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A new approach to uniqueness for inverse Sturm–Liouville problems on finite intervals

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Abstract: In this paper, an approach for studying inverse Sturm–Liouville problems with integrable potentials on finite intervals is presented. We find the relations between Weyl solutions and m_j-functions of Sturm–Liouville problems, and by finding the connection between these and the solutions of second-order partial differential equations for transformation kernels associated with Sturm–Liouville operators, we prove the uniqueness of the solution of inverse problems.

Key words: Inverse Sturm–Liouville problem, Weyl solutions, m_j-functions, transformation kernel

1. Introduction

We consider the following Sturm–Liouville differential equations:

$$\ell_j(y) := -y''(x) + q_j(kx/2)y(x) = \lambda y(x), j = 1, 2, x \in [a, b], \tag{1.1}$$

with the boundary conditions $y(a, \lambda) = 0 = y'(b, \lambda)$, where $-\infty < a \leq 0 < b < \infty$, $k \geq 2$ is constant, and q_j, $j = 1, 2$, are real-valued.

Let y_j be the solution of the Problem L_j consisting of the equation (1.1) together with the conditions

$$y_j(a, \lambda) = 0, \qquad y_j'(a, \lambda) = 1. \tag{1.2}$$

Also, let $\widetilde{y}_j(x, \lambda) \in L^2([a, b])$, $j = 1, 2$, be the unique solutions of (1.1) satisfying

$$\widetilde{y}_j(a, \lambda) = A, \qquad \widetilde{y}_j(b, \lambda) = 0, \tag{1.3}$$

which are the so-called *Weyl solutions* of (1.1). Here, $A \neq 0$ is constant.

We denote the *m_j-functions* associated with (1.1) for $j = 1, 2$, by

$$m_j(\lambda) = m(\lambda; q_j) = \frac{1}{A}\widetilde{y}_j'(a, \lambda), \tag{1.4}$$

for $\lambda \in \mathbb{C}\backslash\sigma(\ell_j)$, where $\sigma(\ell_j)$ is the spectrum of ℓ_j. Letting for $j = 1, 2$, $q_j \in L^1([a, b])$, then we know from [8] that $\sigma(\ell_j)$ is real and bounded. Hence, it follows from [14] that there is a positive constant h_0 such that the m_j-functions are defined for each $\lambda \in \mathbb{C}\backslash[-h_0, \infty)$. Moreover, it can be shown that letting $q_j \in L^1([a, b - a/2])$, $j = 1, 2$, and supposing $q_{j,n} \in L^1([a, b - a/2])$, $||q_{j,n} - q_j|| \to 0$ for $n \to \infty$, then

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$$m(\lambda, q_{j,n}) \to m(\lambda, q_j) \tag{1.5}$$

for $n \to \infty$ pointwise for each $\sqrt{\lambda} \in \mathbb{C}$, $Im(\sqrt{\lambda}) > 0$.

By a method similar to that used in [7,11], we can prove the following theorem.

Theorem 1.1 *Let y_1, y_2 be the solutions of Sturm–Liouville problems L_1, L_2, respectively. Then there exists a unique transformation kernel H independent of λ such that*

$$y_2(x,\lambda) = y_1(x,\lambda) + \int_a^x H(x,t)y_1(t,\lambda)dt, \qquad a \le x \le b. \tag{1.6}$$

In the last two decades, several subjects in the inverse Sturm–Liouville problems were investigated, where the uniqueness and the stability of the solutions of inverse problems with multiple conditions received more attention (for example, see [1–4,9,10,12,13,15–17]).

In [14], the author introduced a new object in Sturm–Liouville problems with differential operators on either $L^2(0,b)$, $b < \infty$, or $L^2(0,\infty)$ and proved a local version of the Borg–Marchenko uniqueness theorem by this new formalism. He investigated necessary and sufficient conditions on the associated m-function for determining the potential of a Sturm–Liouville operator. For another example, in [5], the author proved the existence of a transmutation operator between two Schrödinger equations with perturbed exactly solvable potential. Moreover, by using Varsha and Jafari's method, an explicit formula for the solution of the nucleus function was provided.

