1)^n\,\chi(n)\,\overline{E}_p\left(\frac{bn}{c}\right)$$

and prove the following reciprocity formula.

Theorem 1.5 *Let b and c be positive integers with $(b+c)$ odd and $\chi(-1)(-1)^p=1$. Then the following reciprocity formula holds:*

$$c^pS_p^{(1)}(b,c:\overline{\chi})+b^pS_p^{(2)}(c,b:\chi)=2\sum_{j=0}^{p}\binom{p}{j}c^jb^{p-j}\overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-j}(0).$$

In fact, these sums are generalizations of the Hardy–Berndt sum [5]

$$S(b,c)=\sum_{n=1}^{c-1}(-1)^{n+1+[bn/c]}.$$

For various generalizations and properties of Hardy–Berndt sums, the reader may consult [5, 6, 8, 10, 13, 16, 17, 20, 21, 23, 26–30] and [29, 30] for the relation between $S(b,c)$ and the Dirichlet L-function. One of the generalizations of $S(b,c)$ has been given by [10]

$$S_p(b,c:\chi)=\sum_{n=1}^{ck}\chi(n)\overline{B}_{p,\overline{\chi}}\left(\frac{b+ck}{2c}n\right),$$

and the corresponding reciprocity formula is proved via transformation formulas. Here, utilizing Theorem 1.3, we give a new proof for the following reciprocity formula by refining the conditions.

Theorem 1.6 *Let $p>1$ be odd and let b and c be positive integers with $(b+c)$ odd. Then the following reciprocity formula holds:*

$$\begin{aligned}&\overline{\chi}(-2)\,bc^pS_p(b,c:\chi)+\chi(-2)\,cb^pS_p(c,b:\overline{\chi})\\&=\frac{p}{2^{p+1}}\sum_{j=1}^{p}(-1)^j\binom{p-1}{j-1}c^jb^{p+1-j}\overline{E}_{j-1,\chi}(0)\,\overline{E}_{p-j,\overline{\chi}}(0).\end{aligned}$$

The remainder of this paper is organized as follows: Section 2 is the preliminary section where we give definitions and known results needed. In Section 3, we prove Theorem 1.3 and Theorem 1.4. Some applications of the integral representation and the character analogue of the Boole summation formula are given in Section 4. The final section is devoted to proving the reciprocity formulas for the Hardy–Berndt sums mentioned above via summation formulas.

2. Preliminaries

The Bernoulli and Euler polynomials $B_n(x)$ and $E_n(x)$ are defined by means of the generating functions [2]

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (|t|<2\pi),$$

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (|t|<\pi),$$

respectively. In particular, the rational numbers $B_n=B_n(0)$ and integers $E_n=2^nE_n(1/2)$ are called classical Bernoulli numbers and Euler numbers, respectively.

For $0\le x<1$ and $m\in\mathbb{Z}$, the Bernoulli functions $\overline{B}_n(x)$ are defined by

$$\overline{B}_n(x+m)=B_n(x)\ \text{when}\ n\neq 1\ \text{or}\ x\neq 0,\ \text{and}\ \overline{B}_1(m)=\overline{B}_1(0)=0$$

and the Euler functions $\overline{E}_n(x)$ are defined by [11]

$$\overline{E}_n(x+m)=(-1)^m\,\overline{E}_n(x)\ \text{and}\ \overline{E}_n(x)=E_n(x). \tag{2.1}$$

The Bernoulli functions satisfy the Raabe or multiplication formula

$$r^{n-1}\sum_{j=0}^{r-1}\overline{B}_n\left(x+\frac{j}{r}\right)=\overline{B}_n(rx)$$

and also the following property is valid for even r:

$$r^{n-1}\sum_{j=0}^{r-1}(-1)^j\,\overline{B}_n\left(\frac{x+j}{r}\right)=-\frac{n}{2}\overline{E}_{n-1}(x)\,. \tag{2.2}$$

$\overline{B}_{m,\chi}(x)$ denotes the generalized Bernoulli function, with period k, defined by Berndt [4]. We will often use the following property that can confer as a definition:

$$\overline{B}_{m,\chi}(x)=k^{m-1}\sum_{j=0}^{k-1}\overline{\chi}(j)\,\overline{B}_m\left(\frac{j+x}{k}\right),\ m\geq 1. \tag{2.3}$$

For convenience with the definition of $\overline{B}_{m,\chi}(x)$, let the character Euler function $\overline{E}_{m,\chi}(x)$ be defined by

$$\overline{E}_{m,\chi}(x)=k^{m}\sum_{j=0}^{k-1}(-1)^j\,\overline{\chi}(j)\,\overline{E}_m\left(\frac{j+x}{k}\right),\ m\geq 0, \tag{2.4}$$

for odd k, the modulus of χ.

We list some properties that we need in the sequel:

$$\frac{d}{dx}\overline{E}_m(x)=m\overline{E}_{m-1}(x)\,,\ m>1, \tag{2.5}$$

$$\frac{d}{dx}\overline{E}_{m,\chi}(x)=m\overline{E}_{m-1,\chi}(x)\,,\ m\geq 1, \tag{2.6}$$

$$\overline{E}_{m,\chi}(-x)=(-1)^{m-1}\chi(-1)\,\overline{E}_{m,\chi}(x)\,, \tag{2.7}$$

$$\overline{E}_{m,\chi}(x+nk)=(-1)^{n}\,\overline{E}_{m,\chi}(x)\,. \tag{2.8}$$

In the sequel, unless otherwise stated, we assume that χ is a primitive character of modulus k with $k>1$ odd.

