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Stability analysis of nonlinear fractional differential order systems with Caputo and Riemann–Liouville derivatives

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Abstract: In this paper we establish stability theorems for nonlinear fractional orders systems (FDEs) with Caputo and Riemann–Liouville derivatives. In particular, we derive conditions for $\mathcal{F}$-stability of nonlinear FDEs. By numerical simulations, we verify numerically our theoretical results on a test example.

Key words: Stability, Riemann–Liouville derivative, Caputo derivative, $\mathcal{F}$-asymptotically stable

1. Introduction

Fractional-order models are found to be more adequate than integer-order models in some real-world problems. In fact, fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, and fitting of experimental data. For examples and details, see [1, 8, 13] and the references therein.

Stability analysis is a central task in the study of fractional differential systems. The stability analysis of FDEs is more complex than that of classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Most of the known results on stability analysis of fractional differential systems concentrate on the stability of linear fractional differential systems. In [15], Matignon has given a well-known stability criterion for a linear fractional autonomous differential system with constant coefficient matrix A. The criterion is that the stability is guaranteed if and only if the roots of the eigenfunction of the system lie outside the closed angular sector $|\arg(\lambda(A))| < \frac{\pi}{2}\alpha$, which generalized the result for the integer case $\alpha = 1$. Later, Matignon's stability criterion was developed by several authors. Deng et al. [7] generalized the system to a linear fractional differential system with multiorders and multiple delays, in which the characteristic polynomial is introduced by the Laplace transform method. In [18], a linear matrix inequality (LMI) was used in the stability analysis of the linear fractional differential system.

Compared with the stability criteria for nonlinear integer-order differential systems, the developments of nonlinear fractional differential systems are unsatisfactory. Lyapunov's second method is an effective tool to analyze the stability of nonlinear integer-order differential systems without solving state equations. Recently, the nonlinear fractional differential systems have been discussed in several refs. [3, 4, 19, 20] and some results have been derived by using Lyapunov's method.

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The structural stability of a system with Riemann–Liouville derivative has been presented in [9]. In [5] authors investigated the system of nonautonomous FDEs involving Caputo derivative and derived the result on continuous dependence of solution on initial conditions. In [14], the Mittag–Leffler stability and the fractional Lyapunov of the second method were proposed. Deng [6] derived a sufficient stability condition of nonlinear FDEs. In this paper we introduce some developments of the stability of nonlinear fractional differential systems in detail.

The paper is organized as follows. In section 2, we present some basic materials on fractional calculus and prove a theorem to investigate the asymptotic expansions of the Mittag-Leffler function. Some stability results of the system ${}_{C}D^{\alpha}_{0,t}x(t) = Ax(t) + f(t, x(t))$ are presented in section 3. In section 4, some stability results of fractional differential systems ${}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + f(t, x(t))$ are derived and $\mathcal{F}$-asymptotic stability of the system is presented. In section 5, we present a numerical example, for which we compute different orbits of the corresponding system by means of numerical simulations, to reveal validity of our analytical results. In section 6, we conclude the paper.

2. Preliminaries

Two types of fractional derivatives of Riemann–Liouville and Caputo derivatives have been often used in fractional differential systems. We briefly recall these two definitions.

Definition 1 *The Riemann–Liouville integral $J^{\alpha}_{t_0,t}$ with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined as*

$$J^{\alpha}_{t_0,t}x(t) := D^{-\alpha}_{t_0,t}x(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau,$$

where $\Gamma(.)$ is Euler's gamma function, for $\alpha = 0$ we set $J^{0}_{t_0,t} := Id$, the identity operator.

Definition 2 *The Riemann–Liouville derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by*

$${}_{RL}D^{\alpha}_{t_0,t}x(t) := \frac{d^m}{dt^m} J^{(m-\alpha)}_{t_0,t} x(t),$$

where $m-1 < \alpha \leq m \in \mathbb{Z}_+$.

The Laplace transform of the Riemann–Liouville fractional derivative ${}_{RL}D^{\alpha}_{0,t}x(t)$ for $0 < \alpha \leq 1$ is

$$\mathcal{L}\{{}_{RL}D^{\alpha}_{0,t}x(t)\} = s^{\alpha}X(s) - (D^{\alpha-1}_{0}x(t))_{t=0}$$

Here $X(s)$ is the Laplace transform of $x(t)$.

Definition 3 *The Caputo derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by*

$${}_{C}D^{\alpha}_{t_0,t}x(t) := J^{(m-\alpha)}_{t_0,t} \frac{d^m}{dt^m} x(t),$$

where $m-1 < \alpha \leq m \in \mathbb{Z}_+$.

The Laplace transform of the Caputo fractional derivative ${}_{C}D^{\alpha}_{0,t}x(t)$ is

$$\mathcal{L}\{{}_{C}D^{\alpha}_{0,t}x(t)\} = s^{\alpha}X(s) - \sum_{k=1}^{m} s^{\alpha-k}x^{(k-1)}(0), \qquad (m-1<\alpha\le m).$$

If $0<\alpha\le 1$ we have

$$\mathcal{L}\{{}_{C}D^{\alpha}_{0,t}x(t)\} = s^{\alpha}X(s) - s^{\alpha-1}x(0).$$

Definition 4 *[16] The Mittag-Leffler function is defined by*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

where $\alpha>0, z\in\mathbb{C}$. *The two-parameter Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta>0, z\in\mathbb{C}$. *Clearly* $E_{\alpha}(z)=E_{\alpha,1}(z)$.

For $j\in\mathbb{N}_0$, $\lambda\in\mathbb{R}$, and $\alpha,\beta>0$ the Laplace transform of the function $f(t)=t^{j\alpha+\beta-1}E^{(j)}_{\alpha,\beta}(\pm\lambda t^{\alpha})$ can be easily found to be

$$\mathcal{L}\{f(t)\} = \frac{j!s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{j+1}}, \qquad (|s|>|\lambda|^{\frac{1}{\alpha}}, Re(s)>0)$$

If $\beta=\alpha$ and $j=0$ we have

$$\mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda t^{\alpha})\} = \frac{1}{s^{\alpha}\mp\lambda} \qquad (|s|>|\lambda|^{\frac{1}{\alpha}}, Re(s)>0)$$

and if $\beta=1, j=0$ we have

$$\mathcal{L}\{E_{\alpha}(\pm\lambda t^{\alpha})\} = \frac{s^{\alpha-1}}{s^{\alpha}\mp\lambda} \qquad (|s|>|\lambda|^{\frac{1}{\alpha}}, Re(s)>0).$$

The Mittag-Leffler function has the following asymptotic expression.

Lemma 1 *[16] If* $0<\alpha<2$ *and* β *is an arbitrary complex number, then for an arbitrary integer* $p\ge 1$ *the following expansions hold:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{\frac{(1-\beta)}{\alpha}}\exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^{p}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\Big(\frac{1}{|z|^{p+1}}\Big),$$

with $|z|\to\infty, |\arg(z)|\le\frac{\alpha\pi}{2}$, *and*

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\Big(\frac{1}{|z|^{p+1}}\Big),$$

with $|z|\to\infty, |\arg(z)|>\frac{\alpha\pi}{2}$.