In the present paper, we present a new approach (distinct from [14]) to prove the uniqueness theorem for regular Sturm–Liouville problems on the finite interval $[a,b]$, $-\infty < a \le 0 < b < \infty$. The main role in our approach is played by the transformation kernel $H(x,t)$ (which is defined in Theorem 1.1) and its associated second-order partial differential equation in two variables. In Section 2, we prove several estimates for the kernel H and some its associated operators. In Section 3, we obtain the relations between the Weyl solutions $\widetilde{y}_j$, m_j-functions, and transformation kernels. These relations play important roles in the proof of the uniqueness theorem. Then, by a relation between the different kernels, we prove the uniqueness theorem (see Section 4).

2. Transformation kernels and preliminary results

It follows from substituting (1.6) into (1.1) that for $(x,t) \in \Gamma$, the kernel H solves the following problem:

$$\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial t^2} + \frac{k^2}{4}(q_1(kt/2) - q_2(kx/2))H(x,t) = 0, \tag{2.1}$$

$$H(x,a) = 0, k\frac{\partial}{\partial x}H(x,x) = q_2(kx/2) - q_1(kx/2), a \le x \le b, \tag{2.2}$$

where $\Gamma = \{(x,t) \in \mathbb{R}^2 \mid a < t < x < b\}$.

Lemma 2.1 *Let $q_1, q_2 \in L^1([ka/2, k(2b-a)/4])$. Then:*

a) the problem (2.1)–(2.2) has a unique solution H, which is compactly supported in $[a,b] \times [a,b]$. Moreover, if $q_1, q_2 \in C^m([ka/2,\ k(2b-a)/4])$, then $H \in C^{m+1}(\overline{\Gamma})$.

b) For $a \le t \le x \le b$, the following estimate is valid:

$$|H(x,t)| \le \int_{a/k}^{(x+t-a)/k} |q_2(z) - q_1(z)|dz \times \exp(\int_{a/k}^{(x-t+a)/k} \int_{\eta}^{(x+t-a)/k} |q_2(\zeta+\eta) - q_1(\zeta-\eta)|d\zeta d\eta). \tag{2.3}$$

Proof Denote the variables

$$\tau = \frac{x+t-a}{k}, \quad \theta = \frac{x-t+a}{k}. \tag{2.4}$$

Thus,

$$x = x(\tau,\theta) = k(\tau+\theta)/2, \quad t = t(\tau,\theta) = k(\tau-\theta)/2 + a.$$

For $a/k \le \theta \le \tau \le (2b-a)/k$ we define

$$h(\tau,\theta) = H(x(\tau,\theta), t(\tau,\theta)). \tag{2.5}$$

Therefore, for $a/k < \theta < \tau < \frac{2b-a}{k}$, the function $h(\tau,\theta)$ solves the following problem:

$$\frac{\partial^2 h}{\partial\tau\partial\theta}(\tau,\theta) = f(x(\tau,\theta), t(\tau,\theta))h(\tau,\theta), \tag{2.6}$$

$$h(\tau_0,\tau_0) = 0, \quad (\frac{\partial}{\partial\tau}h(\tau,\tau))|_{\tau=\tau_1} = g(\tau_1), \tag{2.7}$$

where $\tau_0 = x/k$, $\tau_1 = (2x-a)/k$, and

$$f(x,t) = q_2(kx/2) - q_1(kt/2), \quad g(x) = f(x,x).$$

Integration with respect to θ from a/k to θ and then integration with respect to τ from θ to τ yields the following second kind of Volterra integral equation:

$$h(\tau,\theta) = \int_{\theta}^{\tau} \int_{\frac{a}{k}}^{\theta} f(x'(\tau',\theta'), t'(\tau',\theta'))h(\tau',\theta')d\theta' d\tau' + \int_{\theta}^{\tau} g(r)dr. \tag{2.8}$$

Denote the operator T on $C(\Gamma)$ by

$$Th(\tau,\theta) = \int_{\theta}^{\tau} \int_{\frac{a}{k}}^{\theta} f(x'(\tau',\theta'), t'(\tau',\theta'))h(\tau',\theta')d\theta' d\tau',$$

and $G(\tau,\theta) = \int_{\theta}^{\tau} g(r)dr$. Thus, (2.8) has the form

$$(I-T)h(\tau,\theta) = G(\tau,\theta). \tag{2.9}$$

By induction, for each $\widetilde{h} \in C(\Gamma)$, we can establish

$$|T^n \widetilde{h}(\tau,\theta)| \le \sup_{\frac{a}{k} \le \theta' \le \tau' \le \tau} |\widetilde{h}(\tau',\theta')| \frac{1}{n!} (\int_{\frac{a}{k}}^{\theta} \int_{\eta}^{\tau} |f(\zeta+\eta, \zeta-\eta)|d\zeta d\eta)^n.$$