3. Proofs of Theorems 1.3 and 1.4

3.1. Proof of Theorem 1.3

First we write

$$\begin{aligned}\sum_{\alpha<n<\beta}(-1)^n\chi(n)\,f(n)&=\sum_{\alpha<2n<\beta}\chi(2n)\,f(2n)-\sum_{\alpha<2n+1<\beta}\chi(2n+1)\,f(2n+1)\\&=2\chi(2)\sum_{\alpha/2<n<\beta/2}\chi(n)\,f(2n)-\sum_{\alpha<n<\beta}\chi(n)\,f(n).\end{aligned}$$

Applying Theorem 1.1 on the right-hand side, one has

$$\begin{aligned}
&\sum_{\alpha<n<\beta}(-1)^n\chi(n)f(n)\\
&=\chi(-1)\sum_{j=0}^{l}\frac{(-1)^{j+1}}{(j+1)!}\left(\left(2^{j+1}\chi(2)\overline{B}_{j+1,\overline{\chi}}\left(\frac{\beta}{2}\right)-\overline{B}_{j+1,\overline{\chi}}(\beta)\right)f^{(j)}(\beta)\right.\\
&\qquad\left.-\left(2^{j+1}\chi(2)\overline{B}_{j+1,\overline{\chi}}\left(\frac{\alpha}{2}\right)-\overline{B}_{j+1,\overline{\chi}}(\alpha)\right)f^{(j)}(\alpha)\right)\\
&\quad+\chi(-1)\frac{(-1)^l}{(l+1)!}\int_{\alpha}^{\beta}\left(2^{l+1}\chi(2)\overline{B}_{l+1,\overline{\chi}}\left(\frac{u}{2}\right)-\overline{B}_{l+1,\overline{\chi}}(u)\right)f^{(l+1)}(u)du.
\end{aligned}\tag{3.1}$$

On the other hand, taking $x\to x/2$ and $r=2$ in

$$\chi(r)r^{1-m}\overline{B}_{m,\chi}(rx)=\sum_{j=0}^{r-1}\overline{B}_{m,\chi}\left(x+\frac{jk}{r}\right),\quad (r,k)=1$$

([10, Eq. (3.13)]) gives

$$\overline{B}_{m,\chi}\left(\frac{x}{2}\right)+\overline{B}_{m,\chi}\left(\frac{x+k}{2}\right)=2^{1-m}\chi(2)\overline{B}_{m,\chi}(x).\tag{3.2}$$

By using (2.3) and (2.2) for $r=2$, we can write

$$\overline{B}_{m,\chi}\left(\frac{x}{2}\right)-\overline{B}_{m,\chi}\left(\frac{x+k}{2}\right)=-\frac{m}{2^m}k^{m-1}\sum_{v=0}^{k-1}\overline{\chi}(v)\overline{E}_{m-1}\left(\frac{2v+x}{k}\right).\tag{3.3}$$

Employing basic manipulations, (3.3) becomes

$$\begin{aligned}
&k^{m-1}\left\{\sum_{v=0}^{\frac{k-1}{2}}\overline{\chi}(2v)\overline{E}_{m-1}\left(\frac{2v+x}{k}\right)+\sum_{v=\frac{k+1}{2}}^{k-1}\overline{\chi}(2v)\overline{E}_{m-1}\left(\frac{2v+x}{k}\right)\right\}\\
&=k^{m-1}\left\{\sum_{v=0}^{\frac{k-1}{2}}\overline{\chi}(2v)\overline{E}_{m-1}\left(\frac{2v+x}{k}\right)-\sum_{v=0}^{\frac{k-3}{2}}\overline{\chi}(2v+1)\overline{E}_{m-1}\left(\frac{2v+1+x}{k}\right)\right\}\\
&=k^{m-1}\sum_{v=0}^{k-1}(-1)^v\overline{\chi}(v)\overline{E}_{m-1}\left(\frac{v+x}{k}\right)\\
&=\overline{E}_{m-1,\chi}(x),
\end{aligned}\tag{3.4}$$

where we have used (2.1) and (2.4). Thus, combining (3.2) and (3.4) leads to

$$2^m\chi(2)\overline{B}_{m,\overline{\chi}}\left(\frac{x}{2}\right)-\overline{B}_{m,\overline{\chi}}(x)=-\frac{m}{2}\overline{E}_{m-1,\overline{\chi}}(x).\tag{3.5}$$

Substituting (3.5) in (3.1) completes the proof.

3.2. Proof of Theorem 1.4

The method used here have already been employed by Kanemitsu et al. [19] for the Euler–MacLaurin summation formula to obtain integral representations for the Hurwitz zeta function and its partial sum.

For $\alpha = 0$ and $\beta = x$, let $f(t) = (t+a)^s$ in Theorem 1.3. Then, from (2.7),

$$
\begin{aligned}
2\ell_s(x,a,\chi) &= 2 \sum_{0\le n\le x}{}^{'} (-1)^n \chi(n)(n+a)^s \\
&= \chi(-1) \sum_{j=0}^{l} (-1)^j \frac{(s)_j}{j!} \left(\overline{E}_{j,\overline{\chi}}(x)(x+a)^{s-j} - \overline{E}_{j,\overline{\chi}}(0)a^{s-j} \right) \\
&\quad + \frac{(s)_{l+1}}{l!} \int_0^x \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt.
\end{aligned}
$$

Since

$$\left|\overline{E}_{l,\chi}(t)\right| \le 4\frac{l!}{(\pi/k)^{l+1}}\zeta(l+1), \;\; l \ge 1,$$

the integral

$$\int_0^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt$$

is absolutely convergent for $\mathrm{Re}(s) < l$. Thus, we may write

$$
\begin{aligned}
&2\ell_s(x,a,\chi) \\
&= \chi(-1)\sum_{j=0}^{l}(-1)^j \frac{(s)_j}{j!}\overline{E}_{j,\overline{\chi}}(x)(x+a)^{s-j} - \chi(-1)\sum_{j=0}^{l}(-1)^j\frac{(s)_j}{j!}E_{j,\overline{\chi}}(0)a^{s-j} \\
&\quad + \frac{(s)_{l+1}}{l!}\int_0^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt - \frac{(s)_{l+1}}{l!}\int_x^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt. \qquad (3.6)
\end{aligned}
$$

Now, for $\mathrm{Re}(s) < 0$, letting x tend to ∞ in (3.6), we arrive at

$$
\begin{aligned}
2\ell(-s,a,\chi) &= -\chi(-1)\sum_{j=0}^{l}(-1)^j\frac{(s)_j}{j!}E_{j,\overline{\chi}}(0)a^{s-j} \\
&\quad + \frac{(s)_{l+1}}{l!}\int_0^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt, \qquad (3.7)
\end{aligned}
$$

where the integral converges absolutely for $\mathrm{Re}(s) < l$ and represents an analytic function. Substituting (3.7) in

(3.6) gives

$$2\ell_s(x,a,\chi) = \chi(-1)\sum_{j=0}^{l}(-1)^j\frac{(s)_j}{j!}\overline{E}_{j,\overline{\chi}}(x)(x+a)^{s-j} + 2\ell(-s,a,\chi) - \frac{(s)_{l+1}}{l!}\int_x^\infty \overline{E}_{l,\overline{\chi}}(-t)(t+a)^{s-l-1}\,dt, \tag{3.8}$$

for $\mathrm{Re}(s) < l$ and $x \geq 0$.