Definition 5 *Consider the following fractional differential system:*

$$ {}_{C}D^{\alpha}_{t_0,t}x(t) = f(t, x(t)), \tag{1} $$

with initial condition $x^{(k)}(t)|_{t=t_0} = x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 0, 1, ..., m-1)$ *where* $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $m-1 < \alpha \leq m \in \mathbb{Z}_+$, *and* $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.
The system (1) *is said to be stable if, for any initial values* $x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 0, 1, ..., m-1)$, *there exists* $\epsilon > 0$ *such that any solution* $x(t)$ *of* (1) *satisfies* $\|x(t)\| < \epsilon$ *for all* $t > t_0$. *The system* (1) *is said to be asymptotically stable if* $\|x(t)\| \to 0$ *as* $t \to \infty$.

Next we consider the following general type of fractional differential equations involving Riemann–Liouville derivative:

$$ {}_{RL}D^{\alpha}_{t_0,t}x(t) = f(t, x(t)), \tag{2} $$

with suitable initial values ${}_{RL}D^{\alpha-k}_{t_0,t}x(t)|_{t=t_0} = x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 1, ..., m)$, where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $m-1 < \alpha \leq m \in \mathbb{Z}_+$, and $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 6 *The system* (2) *is said to be stable if, for any initial values* $x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 1, ..., m)$, *there exists* $\epsilon > 0$ *such that any solution* $x(t)$ *of* (2) *satisfies* $\|x(t)\| < \epsilon$ *for all* $t > t_0$. *The system* (2) *is said to be asymptotically stable if* $\|x(t)\| \to 0$ *as* $t \to \infty$.

Recently Qian et al. [17] studied the case of the following linear system of FDEs with Riemann–Liouville derivative:

$$ {}_{RL}D^{\alpha}_{t_0,t}x(t) = Ax(t), \quad (0 < \alpha < 1), \tag{3} $$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$. We recall the following theorem from [17].

Theorem 1 *The system* (3) *with initial value* ${}_{RL}D^{\alpha-1}_{t_0,t}x(t)|_{t=t_0}$, *where* $0 < \alpha < 1$ *and* $t_0 = 0$, *is:*

i) *asymptotically stable if all the nonzero eigenvalues of* A *satisfy* $|\arg(spec(A))| > \frac{\alpha\pi}{2}$, *or* A *has k-multiple zero eigenvalues corresponding to a Jordan block* $diag(J_1, J_2, ..., J_i)$, *where* J_l *is a Jordan canonical form with order* $n_l \times n_l$, $\sum_{l=1}^{i} n_l = k$, *and* $n_l\alpha < 1$ *for each* $1 \leq l \leq i$.

ii) *stable if all the nonzero eigenvalues of* A *satisfy* $|\arg(spec(A))| \geq \frac{\alpha\pi}{2}$ *and the critical eigenvalues satisfying* $|\arg(spec(A))| = \frac{\alpha\pi}{2}$ *have the same algebraic and geometric multiplicities, or* A *has* k*-multiple zero eigenvalues corresponding to a Jordan block matrix* $diag(J_1, J_2, ..., J_i)$, *where* J_l *is a Jordan canonical form with order* $n_l \times n_l, \sum_{l=1}^{i} n_l = k$, *and* $n_l\alpha \leq 1$ *for each* $1 \leq l \leq i$.

We present the following theorem needed for the stability of the nonlinear system to be discussed in the next section.

Theorem 2 *Suppose* $0 < \alpha < 2$ *and* $A_{n\times n}$ *is a matrix with* $|\arg(spec(A))| > \frac{\alpha\pi}{2}$. *Then there exists an invertible matrix T, an* $n \times n$ *matrix* M, *and a real constant* $n_0 > 0$ *such that* $\|T^{-1}E_{\alpha,\alpha}(At^{\alpha})T - \frac{M}{t^{2\alpha}}\| \leq \frac{n_0}{t^{3\alpha}}$. *Moreover, there exists a constant* $m_0 > 0$ *such that* $\|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\| \leq \frac{m_0}{t^{\alpha+1}}$ *for all* $t > 0$.

Proof.
First we need the derivatives of the Mittag-Leffler function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\quad \alpha,\beta>0, z\in\mathbb{C}.$$

Hence, we use the integral representations of the Mittag-Leffler function in the form of an improper integral along the Hankel loop, which have been treated with arbitrary β by Erdelyi et al. [12] and Dzherbashyan [10, 11] as

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i\alpha}\int_{\gamma(\epsilon;\delta)}\frac{\exp(\zeta^{\frac{1}{\alpha}})\zeta^{\frac{1-\beta}{\alpha}}}{\zeta-z}d\zeta,\quad z\in G^{(-)}(\epsilon;\delta) \tag{4}$$

and

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{\frac{1-\beta}{\alpha}}\exp(z^{\frac{1}{\alpha}})+\frac{1}{2\pi i\alpha}\int_{\gamma(\epsilon;\delta)}\frac{\exp(\zeta^{\frac{1}{\alpha}})\zeta^{\frac{1-\beta}{\alpha}}}{\zeta-z}d\zeta,\quad z\in G^{(+)}(\epsilon;\delta) \tag{5}$$

under the conditions

$$0<\alpha<2,\ \frac{\pi\alpha}{2}<\delta\le\min\{\pi,\pi\alpha\}. \tag{6}$$

The contour $\gamma(\epsilon;\delta)$ is depicted in Figure 1, which consists of two rays $S_{-\delta}$ and S_δ, $\arg(\zeta)=-\delta,|\zeta|\ge\epsilon$ and $\arg(\zeta)=\delta,|\zeta|\ge\epsilon$, respectively, a circular $C_\delta(0;\epsilon)$, $|\zeta|=\epsilon,-\delta\le\arg(\zeta)\le\delta$, the region $G^{-}(\epsilon;\delta)$ on the left side and the region $G^{+}(\epsilon;\delta)$, on the right side. Using the integral representations in (4) and (5), it is not difficult to get asymptotic expansions for the Mittag-Leffler function in the complex plane. Let $0<\alpha<2$, and δ be chosen to satisfy the condition (6). Then for any constants $p\in\mathbb{N}$ and $\beta=\alpha$ we have

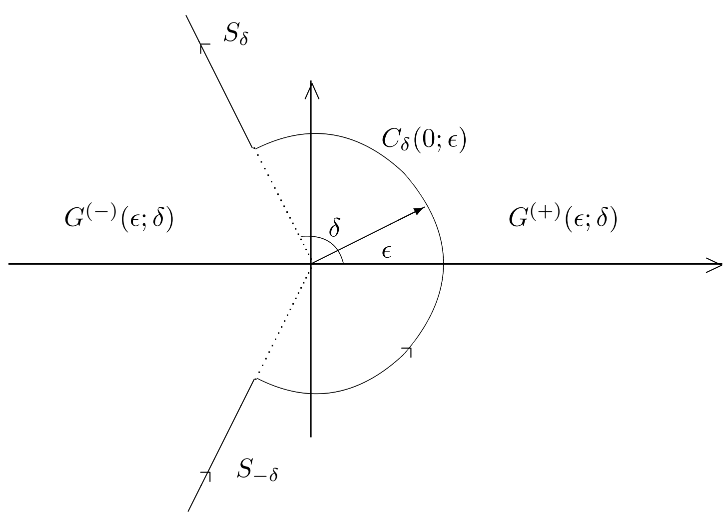

Figure 1. The contour of $\gamma(\epsilon;\delta)$.