Hence, from [6], the Neumann series $\sum_{n=0}^{\infty} T^n$ converges to the operator $I-T$, and the unique solution h is obtained from (2.9). Moreover,

$$|h(\tau,\theta)| \leq \sup_{\frac{a}{k}\leq\theta'\leq\tau'\leq\tau} |G(\tau',\theta')| \times \exp(\int_{\frac{a}{k}}^{\theta}\int_{\eta}^{\tau} |f(\zeta+\eta,\zeta-\eta)|d\zeta d\eta), \tag{2.10}$$

and thus we arrive at (2.3). □

In the same way as in the proof of Lemma 2.1, we can prove the following lemma.

Lemma 2.2 *Let $\delta\in[a,b-a/2]$, $g_1\in C[a,a+\delta]$, $q_1,q_2,g_2\in L^1([a,a+\delta])$. Then, for $(x,t)\in\Gamma_\delta:=\{(x,t)\in\mathbb{R}^2 \mid a<t<x<a+\delta,\ t+x\leq a+\delta\}$, the problem*

$$\begin{cases} \frac{\partial^2 H}{\partial x^2}-\frac{\partial^2 H}{\partial t^2}+\frac{k^2}{4}(q_1(kt/2)-q_2(kx/2))H(x,t)=0, & (x,t)\in\Gamma, \\ H(x,a)=g_1(x), \quad \frac{\partial H}{\partial t}(x,a)=g_2(x), & x\in[a,a+\delta], \end{cases}$$

has a unique solution, $H\in C(\overline{\Gamma}_\delta)$.

In the special case $q_1=q$, $q_2\equiv 0$, the solution y_2 of the regular problem L_2 is $y_2(x,\lambda)=\sin(\sqrt{\lambda}x)/\sqrt{\lambda}$, and thus according to Theorem 1.1, there exists a unique kernel H_1 such that

$$\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} = y_1(x,\lambda) - \int_a^x H_1(x,t)y_1(t,\lambda)dt. \tag{2.11}$$

Moreover, by (2.1)–(2.2), we give the following partial differential equation associated with H_1,

$$\frac{\partial^2 H_1}{\partial x^2}-\frac{\partial^2 H_1}{\partial t^2}+\frac{k^2}{4}q(kt/2)H_1(x,t)=0, \quad (x,t)\in\Gamma, \tag{2.12}$$

together with the conditions

$$H_1(x,a)=0, \quad k\frac{\partial}{\partial x}H_1(x,x)=-q(kx/2), \quad x\geq a. \tag{2.13}$$

By changing variables (2.4), we define the function

$$h_1(\tau,\theta)=H_1(x(\tau,\theta),t(\tau,\theta)). \tag{2.14}$$

Hence,

$$H_1(x,a)=h_1(\tau,\tau), \quad H_1(x,x)=h_1(\tau,a/k), \quad H_1(kx,a)=h_1(x,x). \tag{2.15}$$

Moreover,

$$\frac{\partial h_1}{\partial \tau}(\tau,\theta)-\frac{\partial h_1}{\partial \theta}(\tau,\theta)=k\frac{\partial H_1}{\partial t}(x(\tau,\theta),t(\tau,\theta)). \tag{2.16}$$

In the following theorem, we estimate the kernel H_1 and its partial derivatives.