Writing $x = 0$ in (3.8) yields

$$2\ell(-s,a,\chi) = \sum_{j=0}^{l}\frac{(s)_j}{j!}E_{j,\overline{\chi}}(0)\,a^{s-j} + \frac{(s)_{l+1}}{l!}\int_0^\infty \frac{\overline{E}_{l,\overline{\chi}}(-t)}{(t+a)^{l+1-s}}dt, \tag{3.9}$$

which is valid for $\mathrm{Re}(s) < l$.

4. Some consequences

This section is concerned with some formulas about the alternating Dirichlet L-function and counterparts of Examples 6–10 of [4].

4.1. Around the alternating Dirichlet L-function

It is clear from (3.9) that for $l = p$ and $s = p-1$ with $0 < a < 1$,

$$2\ell(1-p,a,\chi) = \sum_{j=0}^{p-1}\binom{p-1}{j}E_{j,\overline{\chi}}(0)\,a^{p-1-j} = E_{p-1,\overline{\chi}}(a),\ p \geq 1.$$

Also, for $\mathrm{Re}(s) > 0 = l$

$$2\ell(s,a,\chi) = -\chi(-1)E_{0,\overline{\chi}}(0)\,a^{-s} + \chi(-1)\,s\int_0^\infty \frac{\overline{E}_{0,\overline{\chi}}(t)}{(t+a)^{1+s}}dt$$

and for $\mathrm{Re}(s) > -1$, $(l = 1)$

$$2\ell(s,a,\chi) = E_{0,\overline{\chi}}(0)\,a^{-s} - sE_{1,\overline{\chi}}(0)\,a^{-s-1} + s(s+1)\chi(-1)\int_0^\infty \frac{\overline{E}_{1,\overline{\chi}}(t)}{(t+a)^{2+s}}dt. \tag{4.1}$$

Differentiating both sides of (4.1) with respect to s at $s = 0$ gives

$$2\frac{d}{ds}\ell(s,a,\chi)\big|_{s=0} = 2\ell'(0,a,\chi) = -E_{0,\overline{\chi}}(0)\log a - \frac{1}{a}\overline{E}_{1,\overline{\chi}}(0) + \chi(-1)\int_0^\infty \frac{\overline{E}_{1,\overline{\chi}}(x)}{(x+a)^2}dx. \tag{4.2}$$

Similar results for $\ell(s,\chi)=\ell(s,0,\chi)$ can be achieved by applying Theorem 1.3 to $f(x)=x^{-s}$, $\operatorname{Re}(s)>0$, where $\alpha=1$ and $\beta=2kN$, $N\in\mathbb{N}$. Following the arguments in the proof of (3.9) and then letting $N\to\infty$ give rise to

$$2\ell(s,\chi)+2=-\chi(-1)\sum_{j=0}^{l}\frac{s(s+1)\ldots(s+j-1)}{j!}\overline{E}_{j,\overline{\chi}}(1)$$
$$+\chi(-1)\frac{s(s+1)\ldots(s+l)}{l!}\int_1^\infty\frac{\overline{E}_{l,\overline{\chi}}(x)}{x^{s+l+1}}dx,$$

where the integral is analytic for $\operatorname{Re}(s)>-l$. In particular, for $l=1$,

$$2\ell(s,\chi)=-2-\chi(-1)\overline{E}_{0,\overline{\chi}}(1)-\chi(-1)s\overline{E}_{1,\overline{\chi}}(1)+\chi(-1)s(s+1)\int_1^\infty\frac{\overline{E}_{1,\overline{\chi}}(x)}{x^{s+2}}dx.$$

Differentiating both sides of the equality above with respect to s at $s=0$ gives

$$2\ell'(0,\chi)=-\chi(-1)\overline{E}_{1,\overline{\chi}}(1)+\chi(-1)\int_1^\infty\frac{\overline{E}_{1,\overline{\chi}}(x)}{x^2}dx. \tag{4.3}$$

Observe that the integrals in (4.2) and (4.3) can be obtained from Theorem 1.3 by taking the logarithm function. Thus, we may establish a connection between generalized Euler functions and some identities for logarithmic means.

Proposition 4.1 *As t tends to $+\infty$, we have the following asymptotic expansion:*

$$2\sum_{1\le n<t}(-1)^n\chi(n)\log(t/n)\sim 2\ell'(0,\chi)+\ell(0,\chi)\log t+\chi(-1)\sum_{j=1}^{\infty}\frac{\overline{E}_{j,\overline{\chi}}(t)}{jt^j}.$$

Proof Let $f(x)=\log(t/x)$, $\alpha=1$ and $\beta=t$ and $l=1$ in Theorem 1.3. Then

$$\begin{aligned}&2\chi(-1)\sum_{1<n<t}(-1)^n\chi(n)\log(t/n)\\&=2\chi(-1)\sum_{1\le n<t}(-1)^n\chi(n)\log(t/n)+2\chi(-1)\log t\\&=-\overline{E}_{1,\overline{\chi}}(1)-\overline{E}_{0,\overline{\chi}}(1)\log t+\frac{\overline{E}_{1,\overline{\chi}}(t)}{t}+\int_1^t\frac{\overline{E}_{1,\overline{\chi}}(x)}{x^2}dx\\&=2\chi(-1)\ell'(0,\chi)-\overline{E}_{0,\overline{\chi}}(1)\log t+\frac{\overline{E}_{1,\overline{\chi}}(t)}{t}-\int_t^\infty\frac{\overline{E}_{1,\overline{\chi}}(x)}{x^2}dx,\end{aligned} \tag{4.4}$$

where we have used (4.3). Using that $\overline{E}_{0,\overline{\chi}}(1)=\overline{E}_{0,\overline{\chi}}(0)-2\chi(-1)$ and $\ell(0,\chi)=\overline{E}_{0,\overline{\chi}}(0)$, and integrating by parts repeatedly with the use of (4.4), one arrives at the asymptotic formula. □