$$E_{\alpha,\alpha}(z)=-\sum_{k=2}^{p}\frac{1}{\Gamma(\alpha-\alpha k)}\frac{1}{z^k}+I_p(z),\quad z\in G^{(-)}(1;\delta) \tag{7}$$

in which

$$I_p(z) = \frac{1}{2\pi i\alpha z^p}\int_{\gamma(1;\delta)} \frac{\exp(\zeta^{\frac{1}{\alpha}})\zeta^{\frac{1-\alpha}{\alpha}+p}}{\zeta - z}d\zeta.$$

To proceed the proof first suppose that $1 \leq \alpha < 2$ and $\arg(z) > \frac{\alpha\pi}{2}$; we choose δ such that $\frac{\alpha\pi}{2} < \delta \leq \pi$ and then a simple calculation shows that for all $z \in G^{(-)}(1;\delta)$, $|z| \geq |\sec\delta|$,

$$dist\{z, \gamma(1;\delta)\} = |z|\sin(\theta - \delta),$$

where $\theta = |\arg(z)|$. Now by setting $z = \lambda t^\alpha$ and $\phi(\zeta) = \exp(\zeta^{\frac{1}{\alpha}})\zeta^{\frac{1-\alpha}{\alpha}+p}$, and by using

$$|\frac{\partial I_p(z)}{\partial\lambda}| = t^\alpha|\frac{\partial I_p(z)}{\partial z}|.$$

We get

$$|\frac{\partial I_p(\lambda t^\alpha)}{\partial\lambda}| \leq \frac{pt^\alpha}{\pi\alpha|\lambda|^{p+2}t^{(p+2)\alpha}}(I_1 + I_2),$$

in which

$$I_1 = \int_{\gamma(1;\delta)} \frac{|\phi(\zeta)|}{\sin(\theta - \delta)}d|\zeta|,$$

and

$$I_2 = \int_{\gamma(1;\delta)} \frac{|\phi(\zeta)|}{\sin^2(\theta - \delta)}d|\zeta|.$$

We now take $\zeta = re^{i\delta}$, $1 \leq r < \infty$ on the ray S_δ and then get $|\exp(\zeta^{\frac{1}{\alpha}})| = \exp(|\zeta|^{\frac{1}{\alpha}}\cos(\frac{\delta}{\alpha}))$. We have $\frac{\pi}{2} < \frac{\delta}{\alpha} \leq \frac{\pi}{\alpha} < \pi$ and then $\cos(\frac{\delta}{\alpha}) < 0$. Setting $\cos(\frac{\delta}{\alpha}) = -\gamma$, where γ is a positive constant, leads to

$$I_1|_{S_\delta} = \frac{1}{\sin(\theta - \delta)}\int_1^{+\infty} \exp(-\gamma r^{\frac{1}{\alpha}})r^{\frac{1-\alpha}{\alpha}+p}dr,$$

and

$$I_2|_{S_\delta} = \frac{1}{\sin^2(\theta - \delta)}\int_1^{+\infty} \exp(-\gamma r^{\frac{1}{\alpha}})r^{\frac{1-\alpha}{\alpha}+p}dr,$$

in which the right-hand sides of both inequalities are finite. A similar argument on the ray $S_{-\delta}$ shows that these integrals are finite. We notice that on the circular $C_\delta(0;1)$ (that is a compact set) integrals are finite. Thus there exists a constant c such that

$$|\frac{\partial I_p(\lambda t^\alpha)}{\partial\lambda}| \leq \frac{ct^\alpha}{|\lambda|^{p+2}t^{\alpha(p+2)}}.$$

Thus

$$\frac{\partial I_p(\lambda t^\alpha)}{\partial\lambda} = O\Big(\frac{1}{t^{\alpha(p+1)}|\lambda|^{p+2}}\Big),$$

and by induction we get

$$\frac{1}{j!}\Big(\frac{\partial}{\partial\lambda}\Big)^{(j)} I_p(\lambda t^\alpha) = O\Big(\frac{1}{t^{\alpha(p+1)}|\lambda|^{p+j+1}}\Big). \tag{8}$$

Moreover, by using (7) and (8) and by choosing $p=2$ we get

$$\frac{1}{j!}\Big(\frac{\partial}{\partial\lambda}\Big)^{(j)}E_{\alpha,\alpha}(\lambda t^{\alpha}) = \frac{1}{j!}\Big(\frac{\partial}{\partial\lambda}\Big)^{(j)}\Big\{-\frac{1}{\Gamma(-\alpha)}\frac{1}{t^{2\alpha}\lambda^2}+I_2(\lambda t^{\alpha})\Big\} \tag{9}$$

$$= \frac{1}{j!}\Big(\frac{\partial}{\partial\lambda}\Big)^{(j)}\Big\{-\frac{1}{\Gamma(-\alpha)}\frac{1}{t^{2\alpha}\lambda^2}\Big\}+\frac{1}{j!}\Big(\frac{\partial}{\partial\lambda}\Big)^{(j)}\Big\{I_2(\lambda t^{\alpha})\Big\}$$

$$= -\frac{(j+1)(-1)^j}{\Gamma(-\alpha)\lambda^{j+2}t^{2\alpha}}+O\Big(\frac{1}{t^{3\alpha}|\lambda|^{j+3}}\Big).$$

Now suppose that $0<\alpha<1$ and $\arg(z)>\frac{\alpha\pi}{2}$, in this case by choosing δ such that $\frac{\alpha\pi}{2}<\delta<\min\{\arg(z),\frac{\pi}{2},\frac{3\pi}{2}\alpha\}$; then a simple calculation shows that for some $z\in G^{(-)}(1;\delta)$,

$$dist\{z,\gamma(1;\delta)\}=|z|\sin(\theta-\delta),$$

where $\theta=|\arg(z)|$, that by a similar argument we get equation (9), and for other some $z\in G^{(-)}(1;\delta)$, with $|z|$ sufficient large we have

$$dist\{z,\gamma(1;\delta)\}\geq|z|-1,$$

then

$$|\frac{\partial I_p(\lambda t^{\alpha})}{\partial\lambda}|\leq\frac{pt^{\alpha}}{\pi\alpha|\lambda|^{p+2}t^{(p+2)\alpha}}(I_3+I_4),$$

in which

$$I_3=(\frac{1}{1-\frac{1}{|z|}})\int_{\gamma(1;\delta)}|\phi(\zeta)|d|\zeta|,$$

and

$$I_4=\frac{1}{(1-\frac{1}{|z|})^2}\int_{\gamma(1;\delta)}|\phi(\zeta)|d|\zeta|.$$