Theorem 2.3 *Let $q\in L^1([a,b-a/2])$. Then:*

(i) for $a \le t \le x \le b$*, the following inequality holds:*

$$|H_1(x,t)| \le \frac{2}{k}||q||_* \exp(\frac{2}{k}||q||^* \max\{|x|-\frac{a}{k},|t|-\frac{a}{k}\}), \tag{2.17}$$

where $||q||_*, ||q||^*$ *are the norm of* q *in* $L^1([a/2,b-a/2])$ *and* $L^1([a,b-a/2])$*, respectively.*

(ii) The function $\frac{k}{2}\frac{\partial H_1}{\partial t}(kx,a)-q(kx/2)$ *is continuous, and for* $a \le x \le b$*,*

$$|\frac{k}{2}\frac{\partial H_1}{\partial t}(kx,a)-q(kx/2)| \le \alpha_q \exp(\frac{2}{k}(|x|-\frac{a}{k})||q||^*). \tag{2.18}$$

(iii) If q, q' *are compactly supported on* $[a,b]$*, then for* $a \le t \le x \le b$*,*

$$|\frac{\partial H_1}{\partial t}(x,t)| \le \beta_q \exp(\frac{2}{k}\max\{|x|-\frac{a}{k},|t|-\frac{a}{k}\}||q||_{L^1[a,b]}), \tag{2.19}$$

$$|\frac{\partial^2 H_1}{\partial t^2}(x,t)| \le \beta_q \exp(\frac{2}{k}\max\{|x|-\frac{a}{k},|t|-\frac{a}{k}\}||q||_{L^1[a,b]}). \tag{2.20}$$

Here, the constants α_q, β_q *may depend on* q*.*

Proof The problem (2.12)–(2.13) is equal to the problem (2.1)–(2.2) with $H = H_1$, $q_1 = q$, and $q_2 = 0$. Therefore, according to (2.5), (2.8), and (2.14), h_1 satisfies the following integral equation:

$$h_1(\tau,\theta) = -\int_\theta^\tau \int_{\frac{a}{k}}^\theta q(t'(\tau',\theta'))h_1(\tau',\theta')d\theta' d\tau' + \int_\theta^\tau q(kr/2)dr. \tag{2.21}$$

On the other hand, by (2.10) we get

$$|h_1(\tau,\theta)| \le \int_{\frac{a}{k}}^\tau |q(kr/2)|dr.\exp(\int_{\frac{a}{k}}^\theta \int_s^\tau |q(\frac{k(\zeta-\eta)}{2}+a)|d\zeta d\eta). \tag{2.22}$$

Since for $x \ge t$,

$$\begin{aligned}\int_{\frac{a}{k}}^\theta \int_s^\tau |q(\frac{k(\zeta-\eta)}{2}+a)|d\zeta d\eta &= \int_{\tau-\theta}^{\tau-\frac{a}{k}} \int_0^u |q(\frac{kr}{2}+a)|dr du \\ &\le \frac{2}{k}(\theta-\frac{a}{k})\int_a^{(k\tau+a)/2}|q(s)|ds,\end{aligned}$$

this together with (2.22) yields

$$|h_1(\tau,\theta)| \le \int_{\frac{a}{k}}^\tau |q(kr/2)|dr.\exp(\frac{2}{k}(\theta-\frac{a}{k})\int_a^{(k\tau+a)/2}|q(s)|ds). \tag{2.23}$$

Also, since $\tau \le (2b-a)/k$,

$$\begin{aligned}\int_{\frac{a}{k}}^\tau |q(kr/2)|dr &\le \frac{2}{k}\int_{a/2}^{b-a/2}|q(r')|dr' \\ &= \frac{2}{k}||q||_*,\end{aligned} \tag{2.24}$$

and moreover,

$$\exp(\frac{2}{k}(\theta-\frac{a}{k})\int_a^{\frac{k\tau+a}{2}}|q(s)|ds)\leq \exp(\frac{2}{k}||q||^*\max\{|x|-\frac{a}{k},|t|-\frac{a}{k}\}). \tag{2.25}$$

According to (2.23)–(2.25), we arrive at (2.17).

Now, differentiating (2.21) with respect to τ and θ, respectively, yields

$$\begin{aligned}\frac{\partial h_1}{\partial\tau}(\tau,\theta)&=-\int_{\frac{a}{k}}^{\theta}q(t'(\tau,\theta'))h_1(\tau,\theta')d\theta'+q(k\tau/2)\\&=\int_{\tau-\theta}^{\tau-\frac{a}{k}}q(kr/2+a)h_1(\tau,\tau-r)dr+q(k\tau/2),\end{aligned}\tag{2.26}$$

$$\begin{aligned}\frac{\partial h_1}{\partial\theta}(\tau,\theta)&=\int_{\frac{a}{k}}^{\theta}q(t'(\theta,\theta'))h_1(\theta,\theta')d\theta'-\int_{\theta}^{\tau}q(t'(\tau',\theta))h_1(\tau',\theta)d\tau'-q(k\theta/2)\\&=\int_0^{\theta-\frac{a}{k}}q(kr/2+a)h_1(\theta,\theta-r)dr-\int_0^{\tau-\theta}q(kr/2+a)h_1(\theta+r,\theta)dr-q(k\theta/2).\end{aligned}\tag{2.27}$$

