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The t-successive associated Stirling numbers, t-Fibonacci–Stirling numbers, and unimodality

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Abstract: Using a combinatorial approach, we introduce the *t-successive associated Stirling numbers* and we give the recurrence relation and the generating function. We also establish the unimodality of sequence $\left\{ {n-2k \atop k} \right\}_k$ lying over a ray of the second kind's Stirling triangle. Some combinatorial identities are given.

Key words: t-Successive associated Stirling numbers, recurrence relations, generating function, log-concavity, unimodality

1. Introduction

This paper is about some extension of Stirling numbers of second kind and unimodality. We give some introductory tools.

The t-associated Stirling numbers of second kind, denoted by $\left\{ {n \atop k} \right\}^{(t)}$ (see for instance [4] and references therein), count the number of partitions of the set $\{1, 2, \ldots, n\}$ into k subsets such that each subset contains at least t elements. They are generated by the following function:

$$\frac{1}{k!}\left(\exp(x) - \sum_{i=0}^{t-1} \frac{x^i}{i!}\right)^k = \sum_{n \geq tk} \left\{ {n \atop k} \right\}^{(t)} \frac{x^n}{n!}.$$

In this work, we focus on a special situation of t-associated Stirling numbers of second kind.

Our aim is to study combinatorial aspects of these sequences and for $t = 3$ prove the unimodality.

For doing so, we give some results and definitions linked to log-concavity and unimodality.

A finite sequence $(a_n)_{k=0}^n$ is unimodal if it increases to a maximum and then decreases. That is, there exists k such that

$$a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

The sequence $(a_n)_{k=0}^n$ is log-concave (LC for short) if for $k = 2, \ldots, n-1$,

$$a_k^2 \geq a_{k+1} a_{k-1}, \tag{1}$$

it is strictly log-concave (SLC for short) when (1) holds with the strict inequality.

It is known that log-concavity implies unimodality [17].

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Theorem 1 ***(Newton's inequality [13])*** *If the polynomial* $a_1x + a_2x^2 + \cdots + a_nx^n$ *has only real zeros, then*

$$a_k^2 \geq a_{k-1}a_{k+1}\frac{k}{k-1}\frac{n-k+1}{n-k} \quad (k = 2, \dots, n-1).$$

Theorem 1 implies the strict version of (1), (see Hammersley [11] and Erdös [10]).

The first result dealing with unimodality of Pascal's triangle elements is due to Tanny and Zuker [15], who showed that the sequence of terms $\binom{n-k}{k}$ $(k = 0, \dots, \lfloor n/2 \rfloor)$ is unimodal. They also investigated, for a given α, the unimodality of the sequence $\binom{n-\alpha k}{k}$ in [14, 16]. Belbachir et al. provide in [3] some cases of binomial sequence $\binom{n-\alpha k}{\beta k}$ and in [2] Belbachir and Bencherif proved the unimodality of sequences associated to Pell numbers. In [5], Belbachir and Szalay proved that any ray crossing Pascal's triangle provides a unimodal sequence. Our main interest is to examine combinatorial sequences connected to an arithmetical triangle as the second kind's Stirling triangle and its generalizations.

Harper [12] showed that $\sum_k \left\{{n \atop k}\right\} x^k$ has only real roots. Canfield [9] showed that the sequence $\left(\left\{{n \atop k}\right\}\right)_k$ is unimodal for a fixed n with at most two consecutive modes.

In Section 2, we introduce the *t-successive associated Stirling numbers* $\left\{{n \atop k}\right\}^{[t]}$ $(n \geq tk)$ as an extension of the 2-successive associated Stirling numbers; see [6]. Using a combinatorial approach, we derive their recurrence relation and compute the associated generating function. We prove in Section 3 the strict log-concavity and hence the unimodality of the 3-successive associated Stirling numbers. In Section 4, we link the t-successive associated Stirling numbers to the classical Stirling numbers of the second kind; this allowed us to prove the unimodality of sequence $\left\{{n-2k \atop k}\right\}_k$ lying over a ray of the second kind's Stirling triangle (see Figure). We also introduce the t-Fibonacci–Stirling numbers. We conclude, in Section 5, by establishing combinatorial identities related to the 2-successive associated Stirling numbers, as a complement study of our work [6].

2. The t-successive associated Stirling numbers

We start by giving a combinatorial definition of the t-successive associated Stirling numbers.

Definition 2 *The t-successive associated Stirling numbers, denoted by* $\left\{{n \atop k}\right\}^{[t]}$*, count the number of partitions of the set* $\{1, 2, \dots, n\}$ *into* k *parts, such that each part contains at least* t *consecutive numbers. Moreover, the last element* n *must either form a part with its* $t-1$ *predecessors, or belongs to another part that already contains* t *consecutive numbers.*

For some values of the t-successive Stirling numbers, see Tables 1–3.