A similar argument on the ray S_{δ} and $S_{-\delta}$ and circular $C_{\delta}(0;1)$ shows that these integrals are finite and we get equation (9). Next, suppose that the matrix A is similar to a Jordan canonical form, i.e. there exists an invertible matrix T such that $J=T^{-1}AT=diag(J_1,\cdots,J_r)$, where $J_i, 1\leq i\leq r$, $\sum_{i=1}^{r}n_i=n$, and has the following form

$$\begin{pmatrix}\lambda_i & 1 & 0 & \cdots & 0\\ 0 & \lambda_i & 1 & \cdots & \vdots\\ 0 & 0 & \lambda_i & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & 1\\ 0 & \cdots & & 0 & \lambda_i\end{pmatrix}_{n_i\times n_i}.$$

Obviously,

$$E_{\alpha,\alpha}(At^{\alpha})=Tdiag[E_{\alpha,\alpha}(J_1t^{\alpha}),E_{\alpha,\alpha}(J_2t^{\alpha}),\cdots,E_{\alpha,\alpha}(J_rt^{\alpha})]T^{-1},$$

and

$$E_{\alpha,\alpha}(J_it^{\alpha})=\sum_{k=0}^{\infty}\frac{(J_it^{\alpha})^k}{\Gamma(\alpha k+\alpha)}=\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha k+\alpha)}J_i^k=$$

$$\begin{pmatrix} E_{\alpha,\alpha}(\lambda_i t^\alpha) & \frac{1}{1!}\left(\frac{\partial}{\partial\lambda_i}\right)E_{\alpha,\alpha}(\lambda_i t^\alpha) & \cdots & & \frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{(n_i-1)}E_{\alpha,\alpha}(\lambda_i t^\alpha) \\ 0 & E_{\alpha,\alpha}(\lambda_i t^\alpha) & \ddots & \cdots & \vdots \\ 0 & 0 & E_{\alpha,\alpha}(\lambda_i t^\alpha) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{1!}\left(\frac{\partial}{\partial\lambda_i}\right)E_{\alpha,\alpha}(\lambda_i t^\alpha) \\ 0 & \cdots & & 0 & E_{\alpha,\alpha}(\lambda_i t^\alpha) \end{pmatrix}.$$

We set $M = diag[M_1, \ldots, M_r]_{n\times n}$ in which

$$M_i = \begin{pmatrix} \frac{-1}{\Gamma(-\alpha)\lambda_i^2} & \frac{2}{\Gamma(-\alpha)\lambda_i^3} & \cdots & & \frac{-n_i(-1)^{n_i-1}}{\Gamma(-\alpha)\lambda_i^{2+(n_i-1)}} \\ 0 & \frac{-1}{\Gamma(-\alpha)\lambda_i^2} & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \frac{2}{\Gamma(-\alpha)\lambda_i^3} \\ 0 & \cdots & & 0 & \frac{-1}{\Gamma(-\alpha)\lambda_i^2} \end{pmatrix}_{n_i\times n_i}.$$

Thus by using (9) we get

$$\|E_{\alpha,\alpha}(J_i t^\alpha) - \frac{M_i}{t^{2\alpha}}\| = \left\| \begin{pmatrix} O(\frac{1}{t^{3\alpha}|\lambda_i|^3}) & O(\frac{1}{t^{3\alpha}|\lambda_i|^4}) & \cdots & & O(\frac{1}{t^{3\alpha}|\lambda_i|^{3+(n_i-1)}}) \\ 0 & O(\frac{1}{t^{3\alpha}|\lambda_i|^3}) & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & O(\frac{1}{t^{3\alpha}|\lambda_i|^4}) \\ 0 & \cdots & & 0 & O(\frac{1}{t^{3\alpha}|\lambda_i|^3}) \end{pmatrix}_{n_i\times n_i} \right\|,$$

Now for each $i = 1, \ldots, r$, there exist $k_0, k_1, \ldots, k_{n_i-1} \geq 0$ such that

$$O(\frac{1}{t^{3\alpha}|\lambda_i|^3}) \leq \frac{k_0}{t^{3\alpha}}, \ldots, O(\frac{1}{t^{3\alpha}|\lambda_i|^{3+(n_i-1)}}) \leq \frac{k_{n_i-1}}{t^{3\alpha}},$$

then

$$\|E_{\alpha,\alpha}(J_i t^\alpha) - \frac{M_i}{t^{2\alpha}}\| \leq \left\| \begin{pmatrix} \frac{k_0}{t^{3\alpha}} & \frac{k_1}{t^{3\alpha}} & \cdots & & \frac{k_{n_i-1}}{t^{3\alpha}} \\ 0 & \frac{k_0}{t^{3\alpha}} & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \frac{k_1}{t^{3\alpha}} \\ 0 & \cdots & & 0 & \frac{k_0}{t^{3\alpha}} \end{pmatrix} \right\| \leq \frac{1}{t^{3\alpha}} c_i,$$

where

$$c_i = \left\| \begin{pmatrix} k_0 & k_1 & \cdots & & k_{n_i-1} \\ 0 & k_0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & k_1 \\ 0 & \cdots & & 0 & k_0 \end{pmatrix} \right\|,$$

then

$$\|T^{-1}E_{\alpha,\alpha}(At^{\alpha})T - \frac{M}{t^{2\alpha}}\| = \|diag[E_{\alpha,\alpha}(J_1 t^{\alpha}), \cdots, E_{\alpha,\alpha}(J_r t^{\alpha})] - \frac{M}{t^{2\alpha}}\| \leq \frac{n_0}{t^{3\alpha}}$$

where $n_0 = \max\{c_1, \ldots, c_r\}$. Multiplying by $t^{\alpha-1}$, we get

$$\|T^{-1}t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})T - Mt^{-\alpha-1}\| \leq n_0 t^{-2\alpha-1},$$

and so

$$\|T^{-1}t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})T\| \leq \|M\|t^{-\alpha-1} + n_0 t^{-2\alpha-1} \leq t^{-\alpha-1}\Big[\|M\| + n_0 t^{-\alpha}\Big].$$

Thus there exists a constant $l' > 0$ such that for $t \geq 1$, we have

$$\|T^{-1}t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})T\| \leq \frac{l'}{t^{\alpha+1}}.$$

Then we get

$$\|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\| = \|TT^{-1}t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})TT^{-1}\| \leq \|T\|.\|T^{-1}t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})T\|.\|T^{-1}\|$$

$$\leq \|T\|\frac{l'}{t^{\alpha+1}}\|T^{-1}\| \leq \frac{l}{t^{\alpha+1}},\ l = \max\{l', \|T\|\|T^{-1}\|\}$$