Remark 4.2 *The trivial estimate [4, Eq. (5.28)]*

$$\left|\overline{B}_{l,\chi}(x)\right| \leq 2\frac{l!\zeta(l)}{\sqrt{k}(2\pi/k)^{l}}$$

with (3.5) leads to

$$\left|\overline{E}_{l,\chi}(x)\right| \leq 4\frac{l!\zeta(l+1)}{(\pi/k)^{l+1}},$$

and thus, we have

$$\left|\frac{\overline{E}_{1,\overline{\chi}}(N)}{N}\right| \leq 4\frac{\zeta(2)}{N(\pi/k)^{2}} = \frac{2k^{2}}{3N} \quad \text{and} \quad \left|-\int_{N}^{\infty}\frac{\overline{E}_{1,\overline{\chi}}(x)}{x^{2}}dx\right| \leq \frac{2k^{2}}{3N}.$$

Here $\zeta(z)$ denotes the Riemann zeta function. Using these in (4.4) for $t = N \in \mathbb{N}$ gives

$$\sum_{n=1}^{N-1}(-1)^{n}\chi(n)\log(N/n) = \ell'(0,\chi) + \frac{1}{2}\ell(0,\chi)\log N + O\left(\frac{1}{N}\right), \text{ as } N \to \infty. \tag{4.5}$$

By setting $2kN$ instead of N, (4.5) may be stated as follows:

$$\sum_{n=1}^{2kN-1}(-1)^{n}\chi(n)\log n = -\ell'(0,\chi) + \frac{1}{2}\ell(0,\chi)\log(2kN) + O\left(\frac{1}{N}\right), \text{ as } N \to \infty.$$

We now apply Theorem 1.3 to the function $f(x) = \log(x+a)$, $-\pi < \arg a < \pi$, where $\alpha = 0$, $\beta = 2kN$, $N \in \mathbb{N}$, and $l = 1$ to obtain

$$\begin{aligned} &2\sum_{n=1}^{2kN}(-1)^{n}\chi(n)\log(n+a) \\ &= -\overline{E}_{0,\overline{\chi}}(0)\log(2kN+a) - \frac{\overline{E}_{1,\overline{\chi}}(0)}{2kN+a} \\ &\quad + \overline{E}_{0,\overline{\chi}}(0)\log a + \chi(-1)\frac{\overline{E}_{1,\overline{\chi}}(0)}{a} - \chi(-1)\int_{0}^{2Nk}\frac{\overline{E}_{1,\overline{\chi}}(x)}{(x+a)^{2}}dx. \end{aligned} \tag{4.6}$$

Gathering (4.4) and (4.6) for $t = 2kN$, $N \in \mathbb{N}$, then letting $N \to \infty$ and using (4.3), we find that

$$\begin{aligned} &-2\sum_{n=1}^{\infty}(-1)^{n}\chi(n)(\log n - \log(n+a)) \\ &= 2\ell'(0,\chi) + \chi(-1)\frac{\overline{E}_{1,\overline{\chi}}(0)}{a} + \overline{E}_{0,\overline{\chi}}(0)\log a - \chi(-1)\int_{0}^{\infty}\frac{\overline{E}_{1,\overline{\chi}}(x)}{(x+a)^{2}}dx. \end{aligned} \tag{4.7}$$

Note that the sum in (4.7) is reminiscent of the definition of the character analogue of the gamma function defined by Berndt [4, Definition 4]. This motivates us to give the following definition.

Definition 4.3 *Let χ be a real primitive character. Define*

$$\Gamma^*(a,\chi)=\prod_{n=1}^{\infty}\left(\frac{n}{n+a}\right)^{(-1)^n\chi(n)}.$$

In light of this definition, (4.7) becomes

$$2\log\Gamma^*(a,\chi)=-\overline{E}_{0,\overline{\chi}}(0)\log a-2\ell'(0,\chi)-\chi(-1)\frac{\overline{E}_{1,\overline{\chi}}(0)}{a}+\chi(-1)\int_0^{\infty}\frac{\overline{E}_{1,\overline{\chi}}(x)}{(x+a)^2}dx, \tag{4.8}$$

which shows that $\Gamma^*(a,\chi)$ is well defined and analytic for $-\pi<\arg a<\pi$.

Combining (4.2) and (4.8), we infer *Lerch's formula* for $\ell(s,a,\chi)$ as

$$\ell'(0,a,\chi)=\log\Gamma^*(a,\chi)+\ell'(0,\chi), \tag{4.9}$$

which is the character analogue of the familiar formula

$$\zeta'(0,z)=\log\Gamma(z)+\zeta'(0),$$

where $\Gamma(z)$ is the Euler gamma function.

Furthermore, in (4.8), integrating by parts repeatedly in view of (2.6), we arrive at the following asymptotic formula, the counterpart of [4, Proposition 5.3].

Proposition 4.4 (Stirling's formula for $\log\Gamma^*(a,\chi)$) *For $-\pi<\arg a<\pi$, as a tends to ∞,*

$$\log\Gamma^*(a,\chi)\sim-\frac{1}{2}\ell(0,\chi)\log a-\ell'(0,\chi)-\frac{\chi(-1)}{2}\sum_{j=1}^{\infty}\frac{\overline{E}_{j,\overline{\chi}}(0)}{ja^j},$$

where the principal branch of the logarithm is taken.