Examples 3 *The following examples clarify the above combinatorial definition.*

- $t = 2$*:*
  $\left\{{6 \atop 2}\right\}^{[2]} = 7$*, we have* $\{1, 2, 3, 4\}\{5, 6\}; \{1, 2, 3, 6\}\{4, 5\};$
  $\{1, 2, 3\}\{4, 5, 6\}; \{1, 2, 6\}\{3, 4, 5\}; \{1, 2\}\{3, 4, 5, 6\}; \{1, 2, 5, 6\}\{3, 4\};$
  $\{1, 2, 5\}\{3, 4, 6\}.$
  $\left\{{6 \atop 3}\right\}^{[2]} = 1$*, there is one way to partition six elements into three parts* $\{1, 2\}\{3, 4\}\{5, 6\}.$

- $t=3$:
  $\left\{{7 \atop 2}\right\}^{[3]}=3$, *counts the number of partitions of seven elements into two parts*
  $\{1,2,3,4\}\{5,6,7\};\{1,2,3\}\{4,5,6,7\};\{1,2,3,7\}\ \ \{4,5,6\}$.
  The partition $\{2,3,4\}\{1,5,6,7\}$ *cannot be considered. In fact, the* 7^{th} *element is not in a part that already contains three consecutive numbers.*

Theorem 4 *For* $n \geq tk$, *we have*

$$\left\{{n \atop k}\right\}^{[t]} = k\left\{{n-1 \atop k}\right\}^{[t]} + \left\{{n-t \atop k-1}\right\}^{[t]}, \tag{2}$$

where $\left\{{0 \atop 0}\right\}^{[t]}=1$, $\left\{{n \atop n-1}\right\}^{[t]} = \left\{{n \atop n-2}\right\}^{(t)} = \cdots = \left\{{n \atop n-t+1}\right\}^{[t]} = 0$ *and* $\left\{{n \atop 0}\right\}^{[t]}=0,\ (n \geq 1)$.

Proof Let us focus on the last element n. If it forms a part with its $t-1$ predecessors, then it remains to form $(k-1)$ parts from the $(n-t)$ remaining elements, and we have $\left\{{n-t \atop k-1}\right\}^{[t]}$ ways to do that. Otherwise, we constitute k parts from $(n-1)$ elements; then adding the n^{th} element to one of the k parts, we get $k\left\{{n-1 \atop k}\right\}^{[t]}$ possibilities. This ends the proof. □

The t-successive associated Stirling numbers satisfy the following vertical recurrence relation.

Theorem 5 *Let* $n,k,t \in \mathbb{N}$ *such that* $n \geq tk$. *We have*

$$\left\{{n \atop k}\right\}^{[t]} = \sum_{i=0}^{n-tk} k^i \left\{{n-i-t \atop k-1}\right\}^{[t]}. \tag{3}$$

Proof From the recurrence relation (2), we have, for $n \geq tk$,

$$\begin{aligned}
\left\{{n \atop k}\right\}^{[t]} &= k\left\{{n-1 \atop k}\right\}^{[t]} + \left\{{n-t \atop k-1}\right\}^{[t]}, \\
k\left\{{n-1 \atop k}\right\}^{[t]} &= k^2\left\{{n-2 \atop k}\right\}^{[t]} + k\left\{{n-t-1 \atop k-1}\right\}^{[t]}, \\
&\vdots \\
k^{n-tk-1}\left\{{tk+1 \atop k}\right\}^{[t]} &= k^{n-tk}\left\{{tk \atop k}\right\}^{[t]} + k^{n-tk-1}\left\{{1+t(k-1) \atop k-1}\right\}^{[t]}, \\
k^{n-tk}\left\{{tk \atop k}\right\}^{[t]} &= k^{n-tk+1}\left\{{tk-1 \atop k}\right\}^{[t]} + k^{n-tk}\left\{{t(k-1) \atop k-1}\right\}^{[t]}.
\end{aligned}$$

Summing all the equations side by side and according to the initial conditions, we get the result. □

Now we give the generating function of the t-successive associated Stirling numbers.

Theorem 6 *The generating function of the* t*-successive associated Stirling numbers is given, for* $k \geq 1$, *by*

$$A_k(x) := \sum_{n \geq sk} \left\{{n \atop k}\right\}^{[t]} x^n = \frac{x^{tk}}{(1-x)(1-2x)\cdots(1-kx)}, \tag{4}$$

where $A_0(x) = 1$.

Proof From the recurrence relation (2), we have, for $k = 1, 2, \ldots$, and for a fixed t,

$$\sum_{n \geq tk} \left\{ {n \atop k} \right\}^{[t]} x^n = \sum_{n \geq tk} k \left\{ {n-1 \atop k} \right\}^{[t]} x^n + \sum_{n \geq tk} \left\{ {n-t \atop k-1} \right\}^{[t]} x^n,$$

consequently,

$$A_k(x) = kxA_k(x) + x^t A_{k-1}(x), \quad k = 1, 2, \ldots,$$

thus,

$$A_k(x) = x^t (1 - kx)^{-1} A_{k-1}(x), \quad k = 1, 2, \ldots.$$

Hence,

$$\begin{aligned} A_{k-1}(x) &= x^t (1 - (k-1)x)^{-1} A_{k-2}(x), \\ A_{k-2}(x) &= x^t (1 - (k-2)x)^{-1} A_{k-3}(x), \\ &\vdots \\ A_2(x) &= x^t (1 - 2x)^{-1} A_1(x), \\ A_1(x) &= x^t (1 - x)^{-1} A_0(x). \end{aligned}$$