Otherwise, for $0 \leq t \leq 1$, $E_{\alpha,\alpha}(At^{\alpha})$ is a continuous function and thus $E = \sup_{0\leq t\leq 1} \|E_{\alpha,\alpha}(At^{\alpha})\|$ exists. Then for $0 < t < 1$ we have

$$\|E_{\alpha,\alpha}(At^{\alpha})\| \leq E \leq \frac{E}{t^{2\alpha}},$$

This implies

$$\|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\| \leq \frac{E}{t^{\alpha+1}}.$$

Now, we set $m_0 = \max\{l, E\}$; then for all $t > 0$ we have

$$\|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\| \leq \frac{m_0}{t^{\alpha+1}},$$

which completes the proof. □

Theorem 3 *Suppose* $0 < \alpha < 2$ *and* $A_{n\times n}$ *is a matrix with* $|\arg(spec(A))| > \frac{\alpha\pi}{2}$. *Then there exist an invertible matrix T, a matrix* $M_{n\times n}$, *and a real constant* $n_0 > 0$ *such that* $\|T^{-1}E_\alpha(At^\alpha)T - \frac{M}{t^\alpha}\| \leq \frac{n_0}{t^{2\alpha}}$; *moreover, there exists a constant* $m_0 > 0$ *such that* $\|E_\alpha(At^\alpha)\| \leq \frac{m_0}{t^\alpha}$ *for all* $t > 0$.

Proof. The proof is similar to that of Theorem 2. □

We present the following definition needed for the stability of the nonlinear system to be discussed in the next section.

Definition 7 *A solution of fractional differential system* (2) *is said to be* $\mathcal{F}$*-asymptotically stable if every solution that belongs to set* $\mathcal{F}$ *is asymptotically stable. Moreover, a solution of fractional differential system* (2) *is said to be* $\mathcal{F}$*-stable if every solution that belongs to* $\mathcal{F}$ *is stable.*

3. Stability of ${}_{C}D_{0,t}^{\alpha}x(t) = Ax(t) + f(t, x(t))$

In this section, we study the following fractional differential system with Caputo derivative:

$$ {}_{C}D_{0,t}^{\alpha}x(t) = Ax(t) + f(t, x(t)), \quad (0 < \alpha < 1) \tag{10} $$

under the initial condition $x(0) = x_0$, where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. We can get the solution of (10), by using the Laplace and inverse Laplace transforms, as

$$ x(t) = E_\alpha(At^\alpha)x_0 + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^\alpha)f(\theta, x(\theta))d\theta. \tag{11} $$

We establish the following stability results.

Theorem 4 *Suppose* f *is a continuous vector function for which there exists* $p > \frac{1}{\alpha}$ *such that* $\|f(., x(t))\| \in L^p(\mathbb{R}^+)$. *Then the system* (10) *is stable if all the eigenvalues of* A *satisfy*

$$ |\arg(spec(A))| > \frac{\alpha\pi}{2}, \tag{12} $$

especially if $\|f(., x(t))\| \in L^\infty(\mathbb{R}^+)$ the stability holds.

Proof. Equation (11) implies

$$ \|x(t)\| \leq \|E_\alpha(At^\alpha)\| \, \|x_0\| + \int_0^t \|(t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^\alpha)\| \, \|f(\theta, x(\theta))\|d\theta. $$

According to Theorem 3, there exists $m_0 > 0$ such that $\|E_\alpha(At^\alpha)\| \leq \frac{m_0}{t^\alpha}$; then

$$ \|x(t)\| \leq \frac{m_0}{t^\alpha} \, \|x_0\| + \int_0^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^\alpha)\| \, \|f(t-\theta, x(t-\theta))\|d\theta. \tag{13} $$

We now set

$$ I = \int_0^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^\alpha)\| \, \|f(t-\theta, x(t-\theta))\|d\theta. $$

First suppose that $\frac{1}{\alpha} < p < \infty$; then by applying Hölder's inequality, we obtain

$$I \leq \Big[\int_0^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^\alpha)\|^q d\theta\Big]^{\frac{1}{q}}\Big[\int_0^t \|f(t-\theta,x(t-\theta))\|^p d\theta\Big]^{\frac{1}{p}}, \tag{14}$$

where $\frac{1}{p}+\frac{1}{q}=1$. We have

$$\int_0^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^\alpha)\|^q d\theta = \int_0^1 \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^\alpha)\|^q d\theta + \int_1^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^\alpha)\|^q d\theta,$$

where $t \geq 1$. Furthermore,

$$\int_0^1 \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^\alpha)\|^q d\theta \leq E^q \int_0^1 \theta^{q\alpha-q} d\theta, \tag{15}$$

where $E = \sup_{0\leq t\leq 1}\|E_{\alpha,\alpha}(At^\alpha)\|$, and the right-hand side of (15) is bounded for $p > \frac{1}{\alpha}$.
We also have

$$\int_1^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^\alpha)\|^q d\theta \leq \int_1^t (\frac{m_0}{\theta^{\alpha+1}})^q,$$

which is bounded. On the other hand,

$$\int_0^t \|f(t-\theta,x(t-\theta))\|^p d\theta = \int_0^t \|f(\theta,x(\theta))\|^p d\theta \leq \|f\|_p^p.$$

Hence, the right-hand side of (14) remains bounded as $t \longrightarrow \infty$, and so the right-hand side of (13) remains bounded and the system (10) is stable.
It remains to prove our claim for $p=\infty$. In this case $q=1$ and $\|f(.,x(t))\| \in L^\infty(\mathbb{R}^+)$. Thus f is bounded with the upper bound $F \geq 0$, and

$$\int_0^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^\alpha)\|\|f(t-\theta,x(t-\theta))\|d\theta \leq F\int_0^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^\alpha)\|d\theta$$

$$\leq F\Big(\int_0^1 \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^\alpha)\|d\theta + \int_1^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^\alpha)\|d\theta\Big)$$

$$\leq \frac{F}{\alpha}\Big(E + m_0(1-\frac{1}{t^\alpha})\Big),$$

which remains bounded as $t \longrightarrow \infty$. Thus (10) is stable. □

Corollary 1. Suppose A is an $n\times n$ matrix that satisfied (12), and there exist functions $\varphi:\mathbb{R}^+ \longmapsto \mathbb{R}^+$ and $\psi:\mathbb{R}^n \longmapsto \mathbb{R}^+$ such that ψ is bounded, $\varphi \in L^p(\mathbb{R}^+)$ for $p > \frac{1}{\alpha}$, and $\|f(t,x(t))\| \leq \varphi(t)\psi(x(t))$. Then the solution of (10) is stable. Especially by this assumption the system

$${}_C D_{0,t}^\alpha x(t) = Ax(t) + b(t) \quad (0<\alpha<1),$$

where $\|b(t)\| \in L^p(\mathbb{R}^n)$, $p > \frac{1}{\alpha}$ is stable.