Next we write the integral in (4.2) as in the form

$$\begin{aligned}\int_0^{\infty}\frac{\overline{E}_{1,\overline{\chi}}(x)}{(x+a)^2}dx&=\sum_{n=0}^{\infty}\int_{2nk}^{2(n+1)k}\frac{\overline{E}_{1,\overline{\chi}}(x)}{(x+a)^2}dx\\&=\frac{1}{(2k)^2}\int_0^{2k}\overline{E}_{1,\overline{\chi}}(t)\sum_{n=0}^{\infty}\left(n+\frac{t+a}{2k}\right)^{-2}dt\\&=\frac{1}{(2k)^2}\int_0^{2k}\overline{E}_{1,\overline{\chi}}(t)\,\zeta\left(2,\frac{t+a}{2k}\right)dt.\end{aligned}$$

Therefore, (4.2) becomes

$$2\ell'(0,a,\chi)=-\overline{E}_{0,\overline{\chi}}(0)\log a-\frac{1}{a}\overline{E}_{1,\overline{\chi}}(0)+\frac{\chi(-1)}{(2k)^2}\int_0^{2k}\overline{E}_{1,\overline{\chi}}(t)\,\zeta\left(2,\frac{t+a}{2k}\right)dt. \tag{4.10}$$

Since

$$\frac{d^2}{dz^2}\log\Gamma(z)=\frac{d}{dz}\psi(z)=\zeta(2,z), \tag{4.11}$$

where $\psi(z)$ is the digamma function, the integral in (4.10) may be obtained from Theorem 1.3 by setting $f(x)=\log\Gamma((x+a)/2k)$, $\alpha=0$, $\beta=2k$, and $l=1$. Under these circumstances,

$$\begin{aligned}&2\sum_{n=0}^{2k-1}(-1)^n\chi(n)\log\Gamma\left(\frac{n+a}{2k}\right)\\&=-\overline{E}_{0,\overline{\chi}}(0)\log\frac{a}{2k}-\frac{1}{a}\overline{E}_{1,\overline{\chi}}(0)+\frac{\chi(-1)}{(2k)^2}\int_0^{2k}\overline{E}_{1,\overline{\chi}}(x)\zeta\left(2,\frac{x+a}{2k}\right)dx,\end{aligned} \tag{4.12}$$

where we have used that $\Gamma(z+1)=z\Gamma(z)$ and $\psi(z+1)-\psi(z)=1/z$. Assembling (4.10) and (4.12), we have

$$2\ell'(0,a,\chi)=-\overline{E}_{0,\overline{\chi}}(0)\log(2k)+2\sum_{n=1}^{2k-1}(-1)^n\chi(n)\log\Gamma\left(\frac{n+a}{2k}\right). \tag{4.13}$$

The following proposition shows that $\Gamma^*(a,\chi)$ is a quotient of ordinary gamma functions.

Proposition 4.5 *We have*

$$\Gamma^*(a,\chi)=\prod_{n=1}^{2k-1}\left(\frac{\Gamma((n+a)/2k)}{\Gamma(n/2k)}\right)^{(-1)^n\chi(n)}.$$

Proof From

$$\ell(s,2k,\chi)=\ell(s,\chi)-\sum_{n=1}^{2k-1}(-1)^n\chi(n)n^{-s},$$

it is seen that

$$\ell'(0,2k,\chi)=\ell'(0,\chi)+\sum_{n=1}^{2k-1}(-1)^n\chi(n)\log n. \tag{4.14}$$

Setting $a=2k$ in (4.13) and then comparing with (4.14) gives

$$\ell'(0,\chi)=-\frac{1}{2}\ell(0,\chi)\log(2k)+\sum_{n=1}^{2k-1}(-1)^n\chi(n)\log\Gamma\left(\frac{n}{2k}\right).$$

Substituting this in (4.9) and then combining with (4.13) leads to

$$\begin{aligned}\log\Gamma^*(a,\chi)&=\sum_{n=1}^{2k-1}(-1)^n\chi(n)\left(\log\Gamma\left(\frac{n+a}{2k}\right)-\log\Gamma\left(\frac{n}{2k}\right)\right)\\&=\sum_{n=1}^{2k-1}(-1)^n\chi(n)\log\left(\frac{\Gamma((n+a)/2k)}{\Gamma(n/2k)}\right),\end{aligned} \tag{4.15}$$

which is the desired result. □

Let us continue by differentiating both sides of (4.15) with respect to a. Then we have

$$\frac{d}{da}\log\Gamma^*(a,\chi)=\frac{1}{2k}\sum_{n=1}^{2k-1}(-1)^n\chi(n)\,\psi\left(\frac{n+a}{2k}\right), \tag{4.16}$$

by (4.11). For convenience with (4.11), the right-hand side of (4.16) can be denoted by $\psi^*(a,\chi)$, i.e.

$$\psi^*(a,\chi)=\frac{1}{2k}\sum_{n=1}^{2k-1}(-1)^n\chi(n)\,\psi\left(\frac{n+a}{2k}\right).$$

On the other hand, in light of (4.9), differentiating both sides of (4.2) with respect to a and then comparing with (4.1) for $s=1$, we see that

$$\ell(1,a,\chi)=-\psi^*(a,\chi)=-\frac{1}{2k}\sum_{n=1}^{2k-1}(-1)^n\chi(n)\,\psi\left(\frac{n+a}{2k}\right). \tag{4.17}$$

In general, for $m\geq 0$ we have

$$\frac{d^m}{da^m}\psi^*(a,\chi)=(-1)^{m+1}m!\,\ell(m+1,a,\chi), \tag{4.18}$$

which implies the following identity, viewed as the Taylor expansion of $\ell(s,a,\chi)$ in the second variable a.

Proposition 4.6 *For $|z|<1$ we have*

$$\sum_{m=2}^{\infty}\ell(m,a,\chi)\,z^{m-1}=\psi^*(a,\chi)-\psi^*(a-z,\chi). \tag{4.19}$$

Proof The statement follows from the Taylor expansion of $\psi^*(z,\chi)$ at $z=a$. □

The character analogue of the Weierstrass product representation of $\Gamma(s)$ can be derived from Definition 4.3 and also from Proposition 4.6.

Proposition 4.7 *We have for all s*

$$\Gamma^*(s,\chi)=e^{-s\ell(1,\chi)}\prod_{n=1}^{\infty}\left[(1+s/n)^{-1}e^{s/n}\right]^{(-1)^n\chi(n)}, \tag{4.20}$$

where the product converges uniformly on any compact set S that avoids the points $s=-n$, where n is a positive integer and $(-1)^n\chi(n)=1$.

Proof The proof from Definition 4.3 is exactly like the proof of Berndt [4, Proposition 5.4], so we omit it.