Hence, from (2.26)–(2.27), we obtain

$$\frac{\partial h_1}{\partial\tau}(x,t)-\frac{\partial h_1}{\partial\theta}(x,t)=2\int_0^{x-\frac{a}{k}}q(kr/2+a)h_1(x,x-r)dr+2q(kx/2).$$

This together with (2.15)–(2.16) yields

$$k\frac{\partial H_1}{\partial t}(kx,a)-2q(kx/2)=2\int_0^{x-\frac{a}{k}}q(kr/2+a)h_1(x,x-r)dr.$$

Therefore, we arrive at (ii).

If q,q' are compactly supported on $[a,b]$, then there is a positive number β_q (which may depend on q) such that

$$|\frac{\partial h_1}{\partial\tau}(\tau,\theta)|,|\frac{\partial h_1}{\partial\theta}(\tau,\theta)|\leq\beta_q\exp(\frac{2}{k}(\theta-a/k)\int_a^{\tau}|q(s)|ds). \tag{2.28}$$

Thus, the estimate (2.19) follows from (2.15) and (2.28). Similarly, since

$$\frac{\partial^2 H_1}{\partial t^2}(x,t)=\frac{1}{k^2}\{\frac{\partial^2 h_1}{\partial\tau^2}(\tau(x,t),\theta(x,t))-2\frac{\partial^2 h_1}{\partial\tau\partial\theta}(\tau(x,t),\theta(x,t))+\frac{\partial^2 h_1}{\partial\theta^2}(\tau(x,t),\theta(x,t))\},$$

and q,q' are compactly supported on $[a,b]$, we arrive at (2.20). □

3. Relations between the Weyl solutions, m_j-functions, and the kernels

In this section, first we derive the relations between the Weyl solutions $\widetilde{y}_j$ and the kernels H_j, $j=1,2$. Then, with these, we prove the connection between H_j and the Weyl functions m_j, which will be used in the proof of the uniqueness theorem in section 4.

First, in the following lemma, we establish a relation between $\widetilde{y}_j$ and the kernels H_j when q_j,q_j' are compactly supported on $[a,b-a/2]$.

Lemma 3.1 *Let q_j, q_j', $j=1,2$, be compactly supported on $[a, b-\frac{a}{2}]$ and H_j be the kernel in (2.11) such that for $a \le t \le b-\frac{a}{2}$,*

$$H_j(b-\frac{a}{2}, t) = 0 = \frac{\partial H_j}{\partial x}(b-\frac{a}{2}, t), \quad j=1,2. \tag{3.1}$$

Assume $\lambda = -c^2\rho^2$, $\rho = \sigma_1 + i\sigma_2$, $c>0$, and σ_1, σ_2 are constants. Then, for $k=2$, $j=1,2$ and $|\rho| > ||q_j||^*$, the function

$$\widetilde{y}_j(t,\lambda) = A\exp(-c\rho(t-a)) - A\int_t^{b-\frac{a}{2}} H_j(x,t)\exp(-c\rho(x-a))dx \tag{3.2}$$

is the Weyl solution of the differential equation

$$-y''(x) + q_j(x)y(x) = \lambda y(x), \quad x \in [a, b-\frac{a}{2}],$$

where A is defined as in (1.3), and $||q_j||^*$ is the norm of q_j in $L^1([a, b-\frac{a}{2}])$.