4. Stability of ${}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + f(t, x(t))$

In this section, we study the following fractional differential system with Riemann–Liouville derivative

$$ {}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + f(t, x(t)), \quad (0 < \alpha < 1) \tag{16} $$

with the initial condition $x_0 = {}_{RL}D^{\alpha-1}_{0,t}x(t)|_{t=0}$, where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ $\in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.
We can get the solution of (16), by using the Laplace and inverse Laplace transforms, as

$$ x(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})x_0 + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})f(\theta, x(\theta))d\theta. \tag{17} $$

Theorem 5 *Suppose f is a continuous vector function for which there exists $p > \frac{1}{\alpha}$ such that $\|f(., x(t))\| \in L^p(\mathbb{R}^+)$. Then the system (16) is stable if all the eigenvalues of A satisfy (12). Especially if $\|f(., x(t))\| \in L^{\infty}(\mathbb{R}^+)$ the stability holds.*

Proof. We have

$$ \|x(t)\| \le \|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\| \ \|x_0\| + \int_0^t (t-\theta)^{\alpha-1}\|E_{\alpha,\alpha}(A(t-\theta)^{\alpha})\| \ \|f(\theta, x(\theta))\|d\theta. $$

According to Theorem 2 there exists $m_0 > 0$ such that $\|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\| \le \frac{m_0}{t^{\alpha+1}}$; thus

$$ \|x(t)\| \le \frac{m_0}{t^{\alpha+1}}\|x_0\| + \int_0^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|\|f(t-\theta, x(t-\theta))\|d\theta. \tag{18} $$

We now set

$$ I = \int_0^t \theta^{\alpha-1}\|E_{\alpha,\alpha}(A\theta^{\alpha})\| \ \|f(t-\theta, x(t-\theta))\|d\theta. $$

First suppose that $\frac{1}{\alpha} < p < \infty$; then similar to theorem 4 by applying Hölder's inequality, we obtain

$$ I \le \Big[\int_0^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|^q d\theta\Big]^{\frac{1}{q}} \Big[\int_0^t \|f(t-\theta, x(t-\theta))\|^p d\theta\Big]^{\frac{1}{p}}, \tag{19} $$

where $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$ \int_0^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|^q d\theta = \int_0^1 \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|^q d\theta + \int_1^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|^q d\theta, $$

and also

$$ \int_0^1 \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|^q d\theta \le E^q \int_0^1 \theta^{q\alpha-q}d\theta, \tag{20} $$

which is bounded for $p > \frac{1}{\alpha}$, and

$$ \int_1^t \theta^{q\alpha-q}\|E_{\alpha,\alpha}(A\theta^{\alpha})\|^q d\theta \le \int_1^t (\frac{m_0}{\theta^{\alpha+1}})^q, $$

which is bounded. Thus the right-hand side of (18) remains bounded and the system (16) is stable.
Our proof claim for $p=\infty$ is similar to the proof of theorem 4.

Corollary 2. Suppose the matrix A satisfies in (12) and $f(.,x(t))$ is a continuous function and there exist functions $\varphi:\mathbb{R}^+\longmapsto\mathbb{R}^+$ and $\psi:\mathbb{R}^n\longmapsto\mathbb{R}^+$ such that ψ is bounded, $\varphi\in L^p(\mathbb{R}^+)$ for some $p>\frac{1}{\alpha}$, and $\|f(t,x(t))\|\leq\varphi(t)\psi(x(t))$. Then the solution of (16) is stable. Especially the system

$$ {}_{RL}D^{\alpha}_{0,t}x(t)=Ax(t)+b(t),\quad (0<\alpha<1)$$

in which $\|b(t)\|\in L^p(\mathbb{R}^n)$ is stable for $p>\frac{1}{\alpha}$.

Theorem 6 *Suppose that all the eigenvalues of A satisfy in (12) and the following conditions hold:*

i) $\|f(t,x(t))\|\leq t^{\gamma}\varphi(\|x(t)\|)$ *where* $\varphi(\|x(t)\|)\in L^p(\mathbb{R}^+)$,

ii) $p>\frac{1}{\alpha}$ *and* $\gamma<\frac{1}{p}-\alpha$.

Then the system (16) is $\mathcal{F}$-asymptotically stable, for

$$\mathcal{F}:=\big\{x(t):[0,\infty)\longmapsto\mathbb{R}^n,\varphi(\|x(t)\|)\in L^p(\mathbb{R}^+)\big\}.$$

Proof. Clearly, we have

$$\|x(t)\|\leq t^{\alpha-1}\|E_{\alpha,\alpha}(At^{\alpha})\|\|x_0\|+E\int_0^t(t-\theta)^{\alpha-1}\theta^{\gamma}\varphi(\|x(\theta)\|)d\theta.$$

such that $E=\sup_{0\leq t<\infty}\|E_{\alpha,\alpha}(At^{\alpha})\|$. Applying Hölder's inequality yields

$$\int_0^t(t-\theta)^{\alpha-1}\theta^{\gamma}\varphi(\|x(\theta)\|)d\theta\leq\Big[\int_0^t(t-\theta)^{q\alpha-q}\theta^{q\gamma}d\theta\Big]^{\frac{1}{q}}\|\varphi(\|x(\theta)\|)\|_p. \tag{21}$$

We observe that if $\theta=ty$, then

$$\begin{aligned}\int_0^t(t-\theta)^{q\alpha-q}\theta^{q\gamma}d\theta&=\int_0^1 t^{q\alpha-q}(1-y)^{q\alpha-q}t^{q\gamma}y^{q\gamma}tdy\\&=t^{q\alpha-q+q\gamma+1}\mathcal{B}(q\alpha-q+1,q\gamma+1),\end{aligned} \tag{22}$$

where $\mathcal{B}(.,.)$ stands for the Beta function. By the assumptions of this theorem (22) tends to zero, and then the right-hand side of (21) tends to zero. This completes the proof. □

Remark 1 In the previous theorem if $\gamma=\frac{1}{p}-\alpha$, then the system (16) is $\mathcal{F}$-stable. In the case $p=\infty$, we have the following theorem:

Theorem 7 *Suppose that all the eigenvalues of A satisfy in (12) and the following conditions hold:*

i) $\|f(t,x(t))\|\leq t^{\gamma}\varphi(\|x(t)\|)$, *where* $\varphi(\|x(t)\|)\in L^{\infty}(\mathbb{R}^+)$,

ii) $\gamma<-\alpha$.

Then the system (16) is $\mathcal{F}$-asymptotically stable, for $\mathcal{F}:=L^{\infty}(\mathbb{R}^+)$.

Proof. We have

$$\|x(t)\| \leq t^{\alpha-1}\|E_{\alpha,\alpha}(At^{\alpha})\|\|x_0\| + E\int_0^t (t-\theta)^{\alpha-1}\theta^{\gamma}\varphi(\|x(\theta)\|)d\theta,$$

and

$$\Big|\int_0^t (t-\theta)^{\alpha-1}\theta^{\gamma}\varphi(\|x(\theta)\|)d\theta\Big| \leq \Big[\int_0^t (t-\theta)^{\alpha-1}\theta^{\gamma}d\theta\Big]\big\|\varphi(\|x(\theta)\|)\big\|_{\infty}. \tag{23}$$

We set $\theta = ty$; then

$$\int_0^t (t-\theta)^{\alpha-1}\theta^{\gamma}d\theta = t^{\alpha+\gamma}\mathcal{B}(\alpha,\gamma+1).$$

By using the assumptions the right-hand side of (23) tends to zero. This completes the proof. □

Remark 2 It is easily verified that in Theorem 7, for $\gamma = -\alpha$, the system (16) is $\mathcal{F}$-stable.