For the proof via Proposition 4.6, integrating (4.19) from 0 to s, we see that

$$\sum_{m=2}^{\infty}\ell(m,a,\chi)\frac{s^m}{m}=\log\Gamma^*(a-s,\chi)-\log\Gamma^*(a,\chi)+s\psi^*(a,\chi).$$

Taking $s \to -s$ and $a = 0$, we have

$$\sum_{m=2}^{\infty} \ell(m, 0, \chi) \frac{(-s)^m}{m} = \log \Gamma^*(s, \chi) - \log \Gamma^*(0, \chi) - s\psi^*(0, \chi)$$
$$= \log \Gamma^*(s, \chi) + s\ell(1, \chi). \tag{4.21}$$

The left-hand side of (4.21) is

$$\sum_{n=1}^{\infty} (-1)^n \chi(n) \sum_{m=2}^{\infty} \frac{1}{m} \left(-\frac{s}{n}\right)^m = \sum_{n=1}^{\infty} (-1)^n \chi(n) \left[\frac{s}{n} - \log\left(1 + \frac{s}{n}\right)\right], \tag{4.22}$$

where we have used that

$$\sum_{m=2}^{\infty} \frac{r^m}{m} = -r - \log(1 - r), \text{ for } |r| < 1.$$

Combining (4.21) and (4.22) gives (4.20). □

Note that another consequence of (4.17) with $\psi(1-x) - \psi(x) = \pi \cot \pi x$ is

$$2\ell(m+1, \chi) = -\frac{(-\pi/2k)^{m+1}}{m!} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \cot^{(m)}\left(\frac{\pi n}{2k}\right), \ m \geq 0, \tag{4.23}$$

when $\chi(-1)(-1)^{m+1} = 1$. Indeed, it is easy to see that for $0 \leq a < 1$,

$$\ell(1, a, \chi) - \chi(-1)\ell(1, -a, \chi)$$
$$= -\frac{1}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \left\{\psi\left(\frac{n+a}{2k}\right) - \psi\left(1 - \frac{n+a}{2k}\right)\right\}$$
$$= \frac{\pi}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \cot\left(\pi \frac{n+a}{2k}\right). \tag{4.24}$$

Now (4.23) follows from (4.18) and (4.24) for $\chi(-1)(-1)^{m+1} = 1$ and $a = 0$. In particular,

$$2\ell(1, \chi) = \frac{\pi}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \cot\left(\frac{\pi n}{2k}\right), \text{ for odd } \chi,$$

$$2\ell(2, \chi) = \left(\frac{\pi}{2k}\right)^2 \sum_{n=1}^{2k-1} (-1)^n \frac{\chi(n)}{\sin^2\left(\frac{\pi n}{2k}\right)}, \text{ for even } \chi,$$

$$2\ell(3, \chi) = \left(\frac{\pi}{2k}\right)^3 \sum_{n=1}^{2k-1} (-1)^n \chi(n) \frac{\cos\left(\frac{\pi n}{2k}\right)}{\sin^3\left(\frac{\pi n}{2k}\right)}, \text{ for odd } \chi,$$

$$2\ell(4, \chi) = \frac{1}{3}\left(\frac{\pi}{2k}\right)^4 \sum_{n=1}^{2k-1} (-1)^n \chi(n) \left(\frac{2}{\sin^4\left(\frac{\pi n}{2k}\right)} + \frac{\cos\left(\frac{\pi n}{k}\right)}{\sin^4\left(\frac{\pi n}{2k}\right)}\right), \text{ for even } \chi,$$

which are analogues of Eqs. (5.9)–(5.12) of Alkan [1]. Such sums and many other ones can be found in [3, 7, 22].

4.2. Counterparts of Examples 6–10 of [4]

In this part, we constitute $f(x)$ in Theorem 1.3 in order to give some formulas, the counterparts of Examples 6–10 of [4].

- Let $f(x) = e^{xt}$, $\alpha = 0$ and $\beta = k$. Then

$$2\sum_{n=0}^{k} (-1)^n \chi(n) e^{nt} = \chi(-1) \sum_{j=0}^{l} (-1)^{j+1} \overline{E}_{j,\overline{\chi}}(0) \frac{t^j}{j!} \left(e^{kt}+1\right) - R_l,$$

where

$$\begin{aligned} |R_l| &\le \frac{\left|t^{l+1}\right|}{l!} \int_0^k \left|\overline{E}_{l,\overline{\chi}}(x) e^{xt}\right| dx \\ &\le 4ke^{kt} \frac{\left|t^{l+1}\right|}{(\pi/k)^{l+1}} \zeta(l+1) \to 0 \text{ as } l \to \infty \text{ for } |t| < \pi/k. \end{aligned} \tag{4.25}$$

Thus, we have the generating function for the number $E_{j,\overline{\chi}}(0)$ as

$$\sum_{n=0}^{k-1} (-1)^n \chi(n) \frac{2e^{nt}}{e^{kt}+1} = \sum_{j=0}^{\infty} E_{j,\overline{\chi}}(0) \frac{t^j}{j!}.$$

- Let $f(x) = \cos(xt)$, $\alpha = 0$ and $\beta = k$. It is obvious from (2.7) that $\overline{E}_{j,\chi}(0) = 0$ if χ and j have the same parity. If χ is odd, then

$$2\sum_{n=0}^{k-1} (-1)^n \chi(n) \cos(nt) = \chi(-1) \sum_{j=0}^{l} \frac{(-1)^{2j+1}}{(2j)!} \overline{E}_{2j,\overline{\chi}}(0) (\cos(kt)+1) t^{2j} (-1)^j - R_l$$

where, as in (4.25), R_l tends to 0 as $l \to \infty$ for $|t| < \pi/k$. Therefore, we have

$$\frac{2\sum_{n=1}^{k-1} (-1)^n \chi(n) \cos(nt)}{\cos(kt)+1} = \sum_{j=0}^{\infty} (-1)^j E_{2j,\overline{\chi}}(0) \frac{t^{2j}}{(2j)!}, \text{ for } |t| < \frac{\pi}{k}.$$

If χ is even, then similarly

$$\frac{2\sum_{n=1}^{k-1} (-1)^n \chi(n) \cos(nt)}{\sin(kt)} = -\sum_{j=0}^{\infty} (-1)^j E_{2j+1,\overline{\chi}}(0) \frac{t^{2j+1}}{(2j+1)!}, \text{ for } |t| < \frac{\pi}{k}.$$