Proof First, it follows from (2.13) and (3.2) that $\widetilde{y}_j(a,\lambda) = A$, $j=1,2$. Second, (2.17) implies that $\widetilde{y}_j(t,\lambda)$ is well defined by (3.2) for $|\rho| > ||q_j||^*$, and moreover, $\widetilde{y}_j(t,\lambda) \in L^2([a, b-a/2])$. Since q_j, q_j' are compactly supported on $[a, b-a/2]$, it follows from Theorem 2.3 that $H_j \in C^2(\Gamma)$. Now, by (3.2) we have for $j=1,2$,

$$\widetilde{y}_j'(t,\lambda) = A\{-c\rho + H_j(t,t)\}\exp(-c\rho(t-a)) - A\int_t^{b-\frac{a}{2}} \frac{\partial H_j}{\partial t}(x,t)\exp(-c\rho(x-a))dx, \tag{3.3}$$

$$\begin{aligned}\widetilde{y}_j''(t,\lambda) =& A\exp(-c\rho(t-a)) \times \{c^2\rho^2 + \frac{\partial}{\partial t}H_j(t,t) - c\rho H_j(t,t) + \frac{\partial H_j}{\partial t}(t,t)\} \\ &- A\int_t^{b-\frac{a}{2}} \frac{\partial^2 H_j}{\partial t^2}(x,t)\exp(-c\rho(x-a))dx.\end{aligned} \tag{3.4}$$

Since H_j solves (2.12)–(2.13), we obtain

$$\int_t^{b-\frac{a}{2}} \frac{\partial^2 H_j}{\partial t^2}(x,t)\exp(-c\rho(x-a))dx = \int_t^{b-\frac{a}{2}} \{\frac{\partial^2 H_j}{\partial x^2}(x,t) + q_j(t)H_j(x,t)\}\exp(-c\rho(x-a))dx. \tag{3.5}$$

From integration by parts and (3.1), we get

$$\begin{aligned}&-\int_t^{b-\frac{a}{2}} \frac{\partial^2 H_j}{\partial x^2}(x,t)\exp(-c\rho(x-a))dx = \frac{\partial H_j}{\partial x}(t,t)\exp(-c\rho(t-a)) - c\rho\int_t^{b-\frac{a}{2}} \frac{\partial H_j}{\partial x}(x,t)\exp(-c\rho(x-a))dx \\ &= \{\frac{\partial H_j}{\partial x}(t,t) + c\rho H_j(t,t)\}\exp(-c\rho(t-a)) - c^2\rho^2\int_t^{b-\frac{a}{2}} H_j(x,t)\exp(-c\rho(x-a))dx.\end{aligned} \tag{3.6}$$

Substituting (3.5)–(3.6) into (3.4) yields

$$\begin{aligned}\widetilde{y}_j''(t,\lambda) &= A\{c^2\rho^2 + \frac{\partial}{\partial t}H_j(t,t) + \frac{\partial H_j}{\partial t}(t,t) + \frac{\partial H_j}{\partial x}(t,t)\}\exp(-c\rho(t-a)) \\ &\quad - A(c^2\rho^2 + q_j(t))\int_t^{b-\frac{a}{2}} H_j(x,t)\exp(-c\rho(x-a))dx \\ &= (c^2\rho^2 + q_j(t))\widetilde{y}_j(t,\lambda).\end{aligned}$$

This completes the proof of Lemma 3.1. □

Now we prove the main result of this section, which allows the connection between the m_j-functions and their associated kernels H_j, which satisfy

$$\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} = y_j(x,\lambda) - \int_a^x H_j(x,t)y_j(t,\lambda)dt, \quad j=1,2. \tag{3.7}$$

Theorem 3.2 *Let for $j=1,2$, $q_j \in L^1([a,b-a/2])$. Then, for $|\rho| > \max\{||q_1||^*, ||q_2||^*\}$ and $j=1,2$, the following relation is valid:*

$$m_j(\lambda) = -c\rho - \int_a^{b-a/2} \frac{\partial H_j}{\partial t}(x,a)\exp(-c\rho(x-a))dx. \tag{3.8}$$

Proof In the case when q_j, q_j', $j=1,2$, are compactly supported on $[a,b-a/2]$, (3.8) can be obtained from (1.4), (3.3), and $H_j(a,a)=0$.