By substituting each term $A_{i-1}(x)$ in $A_i(x)$, $i = 1, \ldots, k$, we obtain

$$A_k(x) = x^t (1 - (k-1)x)^{-1} x^t (1 - (k-2)x)^{-1} \cdots x^t (1 - x)^{-1} A_0(x).$$

According to the initial conditions, we get the result. □

An explicit expression is given in the following result.

Theorem 7 *The t-successive associated Stirling numbers $\left\{ {n \atop k} \right\}^{[t]}$ have the following explicit formula. For $n \geq tk$, we have*

$$\left\{ {n \atop k} \right\}^{[t]} = \frac{1}{k!} \sum_{p=1}^{k} (-1)^{(k-p)} \binom{k}{p} p^{n-(t-1)k}. \tag{5}$$

Proof Formula (5) satisfies the recurrence relation (2),

$$\begin{aligned}
\left\{{n \atop k}\right\}^{[t]} &= k\left\{{n-1 \atop k}\right\}^{[t]} + \left\{{n-t \atop k-1}\right\}^{[t]} \\
&= k\frac{1}{k!}\sum_{p=1}^{k}(-1)^{(k-p)}\binom{k}{p}p^{n-1-(t-1)k} \\
&+ \frac{1}{(k-1)!}\sum_{p=1}^{k-1}(-1)^{(k-p-1)}\binom{k-1}{p}p^{n-1-(t-1)k} \\
&= \frac{1}{(k-1)!}\sum_{p=1}^{k}(-1)^{(k-p)}\binom{k}{p}p^{n-1-(t-1)k} \\
&- \frac{1}{(k-1)!}\sum_{p=1}^{k}(-1)^{(k-p)}\binom{k-1}{p}p^{n-1-(t-1)k} \\
&= \frac{1}{(k-1)!}\left(\sum_{p=1}^{k}(-1)^{(k-p)}p^{n-1-(t-1)k}\left(\binom{k}{p}-\binom{k-1}{p}\right)\right) \\
&= \frac{1}{(k-1)!}\sum_{p=1}^{k}(-1)^{(k-p)}p^{n-1-(t-1)k}\binom{k-1}{p-1} \\
&= \sum_{p=1}^{k}(-1)^{(k-p)}p^{n-(t-1)k}\frac{1}{(k-1)!}\frac{1}{p}\frac{(k-1)!}{(p-1)!(k-p)!} \\
&= \frac{1}{k!}\sum_{p=1}^{k}(-1)^{(k-p)}p^{n-(t-1)k}\frac{k!}{p!(k-p)!} \\
&= \frac{1}{k!}\sum_{p=1}^{k}(-1)^{(k-p)}p^{n-(t-1)k}\binom{k}{p},
\end{aligned}$$

which ends the proof. □

3. Unimodality of 3-successive associated Stirling numbers

Following Bòna's approach [7], we prove the strict log-concavity of the 3-successive associated Stirling numbers. Let

$$P_n(x) := \sum_{j=0}^{n}\left\{{n \atop j}\right\}^{[t]} x^j.$$

Proposition 8 *Let $d \in \mathbb{N}$ be the degree of $P_n(x)$ for a fixed t. If n is a multiple of t then $P_{n-1}(x)$ and $P_{n-2}(x)$,..., $P_{n-t}(x)$ are of degree $d-1$, where $d = n/t$. Moreover, if n is not a multiple of t, $n \equiv i[t]$, $1 \le i \le t-1$ then $P_n(x), \ldots, P_{n-i}(x)$ are of degree d and $P_{n-i-1}(x), \ldots, P_{n-t}$ are of degree $d-1$, where $d = \lfloor n/t \rfloor$.*

Proof The proof is easy; we leave it to the reader (see also the Appendix). □

Expression of $P_n(x)$ for $n \leq 10$ and $t \leq 4$

- $t = 2$:
  $P_0(x) = 1$,
  $P_1(x) = 0$,
  $P_2(x) = x$,
  $P_3(x) = x$,
  $P_4(x) = x + x^2$,
  $P_5(x) = x + 3x^2$,
  $P_6(x) = x + 7x^2 + x^3$,
  $P_7(x) = x + 15x^2 + 6x^3$,
  $P_8(x) = x + 31x^2 + 25x^3 + x^4$,
  $P_9(x) = x + 63x^2 + 90x^3 + 10x^4$,
  $P_{10}(x) = x + 127x^2 + 301x^3 + 65x^4 + x^5$.
- $t = 3$:
  $P_0(x) = 1$,
  $P_1(x) = 0$,
  $P_2(x) = 0$,
  $P_3(x) = x$,
  $P_4(x) = x$,
  $P_5(x) = x$,
  $P_6(x) = x + x^2$,
  $P_7(x) = x + 3x^2$,
  $P_8(x) = x + 7x^2$,
  $P_9(x) = x + 15x^2 + x^3$,
  $P_{10}(x) = x + 31x^2 + 6x^3$.
- $t = 4$:
  $P_0(x) = 1$,
  $P_1(x) = 0$,
  $P_2(x) = 0$,
  $P_3(x) = 0$,
  $P_4(x) = x$,
  $P_5(x) = x$,
  $P_6(x) = x$,
  $P_7(x) = x$,
  $P_8(x) = x + x^2$,
  $P_9(x) = x + 3x^2$,
  $P_{10}(x) = x + 7x^2$.