- Let $f(x) = \sin(xt)$, $\alpha = 0$ and $\beta = k$. If χ is odd, then for $|t| < \pi/k$

$$\frac{2\sum_{n=1}^{k-1} (-1)^n \chi(n) \sin(nt)}{\sin(kt)} = \sum_{j=0}^{\infty} (-1)^j E_{2j,\overline{\chi}}(0) \frac{t^{2j}}{(2j)!}$$

and if χ is even

$$\frac{2\sum_{n=1}^{k-1} (-1)^n \chi(n) \sin(nt)}{\cos(kt)+1} = \sum_{j=0}^{\infty} (-1)^j E_{2j+1,\overline{\chi}}(0) \frac{t^{2j+1}}{(2j+1)!}.$$

- In a similar way, it can be seen that if χ is odd, then for $|t| < \pi/k$

$$\frac{2\sum_{n=0}^{k}(-1)^n\chi(n)\cosh(nt)}{\cosh(kt)+1} = \frac{2\sum_{n=1}^{k-1}(-1)^n\chi(n)\sinh(nt)}{\sinh(kt)} = \sum_{j=0}^{\infty}E_{2j,\overline{\chi}}(0)\frac{t^{2j}}{(2j)!}$$

and if χ is even

$$\frac{2\sum_{n=0}^{k}(-1)^n\chi(n)\cosh(nt)}{\sinh(kt)} = \frac{2\sum_{n=0}^{k}(-1)^n\chi(n)\sinh(nt)}{\cosh(kt)+1} = \sum_{j=0}^{\infty}E_{2j+1,\overline{\chi}}(0)\frac{t^{2j+1}}{(2j+1)!}.$$

5. Proofs of reciprocity theorems

Proof [Proof of Theorem 1.5]Let $f(x) = \overline{E}_{p,\chi}(xb/c)$, $\alpha = 0$, and $\beta = ck$ in Theorem 1.2. By virtue of (2.6), for $1 \le l \le p$, one has

$$2\sum_{n=0}^{ck-1}(-1)^n\overline{E}_{p,\chi}\left(n\frac{b}{c}\right) = \sum_{j=0}^{l-1}\frac{E_j(0)}{j!}\left(\frac{b}{c}\right)^j\frac{p!}{(p-j)!}\left((-1)^{ck-1}\overline{E}_{p-j,\chi}(bk)+\overline{E}_{p-j,\chi}(0)\right) + \frac{p!}{(l-1)!(p-l)!}\left(\frac{b}{c}\right)^l\int_0^{ck}\overline{E}_{p-l,\chi}\left(\frac{b}{c}x\right)\overline{E}_{l-1}(-x)\,dx.$$

For odd $b+c$, with the use of (2.8), one can write

$$2\sum_{n=0}^{ck-1}(-1)^n\overline{E}_{p,\chi}\left(n\frac{b}{c}\right) = 2\sum_{j=0}^{l-1}\left(\frac{b}{c}\right)^j\binom{p}{j}E_j(0)\overline{E}_{p-j,\chi}(0) + l\binom{p}{l}\left(\frac{b}{c}\right)^l(-1)^l c\int_0^{k}\overline{E}_{p-l,\chi}(bx)\overline{E}_{l-1}(cx)\,dx. \tag{5.1}$$

Now, let $f(x)=\overline{E}_p(xc/b)$, $\alpha=0$, and $\beta=bk$ in Theorem 1.3. Using (2.5),

$$\begin{aligned}
&2\chi(-1)\sum_{n=0}^{bk}(-1)^n\chi(n)\overline{E}_p\left(n\frac{c}{b}\right)\\
&=\sum_{j=0}^{l}\frac{(-1)^j}{j!}\left(\frac{c}{b}\right)^j\frac{p!}{(p-j)!}\left(\overline{E}_{j,\overline{\chi}}(bk)\,\overline{E}_{p-j}(ck)-\overline{E}_{j,\overline{\chi}}(0)\overline{E}_{p-j}(0)\right)\\
&\quad-\frac{(-1)^l}{l!}\frac{p!}{(p-l-1)!}\left(\frac{c}{b}\right)^{l+1}\int_0^{bk}\overline{E}_{l,\overline{\chi}}(x)\,\overline{E}_{p-l-1}\left(x\frac{c}{b}\right)dx\\
&=\sum_{j=0}^{l}(-1)^j\binom{p}{j}\left(\frac{c}{b}\right)^j\overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-j}(0)\left((-1)^{b+c}-1\right)\\
&\quad-(-1)^l p\binom{p-1}{l}\left(\frac{c}{b}\right)^{l+1}b\int_0^{k}\overline{E}_{l,\overline{\chi}}(bx)\,\overline{E}_{p-l-1}(cx)\,dx,
\end{aligned}$$

for $0\leq l\leq p-2$. Then, for odd $b+c$, we have

$$\begin{aligned}
2\chi(-1)\sum_{n=0}^{bk}(-1)^n\chi(n)\overline{E}_p\left(n\frac{c}{b}\right)&=2\sum_{j=0}^{l}(-1)^{j+1}\binom{p}{j}\left(\frac{c}{b}\right)^j\overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-j}(0)\\
&\quad-(-1)^l p\binom{p-1}{l}\left(\frac{c}{b}\right)^{l+1}b\int_0^{k}\overline{E}_{l,\overline{\chi}}(bx)\,\overline{E}_{p-l-1}(cx)\,dx.
\end{aligned}\tag{5.2}$$

Taking $\chi\to\overline{\chi}$ and $l=2$ in (5.1) leads to

$$\begin{aligned}
S_p^{(1)}(b,c:\overline{\chi})&=2\sum_{n=0}^{ck-1}(-1)^n\,\overline{E}_{p,\overline{\chi}}\left(n\frac{b}{c}\right)\\
&=2E_0(0)\,\overline{E}_{p,\overline{\chi}}(0)+2\frac{bp}{c}E_1(0)\,\overline{E}_{p-1,\overline{\chi}}(0)\\
&\quad+p(p-1)\left(\frac{b}{c}\right)^2c\int_0^{k}\overline{E}_{p-2,\overline{\chi}}(bx)\,\overline{E}_1(cx)\,dx.
\end{aligned}\tag{5.3}$$