In the general case $q_j \in L^1([a,b-a/2])$, $j=1,2$, let $\{q_{j,n}\}_{n=1}^{\infty}$ be a sequence in which $q_{j,n}, q_{j,n}'$ are compactly supported on $[a,b-a/2]$, $||q_{j,n}||^* \leq \alpha_{q_j} < |\rho|$ and $||q_{j,n} - q_j||^* \to \infty$ for $n \to \infty$. Then, according to (2.17), (2.19), and dominated convergence,

$$\int_a^{b-a/2} \frac{\partial H_j}{\partial t}(x,a;q_{j,n})\exp(-c\rho(x-a))dx$$

converges to

$$\int_a^{b-a/2} \frac{\partial H_j}{\partial t}(x,a;q_j)\exp(-c\rho(x-a))dx$$

as $n \to \infty$. This together with (1.5) completes the proof. □

Remark 3.3 *In the special case $q_1 = 0$, according to Theorem 1.1, there exists a unique transformation kernel $\widetilde{H}$ such that*

$$y_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + \int_a^x \widetilde{H}(x,t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt. \tag{3.9}$$

4. The uniqueness theorem

In this section, we give a result about the connection between the differential kernels H_1, H_2, associated with the problems (1.1)–(1.2) (with $j=1,2$) and the kernel H in (1.6), and we prove a uniqueness result for the potentials of the Sturm–Liouville operators.

Lemma 4.1 *Let for $j=1,2$, $q_j \in L^1([a,b-a/2])$, H_j be the kernels given by (3.7) associated with the following problems:*

$$\begin{cases} -y''(x) + q_j(x)y(x) = \lambda y(x), \\ y(a,\lambda) = 0, \end{cases} \tag{4.1}$$

and let H be the kernel given by (1.6). If $\frac{\partial H_1}{\partial t}(x,a) = \frac{\partial H_2}{\partial t}(x,a)$ in $L^1((a,a_0))$ for some $a_0 \in (a,b-a/2)$, then $\frac{\partial H}{\partial t}(x,a) = 0$ in $L^1((a,a_0))$.

Proof Substituting (3.7) for $j=1$ into (3.9) yields

$$y_2(x,\lambda) = y_1(x,\lambda) + \int_a^x \{\widetilde{H}(x,t) - H_1(x,t) - \int_t^x \widetilde{H}(x,\eta)H_1(\eta,t)d\eta\}y_1(t,\lambda)dt. \tag{4.2}$$

Therefore, by (3.9) and (4.2), the kernel H is uniquely determined as follows:

$$H(x,t) = \widetilde{H}(x,t) - H_1(x,t) - \int_t^x \widetilde{H}(x,\eta)H_1(\eta,t)d\eta.$$

This yields

$$\frac{\partial H}{\partial t}(x,t) = \frac{\partial \widetilde{H}}{\partial t}(x,t) - \frac{\partial H_1}{\partial t}(x,t) + \widetilde{H}(x,t)H_1(t,t) - \int_t^x \widetilde{H}(x,\eta)\frac{\partial H_1}{\partial t}(\eta,t)d\eta,$$

for almost all $(x,t) \in \Gamma$. Hence, since $H_1(a,a) = 0$ and $\frac{\partial H_1}{\partial t}(x,a) = \frac{\partial H_2}{\partial t}(x,a)$ in $L^1((a,a_0))$, we obtain

$$\frac{\partial H}{\partial t}(x,a) = \frac{\partial \widetilde{H}}{\partial t}(x,a) - \frac{\partial H_2}{\partial t}(x,a) - \int_a^x \widetilde{H}(x,\eta)\frac{\partial H_2}{\partial t}(\eta,a)d\eta, \tag{4.3}$$

for almost all $x \in (a,a_0)$.
Similarly, substituting (3.7) for $j=2$ into (3.9) gives us

$$\int_a^x \{\widetilde{H}(x,t) - H_2(x,t) - \int_t^x \widetilde{H}(x,\eta)H_2(\eta,t)d\eta\}y_2(t,\lambda)dt = 0.$$

Hence,

$$\widetilde{H}(x,t) - H_2(x,t) - \int_t^x \widetilde{H}(x,\eta)H_2(\eta,t)d\eta = 0, \quad a \le t \le x \le b - a/2,$$

for almost all $(x,t) \in \Gamma$. Therefore,

$$\frac{\partial \widetilde{H}}{\partial t}(x,a) - \frac{\partial H_2}{\partial t}(x,a) - \int_a^x \widetilde{H}(x,\eta)\frac{\partial H_2}{\partial t}(\eta,a)d\eta = 0,$$

for almost all $x \in (a,a_0)$. This together with (4.3) completes the proof. □

To prove the main theorem (Theorem 4.3), we need the following lemma, which can be proved similarly by a method used in [14].