5. Numerical approach and example

Example 5.1. Consider the Riemann–Liouville fractional-order model presented by

$$\begin{pmatrix} x^{\alpha} \\ y^{\alpha} \\ z^{\alpha} \end{pmatrix} = \begin{pmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-kxz}{1+\|X\|^2}g(t) \\ \frac{hx^2}{1+\|X\|^2}h(t) \end{pmatrix}, \tag{24}$$

in which a, b, c, k, and h are positive parameters, $g(t), h(t) \in L^p(\mathbb{R}^+)$ and $X = (x, y, z)$. If we set

$$A = \begin{pmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{pmatrix}, F(t,X) = \begin{pmatrix} 0 \\ \frac{-kxz}{1+\|X\|^2}g(t) \\ \frac{hx^2}{1+\|X\|^2}h(t) \end{pmatrix},$$

then $\|F(.,X)\| \in L^p(\mathbb{R}^+)$. By choosing appropriate parameter values, we get $|\arg(spec(A))| > \frac{\alpha\pi}{2}$. Therefore, according to Theorem 5 the system (24) becomes stable for $p > \frac{1}{\alpha}$.

To verify the stability results of this example numerically, we perform numerical simulation by means of the method given by Atanackovic and Stankovic [2]. In [2] it was shown that for a function $f(t)$, the Riemann–Liouville derivative of order α with $0 < \alpha < 1$ may be expressed as

$$_{RL}D_{0,t}^{\alpha}f(t) = \frac{1}{\Gamma(2-\alpha)}\times$$
$$\Bigg[\frac{f'(t)}{t^{\alpha-1}}(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}) - \Big(\frac{\alpha-1}{t^{\alpha}}f(t) + \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}(\frac{f(t)}{t^{\alpha}} + \frac{v_p(f)(t)}{t^{p-1+\alpha}})\Big)\Bigg], \tag{25}$$

where

$$v_p(f)(t) = -(p-1)\int_0^t \tau^{p-2}f(\tau)d\tau, \quad p = 2, 3, \cdots$$

For the sake of simplicity, we proceed the computations as follows.
First we approximate ${}_{RL}D^{\alpha}_{0,t}f(t)$ by using the first M terms in the sum appearing in Eq. (25) by

$$
{}_{RL}D^{\alpha}_{0,t}f(t) \simeq \frac{1}{\Gamma(2-\alpha)}\times
\left[\frac{f'(t)}{t^{\alpha-1}}\Big(1+\sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\Big)-\Big(\frac{\alpha-1}{t^{\alpha}}f(t)+\sum_{p=2}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\big(\frac{f(t)}{t^{\alpha}}+\frac{v_p(f)(t)}{t^{p-1+\alpha}}\big)\Big)\right]. \tag{26}
$$

We can rewrite Eq. (26) as follows

$$
{}_{RL}D^{\alpha}_{0,t}f(t) \simeq \Omega(\alpha,t,M)f'(t)+\Phi(\alpha,t,M)f(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{v_p(f)(t)}{t^{p-1+\alpha}},
$$

where

$$
\Omega(\alpha,t,M)=\frac{1+\sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}}{\Gamma(2-\alpha)t^{\alpha-1}},\quad R(\alpha,t)=\frac{1-\alpha}{\Gamma(2-\alpha)t^{\alpha}},
$$

and

$$
\Phi(\alpha,t,M)=R(\alpha,t)+\sum_{p=2}^{M}\frac{A(\alpha,t,p)}{t^{\alpha}},\quad A(\alpha,t,p)=-\frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(p-1)!},
$$

We set

$$
v_p(x)(t)=w_p(t),\ v_p(y)(t)=u_p(t),\ v_p(z)(t)=k_p(t),\quad p=2,3,...
$$

and rewrite system (24) as

$$
\Omega(\alpha,t,M)x'(t)+\Phi(\alpha,t,M)x(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{w_p(t)}{t^{p-1+\alpha}}=a(y-x),
$$

where

$$
w_p(t)=-(p-1)\int_0^t\tau^{p-2}x(\tau)d\tau,\quad p=2,3,...,M.
$$

We also have

$$
\Omega(\alpha,t,M)y'(t)+\Phi(\alpha,t,M)y(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{u_p(t)}{t^{p-1+\alpha}}=bx-\frac{kxz}{1+\|X\|^2}g(t),
$$

where

$$
u_p(t)=-(p-1)\int_0^t\tau^{p-2}y(\tau)d\tau,\quad p=2,3,...,M.
$$

Further

$$\Omega(\alpha,t,M)z'(t)+\Phi(\alpha,t,M)z(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{k_p(t)}{t^{p-1+\alpha}}=-cz+\frac{hx^2}{1+\|X\|^2}h(t),$$

where

$$k_p(t)=-(p-1)\int_0^t \tau^{p-2}z(\tau)d\tau,\quad p=2,3,...,M$$

Now we can rewrite the above equations as the following forms:

$$x'(t)=\frac{1}{\Omega(\alpha,t,M)}\Big[a(y-x)-\Phi(\alpha,t,M)x(t)-\sum_{p=2}^{M}A(\alpha,t,p)\frac{w_p(t)}{t^{p-1+\alpha}}\Big],\tag{27}$$

where

$$w_p'(t)=-(p-1)t^{p-2}x(t),\quad p=2,3,...,M$$

and

$$y'(t)=\frac{1}{\Omega(\alpha,t,M)}\Big[bx-\frac{kxz}{1+\|X\|^2}g(t)-\Phi(\alpha,t,M)y(t)-\sum_{p=2}^{M}A(\alpha,t,p)\frac{u_p(t)}{t^{p-1+\alpha}}\Big],\tag{28}$$

in which

$$u_p'(t)=-(p-1)t^{p-2}y(t),\quad p=2,3,...,M$$

and

$$z'(t)=\frac{1}{\Omega(\alpha,t,M)}\Big[-cz+\frac{hx^2}{1+\|X\|^2}h(t)-\Phi(\alpha,t,M)z(t)-\sum_{p=2}^{M}A(\alpha,t,p)\frac{k_p(t)}{t^{p-1+\alpha}}\Big],\tag{29}$$

with

$$k_p'(t)=-(p-1)t^{p-2}z(t),\quad p=2,3,...,M$$

along with the following initial conditions

$$\begin{aligned}&x(\delta)=x_0,\ w_p(\delta)=0,\ p=2,3,...,M,\\&y(\delta)=y_0,\ u_p(\delta)=0,\ p=2,3,...,M,\\&z(\delta)=z_0,\ k_p(\delta)=0,\ p=2,3,...,M,\end{aligned}\tag{30}$$

where δ is a positive constant. Now we consider the numerical solution of the system of ordinary differential Eqs. (27), (28), (29), with the initial conditions (30) by using the well-known Runge–Kutta method of fourth order and depict orbits of the system (24) for different sets of parameters.