Now we prove the log-concavity and the unimodality of the 3-successive associated Stirling numbers.

Theorem 9 *The roots of $P_n(x)$ are real, distinct, and nonpositive for $n = 1, 2, \ldots$.*

Furthermore, the roots of $P_n(x)$, $P_{n-1}(x)$ and $P_{n-2}(x)$ are interlacing in the following sense:

If $P_n(x), P_{n-1}(x)$ et $P_{n-2}(x)$ are all of degree d and their roots are, respectively: $0 = x_0^{(n)} > x_1^{(n)} > \cdots > x_{d-1}^{(n)} > x_d^{(n)}$, $0 = x_0^{(n-1)} > x_1^{(n-1)} > \cdots > x_{d-1}^{(n-1)}$, $0 = x_0^{(n-2)} > x_1^{(n-2)} > \cdots > x_{d-1}^{(n-2)}$, then

$$0 > x_1^{(n)} > x_1^{(n-1)} > x_1^{(n-2)} > \cdots > x_{d-2}^{(n)} > x_{d-2}^{(n-1)} > x_{d-2}^{(n-2)} >$$

$$> x_{d-1}^{(n)} > x_{d-1}^{(n-1)} > x_{d-1}^{(n-2)} \tag{6}$$

. *If $P_n(x)$ and $P_{n-1}(x)$ are of degree d and $P_{n-2}(x)$ is of degree $d-1$ and their roots are, respectively: $0 = x_0^{(n)} > x_1^{(n)} > \cdots > x_{d-1}^{(n)} > x_d^{(n)}$, $0 = x_0^{(n-1)} > x_1^{(n-1)} > \cdots > x_{d-1}^{(n-1)}$, $0 = x_0^{(n-2)} > x_1^{(n-2)} > \cdots > x_{d-2}^{(n-2)}$, then*

$$0 > x_1^{(n)} > x_1^{(n-1)} > x_1^{(n-2)} > \cdots > x_{d-2}^{(n)} > x_{d-2}^{(n-1)} > x_{d-2}^{(n-2)} > x_{d-1}^{(n)} > x_{d-1}^{(n-1)} \tag{7}$$

. *While if $P_n(x)$ is of degree d and $P_{n-1}(x)$ and $P_{n-2}(x)$ are of degree $d-1$ and their roots are, respectively: $0 = x_0^{(n)} > x_1^{(n)} > \cdots > x_{d-1}^{(n)}$, $0 = x_0^{(n-1)} > x_1^{(n-1)} > \cdots > x_{d-2}^{(n-1)}$, $0 = x_0^{(n-2)} > x_1^{(n-2)} > \cdots > x_{d-2}^{(n-2)}$, then*

$$0 > x_1^{(n)} > x_1^{(n-1)} > x_1^{(n-2)} > \cdots > x_{d-2}^{(n)} > x_{d-2}^{(n-1)} > x_{d-2}^{(n-2)} > x_{d-1}^{(n)}. \tag{8}$$

Proof The recurrence relation of $\left\{ {n \atop j} \right\}^{[t]}$, for $t = 3$, yields

$$P_n(x) = x\left[P'_{n-1}(x) + P_{n-3}(x)\right], \tag{9}$$

where $P_0(x) = 1, P_i(x) = 0$, if $i < 3$ and $P_i(x) = x$, if $3 \le i < 6$.

We prove our theorem by induction on n. For $n \le 4$, the statements are true.

Supposing the theorem true for $n-1$, we prove it for n.

First we consider the case where $P_n(x)$, $P_{n-1}(x)$, and $P_{n-2}(x)$ are of degree d, which means that $n \equiv 2[3]$.

Let $0 = x_0^{(n-1)} > x_1^{(n-1)} > \cdots > x_{d-1}^{(n-1)}$ be the roots of $P_{n-1}(x)$.

Step 1: Consider the two largest roots of $P_{n-1}(x)$, which are 0 and $x_1^{(n-1)}$. By Rolle's theorem, there exists $c \in]x_1^{(n-1)}, 0[$ such that $P'_{n-1}(c) = 0$. Since the coefficients of $P_{n-1}(x)$ are positive, $P_{n-1}(x)$ is monotone decreasing in $]x_1^{(n-1)}, c[$ and monotone increasing in $]c, 0[$. This implies that $P_{n-1}(x) < 0$ for all $x \in]x_1^{(n-1)}, 0[$.