Taking $l=p-2$ in (5.2) yields

$$\begin{aligned}
S_p^{(2)}(c,b:\chi)&=2\sum_{n=1}^{bk}(-1)^n\chi(n)\,\overline{E}_p\left(n\frac{c}{b}\right)\\
&=2\chi(-1)\sum_{j=0}^{p-2}(-1)^{j+1}\binom{p}{j}\left(\frac{c}{b}\right)^j\overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-j}(0)\\
&\quad-(-1)^p\chi(-1)\,p(p-1)\left(\frac{c}{b}\right)^{p-1}b\int_0^{k}\overline{E}_{p-2,\overline{\chi}}(bx)\,\overline{E}_1(cx)\,dx.
\end{aligned}\tag{5.4}$$

Combining (5.3) and (5.4), one obtains that

$$c^p S_p^{(1)}(b,c:\overline{\chi}) + b^p S_p^{(2)}(c,b:\chi) = 2\sum_{j=0}^{p}\binom{p}{j} c^j b^{p-j}\overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-j}(0),$$

for odd $(b+c)$ and $(-1)^p\chi(-1)=1$. □

Proof [Proof of Theorem 1.6]The definition

$$S_p(b,c:\chi) = \sum_{n=1}^{ck}\chi(n)\,\overline{B}_{p,\overline{\chi}}\left(\frac{b+ck}{2c}n\right)$$

in this form is not convenient to prove the reciprocity formula by aid of the Euler–MacLaurin or Boole summation formula. Thus, $S_p(b,c:\chi)$ should be modified to apply summation formulas. For this, using (3.5) in the definition of $S_p(b,c:\chi)$, and then [12, Lemma 5.5], we see that

$$\begin{aligned} S_p(b,c:\chi) &= 2^{-p}\chi(2)\sum_{n=1}^{ck}\chi(n)\,\overline{B}_{p,\overline{\chi}}\left(\frac{bn}{c}\right) - p\frac{\chi(2)}{2^{p+1}}\sum_{n=1}^{ck}\chi(n)\,\overline{E}_{p-1,\chi}\left(\frac{bn}{c}+kn\right) \\ &= \frac{\chi(2c)\,\overline{\chi}(-b)}{2^p c^{p-1}}(k^p-1)\,\overline{B}_p(0) - p\frac{\chi(2)}{2^{p+1}}\sum_{n=1}^{ck}(-1)^n\chi(n)\,\overline{E}_{p-1,\chi}\left(\frac{bn}{c}\right). \end{aligned} \tag{5.5}$$

Now let $f(x) = \overline{E}_{p-1,\chi}(xb/c)$, $\alpha=0$, and $\beta=ck$ in Theorem 1.3. Then, in the light of (2.6), we can write

$$\begin{aligned} &\sum_{n=0}^{ck}(-1)^n\chi(n)\,\overline{E}_{p-1,\chi}\left(n\frac{b}{c}\right) \\ &= \frac{\chi(-1)}{2}\sum_{j=0}^{l}(-1)^j\binom{p-1}{j}\left(\frac{b}{c}\right)^j\left\{\left((-1)^{(b+c)}-1\right)\overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-1-j,\chi}(0)\right\} \\ &- \frac{\chi(-1)}{2}(-1)^l(p-1)\binom{p-2}{l}\left(\frac{b}{c}\right)^{l+1}\int_0^{ck}\overline{E}_{l,\overline{\chi}}(x)\,\overline{E}_{p-2-l,\chi}\left(\frac{b}{c}x\right)dx. \end{aligned} \tag{5.6}$$

Following precisely the method in the proof of Theorem 1.5 and using that $\overline{B}_p(0)=0$ for odd p yields

$$\begin{aligned} &\overline{\chi}(-2)\,bc^p S_p(b,c:\chi) + \chi(-2)\,cb^p S_p(c,b:\overline{\chi}) \\ &= \frac{p}{2^{p+1}}\sum_{j=1}^{p}(-1)^j\binom{p-1}{j-1}c^j b^{p+1-j}\overline{E}_{j-1,\chi}(0)\,\overline{E}_{p-j,\overline{\chi}}(0). \end{aligned}$$

□

Remark 5.1 *Taking into consideration (3.5), this formula coincides with [10, Corollary 4.3] wherein there is the condition b or $c\equiv 0 \pmod{k}$.*

We conclude the study with some results for the integral involving character Euler functions in consequence of (5.6) and (5.5). We first note that the sum on the left-hand side of (5.6) is zero when p and $(b+c)$ have opposite parity. Therefore, if $p>1$ is odd and $(b+c)$ is even, then

$$\int_0^k \overline{E}_{l,\overline{\chi}}(x)\,\overline{E}_{p-2-l,\chi}\left(\frac{b}{c}x\right)dx = 0$$

and if p is even and $(b+c)$ is odd, then

$$\int_0^k \overline{E}_{l,\overline{\chi}}(cx)\,\overline{E}_{p-2-l,\chi}(bx)\,dx$$

$$= \frac{2\,(-c/b)^{l+1}}{c\,(p-1)\binom{p-2}{l}} \sum_{j=0}^{l} (-1)^j \binom{p-1}{j} \left(\frac{b}{c}\right)^j \overline{E}_{j,\overline{\chi}}(0)\,\overline{E}_{p-1-j,\chi}(0).$$

Let p and $(b+c)$ be even. Gathering $S_p(b,c:\chi) = c^{1-p}\chi(2c)\,\overline{\chi}(-b)\,(k^p-1)\,\overline{B}_p(0)$ ([10, Proposition 5.7]) and (5.5), one arrives at

$$\sum_{n=1}^{ck} (-1)^n \chi(n)\,\overline{E}_{p-1,\chi}\left(\frac{bn}{c}\right) = \frac{1}{p} 2\,(1-2^p)\,c^{1-p}\chi(c)\,\overline{\chi}(-b)\,(k^p-1)\,B_p.$$

Thus, from the fact that $2\,(2^p-1)\,B_p = -pE_{p-1}(0)$, we have

$$\int_0^k \overline{E}_{l,\overline{\chi}}(cx)\,\overline{E}_{p-2-l,\chi}(bx)\,dx = 2\,(-1)^{l+1} \frac{\chi(c)\,\overline{\chi}(b)}{c^{p-l-1}b^{l+1}} \frac{(k^p-1)}{\binom{p-2}{l}} \frac{E_{p-1}(0)}{p-1}.$$

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