Phase portrait and numerical values of (24) for the fixed parameter values $\alpha=.98, b=-400, c=2, k=10, h=40, \omega=20, x_0=0.1, y_0=0.1, z_0=0.1$, and the function $g(t)=\sin\omega t$ and $h(t)=\cos\omega t$ that are in $L^\infty(\mathbb{R}^+)$, are depicted in Figures 2–11.

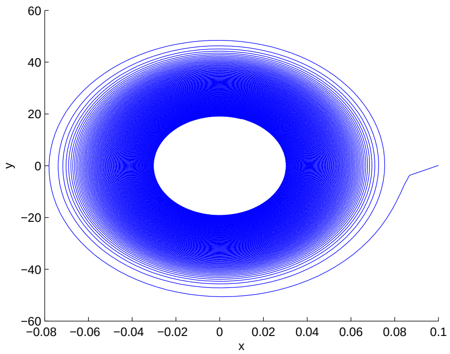

Figure 2. $x-y$ plane of (24), for $a=.01$.

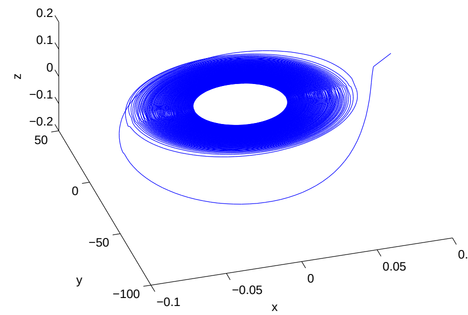

Figure 3. Phase portrait of (24), for $a=.01$.

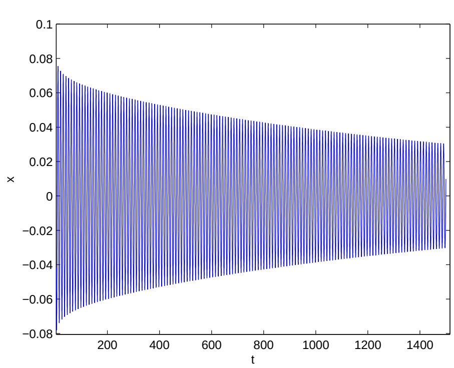

Figure 4. Numerical value of (24), for $a=.01$.

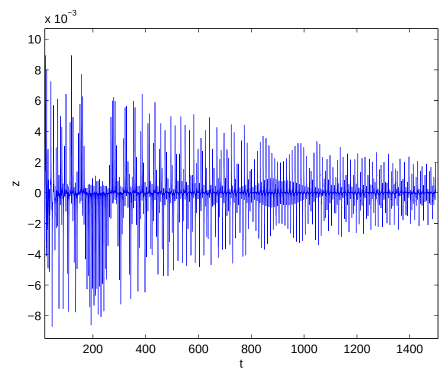

Figure 5. Numerical value of (24), for $a=.01$.

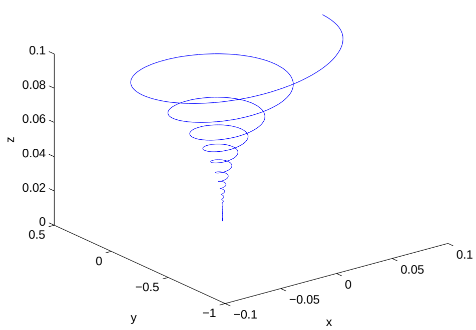

Figure 6. Phase portrait of (24), for $a=10$.

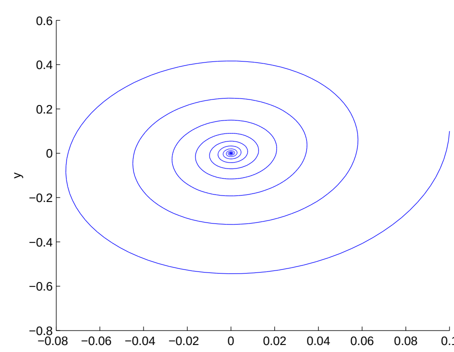

Figure 7. $x-y$ plane of (24), for $a=10$.

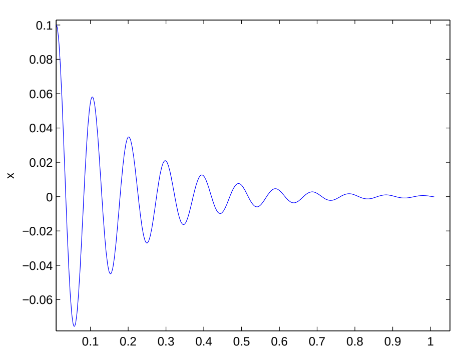

Figure 8. Numerical value of (24), for $a=10$.

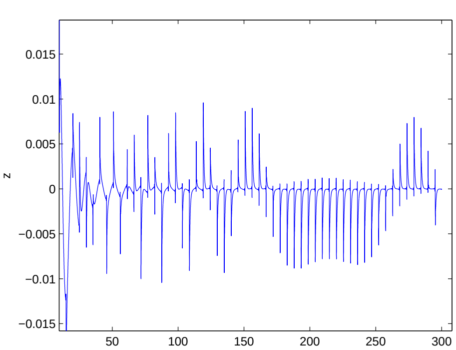

Figure 9. Numerical value of (24), for $a=.02$.

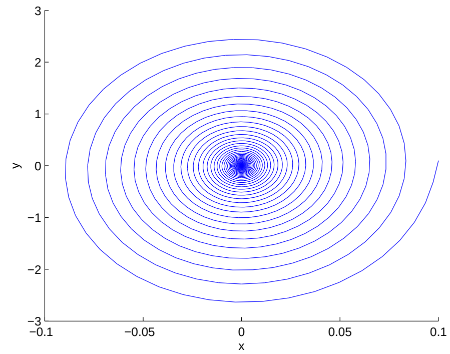

Figure 10. $x - y$ plane of (24), for $a = .5$.

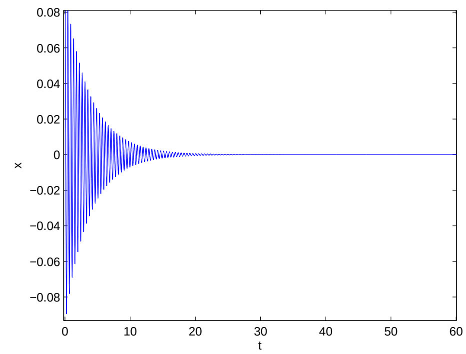

Figure 11. Numerical value of (24), for $a = .5$.

6. Concluding remarks

In this paper, the stability of a class of fractional order nonlinear systems with Caputo and Riemann–Liouville derivatives for the commensurate order $0 < \alpha < 1$ has been studied. We derived sufficient conditions for stability and $\mathcal{F}$-stability of nonlinear fraction systems. The effectiveness of the obtained results has been illustrated by a numerical example.

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