Step 2: Consider the case when $x = x_1^{(n-1)}$ in (9).

- By induction hypothesis, the roots of $P_{n-3}(x)$ are: $0 = x_0^{(n-3)} > x_1^{(n-3)} > \cdots > x_{d-2}^{(n-3)}$, and they are interlacing with the roots of $P_{n-1}(x)$ and $P_{n-2}(x)$ as follows: $0 > x_1^{(n-1)} > x_1^{(n-2)} > x_1^{(n-3)} > \cdots > x_{d-2}^{(n-1)} > x_{d-2}^{(n-2)} > x_{d-2}^{(n-3)} > x_{d-1}^{(n-1)} > x_{d-1}^{(n-2)}$.

  From Step 1, we have $P_{n-3}(x) < 0$ for $x \in]x_1^{(n-3)}, 0[$, in particular for $x = x_1^{(n-1)}$ which implies that $P_{n-3}(x_1^{(n-1)}) < 0$.

- We know that $P_{n-1}(x)$ has d nonpositive real roots and so $P'_{n-1}(x)$ must have $d-1$.

  By Rolle's theorem, we know that there exists a root of $P'_{n-1}(x)$ between any two consecutive roots of $P_{n-1}(x)$, and so there is a root of $P'_{n-1}(x)$ between $]x_1^{(n-1)}, 0[$, and the sign of $P'_{n-1}(x_1^{(n-1)})$ is the opposite sign of $P'_{n-1}(0)$, which is positive, and so $P'_{n-1}(x_1^{(n-1)}) < 0$.

For $x = x_1^{(n-1)}$, we have shown that $x_1^{(n-1)}\left[P'_{n-1}(x_1^{(n-1)}) + P_{n-3}(x_1^{(n-1)})\right]$ is positive as a product of two nonpositive numbers. Hence $P_n(x_1^{(n-1)})$ must be positive as well.

Moreover, we have $P_n(x) < 0$ in $]x_1^{(n-1)}, 0[$ because $P'_{n-1}(x) > 0$ is in $]x_1^{(n-1)}, 0[$ from Step 1 and $P_{n-3}(x) < 0$, and so $P_n(x)$ has a root in $]x_1^{(n-1)}, 0[$.

Now we prove that $P_n(x)$ has a root in each interval $]x_{i+1}^{(n-1)}, x_i^{(n-1)}[$. In order to do so, it is enough to show that $P_n(x_i^{(n-1)})$ and $P_n(x_i^{(n-1)})$ have opposite signs.

- By Rolle's theorem, we conclude that $P'_{n-1}(x_{i+1}^{(n-1)})$ and $P'_{n-1}(x_i^{(n-1)})$ have opposite signs.
- By induction hypothesis, we conclude that $P_{n-3}(x_{i+1}^{(n-1)})$ and $P_{n-3}(x_i^{(n-1)})$ have opposite signs.
- Based on Rolle's theorem, P'_{n-1} changes its sign i times in $]x_i^{(n-1)}, 0[$, and by induction hypothesis, P_{n-3} changes its sign $i-1$ times; however, there exists a small neighborhood of 0 where $P'_{n-1}(x) > 0$ and $P_{n-3} < 0$, and so $P'_{n-1}(x_i^{(n-1)})$ and $P_{n-3}(x_i^{(n-1)})$ have equal signs.

By Equation (9), $P_n(x_i^{(n-1)})$ and $P_n(x_{i+1}^{(n-1)})$ have opposite signs, which implies that $P_n(x)$ has a root in each interval $]x_{i+1}^{(n-1)}, x_i^{(n-1)}[$.

Furthermore, $P_n(x)$ has an odd number of roots in such interval because it has different signs in the limits of such interval, and we know that the number of the roots of $P_n(x)$ is at most one larger than $P_{n-1}(x)$. Then $P_n(x)$ has exactly one root in each interval $]x_{i+1}^{(n-1)}, x_i^{(n-1)}[$.

Now we prove that the polynomial $P_n(x)$ has exactly one root in each interval $]x_{i+1}^{(n-2)}, x_i^{(n-2)}[$. To do this, we just have to prove that $P_n(x_i^{(n-2)})$ and $P_n(x_{i+1}^{(n-2)})$ have opposite signs. We know by induction hypothesis that $P_{n-3}(x_i^{(n-1)})$ and $P_{n-3}(x_i^{(n-2)})$ have equal signs. For this, we just have to show that for $0 \le i \le d-1$, $P'_{n-1}(x_i^{(n-1)})$ and $P'_{n-1}(x_i^{(n-2)})$ have equal signs.