Lemma 4.2 *Let* $p \in L^1([\alpha,\beta])$ *and assume that the function* $s(\zeta) = \int_\alpha^\beta p(r)\exp(-\zeta r)dr$ *satisfies*

$$s(\eta) = o(\exp(-(\beta-\alpha)(1-\varepsilon)c\eta))$$

as $\eta \to +\infty$, *for all* $0 < \varepsilon < 1$. *Then* $p \equiv 0$.

Now, we prove the following uniqueness theorem, which is the main result of this section.

Theorem 4.3 *Let* $q_j \in L^1([a,b-a/2])$ *be a real potential for the problem* (4.1), *and let* m_j *be the associated* m_j*-function. Assume that there is a number* $a_0 \in (a,b-a/2)$ *such that*

$$m_1(\lambda) - m_2(\lambda) = o(\exp(-(a_0-a)(1-\varepsilon)c|\rho|)),$$

as $|\rho| \to \infty$, *for each* $0 < \varepsilon < 1$. *Then* $q_1 = q_2$ *a.e. on* $[a,(a+a_0)/2]$.

Proof From (3.8) and the hypothesis of the theorem, we have

$$m_1(\lambda)-m_2(\lambda)=\int_a^{b-a/2}\{\frac{\partial H_2}{\partial t}(x,a)-\frac{\partial H_1}{\partial t}(x,a)\}\times\exp(-c\rho(x-a))dx \quad (4.4)$$
$$=o(\exp(-(a_0-a)(1-\varepsilon)c|\rho|)),$$

as $|\rho|\to\infty$, for each $0<\varepsilon<1$. On the other hand, by (2.18) with $k=2$ we have

$$|\frac{\partial H_j}{\partial t}(x,a)-q_j(x/2)|\leq\alpha_{q_j}\exp(||q_j||^*(|x|-a)/2).$$

Thus, for $|\rho|>\max\{||q_1||^*,||q_2||^*\}$, we get

$$\int_{a_0}^{b-a/2}|(\frac{\partial H_2}{\partial t}(x,a)-\frac{\partial H_1}{\partial t}(x,a))\exp(-c\rho(x-a))|dx=o(\exp(-(a_0-a)(1-\varepsilon)c|\rho|)),$$

as $|\rho|\to\infty$, for each $0<\varepsilon<1$. This together with (4.4) gives

$$\int_a^{a_0}(\frac{\partial H_2}{\partial t}(x,a)-\frac{\partial H_1}{\partial t}(x,a))\exp(-c\rho(x-a))dx=o(\exp(-(a_0-a)(1-\varepsilon)c|\rho|)),$$

as $|\rho|\to\infty$, for each $0<\varepsilon<1$. Hence, by Lemma 4.2 we get

$$\frac{\partial H_2}{\partial t}(x,a)-\frac{\partial H_1}{\partial t}(x,a)=0,$$

for a.e. $x\in(a,a_0)$. Now it follows from Lemma 4.1 that the kernel H satisfies $\frac{\partial\widetilde{H}}{\partial t}(x,a)=0$, for a.e. $x\in(a,a_0)$. Consequently, since H solves uniquely the problem (2.1)–(2.2) with $k=2$, H specially solves the following problem:

$$\begin{cases}\frac{\partial^2 H}{\partial x^2}-\frac{\partial^2 H}{\partial t^2}+(q_1(t)-q_2(x))H(x,t)=0, & (x,t)\in\Gamma,\\ H(x,a)=0,\quad \frac{\partial H}{\partial t}(x,a)=0, & x\in[a,a_0].\end{cases} \quad (4.5)$$

According to Lemma 2.2, the problem (4.5) has a unique solution $H\in C(\overline{\Gamma}_{a_0})$, where

$$\Gamma_{a_0}=\{(x,t)\in\mathbb{R}^2 \mid a<t<x<a_0,\ t+x\leq a+a_0\}.$$

Therefore,

$$H(x,t)\equiv 0,\quad (x,t)\in\Gamma_{a_0}. \quad (4.6)$$

Now, from (2.2) with $k=2$ and (4.6), we obtain

$$\frac{\partial H}{\partial x}(x,x)=\frac{1}{2}(q_2(x)-q_1(x))=0,$$

for a.e. $x\in[a,(a+a_0)/2]$. The proof is complete. □

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