- Consider the case where $P'_{n-1}(x_i^{(n-1)})$ is nonpositive and so by induction hypothesis $P_{n-1}(x)$ is nonnegative in the interval $]x_{i+1}^{(n-1)}, x_i^{(n-1)}[$, $P_{n-2}(x)$ is nonnegative in the interval $]x_{i+1}^{(n-1)}, x_i^{(n-2)}[$ and nonpositive in $]x_i^{(n-2)}, x_i^{(n-1)}[$, knowing that $P'_{n-2}(x)$ achieves its maximum at $x_i^{(n-2)}$ and so $P'_{n-2}(x_i^{(n-2)})$ is nonpositive; moreover, $P'_{n-2}(x_i^{(n-1)})$ is nonnegative, then P'_{n-2} has a root in such interval, denoted as γ_i. For this, $P'_{n-2}(x)$ is monotone increasing and nonpositive in the interval $]x_i^{(n-2)}, \gamma_i[$ and monotone increasing and nonnegative in the interval $]\gamma_i, x_i^{(n-1)}[$.

Supposing that $P_{n-1}^{'}(x_i^{(n-2)})$ is nonnegative and β_i the root of $P_{n-1}^{'}(x)$, we have $P_{n-1}(x)$ monotone increasing in the interval $]x_{i+1}^{(n-1)}, \beta_i[$ and monotone decreasing in the interval $]\beta_i, x_i^{(n-1)}[$.

From our supposition, we have $P_{n-1}^{'}(x_i^{(n-2)}) > 0$, which means that $\beta_i \in]x_i^{(n-2)}, x_i^{(n-1)}[$, and we discuss two cases:

1. $\beta_i \in]x_i^{(n-2)}, \gamma_i[$, from the supposition, $P_{n-1}(\beta_i) - P_{n-1}(x_i^{(n-2)}) \geq 0$, which implies $\beta_i P_{n-2}^{'}(\beta_i) + \beta_i P_{n-4}(\beta_i) - x_i^{(n-2)} P_{n-2}^{'}(x_i^{(n-2)}) - x_i^{(n-2)} P_{n-4}(x_i^{(n-2)}) \geq 0$; however, by induction hypothesis and from the previous paragraph, we have $\beta_i P_{n-2}^{'}(\beta_i) - x_i^{(n-2)} P_{n-2}^{'}(x_i^{(n-2)}) \leq 0$ and $\beta_i P_{n-4}(\beta_i) - x_i^{(n-2)} P_{n-4}(x_i^{(n-2)}) \leq 0$, a contradiction, and so $P_{n-1}^{'}(x_i^{(n-2)})$ is nonpositive, which means that it has the same sign of $P_{n-1}^{'}(x_i^{(n-1)})$.
2. $\beta_i \in]\gamma_i, x_i^{(n-1)}[$, from the supposition, $P_{n-1}(\beta_i) - P_{n-1}(x_i^{(n-2)}) \geq 0$, which implies $\beta_i P_{n-2}^{'}(\beta_i) + \beta_i P_{n-4}(\beta_i) - \gamma_i P_{n-2}^{'}(\gamma_i) - \gamma_i P_{n-4}(\gamma_i) \geq 0$, $\beta_i P_{n-2}^{'}(\beta_i) + \beta_i P_{n-4}(\beta_i) - \gamma_i P_{n-4}(\gamma_i) \geq 0$; however, by induction hypothesis and from the previous paragraph, we have $\beta_i P_{n-2}^{'}(\beta_i) \leq 0$ and $\beta_i P_{n-4}(\beta_i) - \gamma_i P_{n-4}(\gamma_i) \leq 0$, a contradiction, and so $P_{n-1}^{'}(x_i^{(n-2)})$ is nonpositive, which means that it has the same sign of $P_{n-1}^{'}(x_i^{(n-1)})$.

- For the second case, where $P_{n-1}^{'}(x_i^{(n-1)})$ is nonnegative, the proof is the same.

This completes the proof.

Consider now the case when $P_n(x)$ is of degree d and $P_{n-1}(x)$, $P_{n-2}(x)$ are of degree $d-1$, which means that $n \equiv 0[3]$. We follow the same method of proof presented in the previous case.

For the case when $P_n(x)$ and $P_{n-1}(x)$ are of degree d and $P_{n-2}(x)$ is of degree $d-1$, $n \equiv 1[3]$. We follow the same approach of proof presented in the previous case. We know that the last root of $P_n(x)$ has to be nonpositive and it cannot be in any interval $]x_{i+1}^{(n-2)}, x_i^{(n-1)}[$, which means that it should be in $]-\infty, x_{d-2}^{(n-2)}[$. This concludes the proof. □

Theorem 10 *The sequence* $\left(\left\{ {n \atop k} \right\}^{[3]}\right)_k$ *is strictly log-concave and thus unimodal with at most two consecutive modes.*

Proof It follows by Theorem 1 and Theorem 9. □

4. Link with the Stirling numbers of second kind and the t-Fibonacci–Stirling numbers

In this section, we give the relation between the t-successive associated Stirling numbers and the Stirling numbers of second kind. We also introduce the t-Fibonacci–Stirling numbers.

4.1. The Stirling numbers of second kind

The t-successive associated Stirling numbers are defined as the second kind's Stirling triangle elements of direction $(\alpha, 1)$, $\alpha = t-1$. The sequence $\left\{ {n-\alpha k \atop k} \right\}$ associated with the direction $(\alpha, 1)$ is illustrated by the theorem below.

n/k	0	1	f2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	7	6	1
5	0	1	15	25	10
6	0	1	31	90	65

Figure. Direction $(2,1)$ in second kind's Stirling triangle.

Theorem 11 *For $n \geq tk$, we have*

$$\left\{ {n \atop k} \right\}^{[t]} = \left\{ {n-\alpha k \atop k} \right\}. \tag{10}$$

Proof It is a consequence of (2). □

Remark 12 *For $n \geq tk$, we have*

$$\left\{ {n \atop k} \right\}^{[t]} = \left\{ {n-k \atop k} \right\}^{[t-1]} = \left\{ {n-\alpha k \atop k} \right\}.$$

The following theorem is an analogue of Tanny and Zuker's theorem [14, 16].

Theorem 13 *The sequence $\left(\left\{ {n-2k \atop k} \right\}\right)_k$ is log-concave and thus unimodal with at most two consecutive modes.*

Proof It follows from Theorem 11, for $t=3$, and Theorem 10. □

4.2. The t-Fibonacci–Stirling numbers

It is well known that Fibonacci numbers are defined as the sum of diagonal elements of Pascal's triangle; see for instance [1]. Hence we introduce the t-Fibonacci–Stirling numbers as well.

Definition 14 *We define the t-Fibonacci–Stirling numbers $\left(\varphi_n^{(t)}\right)_n$, for $n \geq tk$, by*

$$\varphi_{n+1}^{(t+1)} := \sum_k \left\{ {n-tk \atop k} \right\}, \tag{11}$$

where, $\varphi_0^{(t)} = 1, \varphi_1^{(t)} = 0.$

For some values of the t-Fibonacci–Stirling numbers, see Tables 4–6.

Corollary 15 *The t-Fibonacci–Stirling numbers are linked to the t-successive associated Stirling numbers by the following expression:*

$$\varphi_{n+1}^{(t+1)} = \sum_k \left\{ {n \atop k} \right\}^{[t]}. \tag{12}$$

Proof It follows by (10) in (11). □

The sequence $\left(\varphi_n^{(t)}\right)$ is called the sequence of *the t-successive associated Bell numbers.*

5. The 2-successive associated Stirling numbers

In this section, we give some complementary identities specific to the 2-successive associated Stirling numbers; see [6].

Remark 16 *For all $n \geq 3$, we have $\left\{ {n \atop 2} \right\}^{[2]} = 2^{n-3} - 1$.*

Corollary 17 *Expression of the generating function in terms of noncentral ascending factorial*

$$\sum_{n \geq 2k} \left\{ {n \atop k} \right\}^{[2]} \frac{1}{z^n} = \frac{1}{z^k (z)_{k+1}}, \tag{13}$$

where $(z)_{k+1} = z(z-1)\cdots(z-k)$.

Proof We have to set $x = 1/z$ in (4), with $t = 2$. □

The 2-successive associated Stirling numbers $\left\{ {n \atop k} \right\}^{[2]}$ are given by the following sum. It is the main result of this section.

Theorem 18 *For $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, we have,*

$$\left\{ {n \atop k} \right\}^{[2]} = \sum 1^{r_1} 2^{r_2} \cdots k^{r_k}, \tag{14}$$

where the summation is extended over all integers $r_j \geq 0$, $j = 1, \ldots, k$, with $r_1 + r_2 + \cdots + r_k = n - 2k$.

Proof Expanding each factor in (4), where $t = 2$, and using the geometric series, we find

$$\begin{aligned} A_k(x) &= \sum_{n \geq 2k} \left\{ {n \atop k} \right\}^{[2]} x^n \\ &= x^{2k} \prod_{j=1}^{k} \left(\sum_{r_j = 0}^{\infty} k^{r_j} x^{r_j} \right) \\ &= \sum_{n \geq 2k} \left(\sum_{r_1 + r_2 + \cdots + r_k = n - 2k} 1^{r_1} 2^{r_2} \cdots k^{r_k} \right) x^n. \end{aligned}$$

The result follows by identification. □

Corollary 19 *The following relation, with symmetric functions, holds:*

$$\left\{ {2n+k \atop n} \right\}^{[2]} = \sum_{1 \le i_1 \le \cdots \le i_k \le n} i_1 i_2 \cdots i_k. \tag{15}$$

Proof See [8, Th. 8]. □

We give now an exponential generating function and double generating function for $\left\{ {n+k \atop k} \right\}^{[2]}$.

Corollary 20 *We have the following generating functions:*

$$\sum_{n \ge k} \left\{ {n+k \atop k} \right\}^{[2]} \frac{x^n}{n!} = \frac{1}{k!} e^x (e^x - 1)^k, \tag{16}$$

$$\sum_n \sum_k \left\{ {n+k \atop k} \right\}^{[2]} \frac{x^n}{n!} y^k = e^{y(e^x - 1)}. \tag{17}$$

Proof See [8, Th. 16]. □

6. Commentaries

In this section, we give some commentaries related to the log-concavity and the unimodality of the t-successive associated Stirling numbers.

Let

$$P_n(x) = x \left[P'_{n-1}(x) + P_{n-t}(x) \right].$$

We are convinced that the roots of $P_n(x)$ are real, distinct, and nonpositive for $n = 1, 2, \ldots$.
Furthermore, the roots of $P_n(x)$, $P_{n-1}(x), \ldots, P_{n-t+1}(x)$ are interlacing in the following sense:
If $P_n(x)$ is of degree d and $P_{n-1}(x)$, $\cdots$, $P_{n-(t-1)}(x)$ are of degree $d-1$ and their roots are, respectively:
$0 = x_0^{(n)} > x_1^{(n)} > \cdots > x_{d-1}^{(n)}$, $0 = x_0^{(n-1)} > x_1^{(n-1)} > \cdots > x_{d-2}^{(n-1)}$, $\cdots$, $0 = x_0^{(n-(t-1))} > x_1^{(n-(t-1))} > \cdots > x_{d-2}^{(n-(t-1))}$, then

$$0 > x_1^{(n)} > x_1^{(n-1)} > \cdots > x_1^{(n-(t-1))} > \cdots > x_{d-2}^{(n)} > x_{d-2}^{(n-1)} > \cdots$$

$$\cdots > x_{d-2}^{(n-(t-1))} > x_{d-1}^{(n)}. \tag{18}$$

While if $P_n(x), \ldots, P_{n-i}(x)$ are of degree d and $P_{n-i-1}(x), \ldots, P_{n-(t-1)}$, $1 \le i \le t-1$, are of degree $d-1$ and their roots are, respectively:
$0 = x_0^{(n)} > x_1^{(n)} > \cdots > x_{d-1}^{(n)} > x_d^{(n)}, \cdots, 0 = x_0^{(n-i)} > x_1^{(n-i)} > \cdots > x_{d-1}^{(n-i)}$, $\cdots$, $0 = x_0^{(n-(t-1))} > x_1^{(n-(t-1))} > \cdots > x_{d-2}^{(n-(t-1))}$, then

$$0 > x_1^{(n)} > \cdots > x_1^{(n-i)} > \cdots > x_1^{(n-(t-1))} > \cdots > x_{d-2}^{(n)} > \cdots > x_{d-2}^{(n-i)} > \cdots$$

$$\cdots > x_{d-2}^{(n-(t-1))} > x_{d-1}^{(n)} > \cdots > x_{d-1}^{(n-i)}. \tag{19}$$

We can then conclude that

1. The sequence $\left(\left\{ {n \atop k} \right\}^{[t]}\right)_k$, for a fixed t, is strictly log-concave and thus unimodal with at most two consecutive modes.

2. The sequence $\left(\left\{ {n-(t-1)k \atop k} \right\}\right)_k$ is log-concave and thus unimodal with at most two consecutive modes.

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Appendix

Table 1. Some values for the 2-successive associated Stirling numbers.

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	0							
2	0	1						
3	0	1						
4	0	1	1					
5	0	1	3					
6	0	1	7	1				
7	0	1	15	6				
8	0	1	31	25	1			
9	0	1	63	90	10			
10	0	1	127	301	65	1		
11	0	1	255	966	350	15		
12	0	1	511	3025	1701	140	1	
13	0	1	1023	9330	7770	1050	21	
14	0	1	2047	28501	34105	6951	266	1
15	0	1	4095	86526	145750	42525	2646	28

Table 2. Some values for the 3-successive associated Stirling numbers.

$n\backslash k$	0	1	2	3	4	5
0	1					
1	0					
2	0					
3	0	1				
4	0	1				
5	0	1				
6	0	1	1			
7	0	1	3			
8	0	1	7			
9	0	1	15	1		
10	0	1	31	6		
11	0	1	63	25		
12	0	1	127	90	1	
13	0	1	255	301	10	
14	0	1	511	966	65	
15	0	1	1023	3025	350	1

Table 3. Some values for the 4-successive associated Stirling numbers.

$n\backslash k$	0	1	2	3
0	1			
1	0			
2	0			
3	0			
4	0	1		
5	0	1		
6	0	1		
7	0	1		
8	0	1	1	
9	0	1	3	
10	0	1	7	
11	0	1	15	
12	0	1	31	1
13	0	1	63	6
14	0	1	127	25
15	0	1	255	90

Table 4. Some values for the 2-Fibonacci–Stirling numbers, (Sloane, A171367).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\varphi_n^{(2)}$	1	0	1	1	2	4	9	22	58	164	495	1587	5379

Table 5. Some values for the 4-Fibonacci-Stirling numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\varphi_n^{(3)}$	1	0	0	1	1	1	2	4	8	17	38	89	219

Table 6. Some values for the 3-Fibonacci–Stirling numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\varphi_n^{(4)}$	1	0	0	0	1	1	1	1	2	4	8	16	33