Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1292 – 1307

doi:10.3906/mat-1510-52

Stationary distribution and global asymptotic stability of a three-species stochastic food-chain system

Hong QIU*, **Wenmin DENG**
College of Science, Civil Aviation University of China, Tianjin, P.R. China

Received: 15.10.2015 • **Accepted/Published Online:** 14.12.2016 • **Final Version:** 28.09.2017

Abstract: This paper intends to study some dynamical properties of a stochastic three-dimensional Lotka–Volterra system. Under some mild assumptions, we first introduce a simple method to show that the model has a global and positive solution almost surely. Secondly, we prove that this model has a stationary distribution. Then we study the global asymptotic stability of the positive solution. Finally, some numerical simulations are introduced to illustrate the theoretical results.

Key words: Stochastic food-chain model, stationary distribution, global asymptotic stability

1. Introduction

When species interact with each other the population dynamics of each species is affected, and the predator–prey interaction for the predation of one species by another is one of the important ecological phenomena. The first predator–prey model is two species, which was proposed by Volterra and Lotka in the mid 1920s to explain the oscillation of certain fish catches in the Adriatic (see [33]). The population dynamics become more complex when the interacting species are three than when there are two. Nonetheless, such models attracted considerable attention; for example, Pande [34] considered the coexistence of three species with one prey and two predators. Krikorian [19] studied the global asymptotic stability and global boundedness of the classical Volterra equations modeling with three-species predator–prey interactions. Farkas and Freedman [7] gave the stability criterion for a system of a three-dimensional case when two predators compete for a single prey species. Chattopadhyay and Arino [4] considered a three-species predator–prey eco-epidemiological system, and derived the persistence and extinction conditions of the populations. There is a large amount of literature on interacting populations of the predator–prey type; we only mention [3, 6, 14] and the references therein.

A classical three-species interacting predator–prey Lotka–Volterra model is the population described by $x_1(t), x_2(t)$ and $x_3(t)$, where $x_1(t)$ is always a prey population, $x_2(t)$ is not only a predator population feeding on $x_1(t)$ but also a prey of $x_3(t)$, and $x_3(t)$ is a predator feeding exclusively on prey $x_1(t), x_2(t)$ within the

*Correspondence: qiuhong1003@163.com
2010 *AMS Mathematics Subject Classification:* 60H10, 34D23, 92B05
This work is supported in part by National Nature Science Foundation of China (Grant No.11401574 and 11326197) and Fundamental Research Funds for the Central Universities (Grant No. 3122017076).

system. Such a food-chain model can be expressed by

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)]\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)]\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)[-r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)]\mathrm{d}t, \end{cases} \tag{1}$$

where $x_1(t), x_2(t), x_3(t)$ are the corresponding population densities, $r_i (i = 1, 2, 3)$ denote the growth rates of the populations, $a_{ii} (i = 1, 2, 3)$ are intraspecific competition rates, and $a_{ij} (i \neq j, i, j = 1, 2, 3)$ are the corresponding interspecific competition rates of the system. Here all $r_i, a_{ij} (i, j = 1, 2, 3)$ are positive parameters. Freedman and Waltman [8] studied a general case of model (1) and discussed persistence for all components of the ecosystem. Freedman and So [9] considered the global stability and persistence of a food-chain model.

Since the population systems are inevitably affected by the environmental white noise, which plays an important role in an ecosystem, several authors (see [1, 5, 16, 17, 20, 21, 24, 25, 29, 35, 39]) studied the stochastic models by supposing that noises in the environment mainly affect the growth rates. In this paper, we incorporate the white noise in each equation of system (1) in the following way:

$$\begin{aligned} r_1 \quad &\to r_1 + \sigma_1 \dot{B}_1(t), \\ -r_2 \quad &\to -r_2 + \sigma_2 \dot{B}_2(t), \\ -r_3 \quad &\to -r_3 + \sigma_3 \dot{B}_3(t), \end{aligned}$$

where $(B_1(t), B_2(t), B_3(t))^T$ (the superscript T on a matrix denotes transpose) is a three-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $\sigma_i^2 (i = 1, 2, 3)$ denote the intensities of the white noise. Then the stochastic three-species interacting predator–prey model corresponding to the above deterministic system (1) is as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = x_2(t)[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}x_3(t) = x_3(t)[-r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}B_3(t), \end{cases} \tag{2}$$

with initial value $x_i(0) > 0 (i = 1, 2, 3)$.

The above stochastic three-species predator–prey Lotka–Volterra model is very interesting and worth studying. Several authors studied the dynamical properties of this stochastic predator–prey model, such as [10, 21–23, 26–28, 36]. However, there is little work on the stationary distribution and global asymptotic stability of the system (2).

The aim of this paper is to consider the dynamic behavior of equation (2). We show that model (2) has a global positive solution almost surely (for short a.s.) by introducing a simple method. The stability of positive equilibrium of model (1) is one of the most interesting problems, while there is no equilibrium for the stochastic system (2). Then we show that system (2) has a stationary distribution under some simple parametric conditions. We also study the global asymptotic stability of the positive solution.

The rest of this paper is organized as follows. In Section 2, we show that system (2) has a global positive solution under some mild conditions. In Section 3, we prove that system (2) has a stationary distribution that is ergodic when the noises are small enough. In Section 4 we consider the global attractivity of model (2). In Section 5, we work out some figures to illustrate the main results.

2. Global positive solutions

Throughout this paper, we denote by $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_i > 0, i = 1,2,3\}$. Now we will show the uniqueness and existence of the global positive solution.

Theorem 1 *If $r_i, a_{ij} > 0 (i,j = 1,2,3)$, then for any given initial value $x(0) = (x_1(0), x_2(0), \ x_3(0))^T \in \mathbb{R}^3_+$, system (2) has a unique solution $x(t) = (x_1(t), x_2(t), x_3(t))^T$ on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ almost surely (a.s.).*

Proof For the positiveness of $x(t)$, we first consider the following auxiliary equation:

$$\begin{cases} \mathrm{d}u_1(t) = \left[r_1 - \frac{1}{2}\sigma_1^2 - a_{11}e^{u_1(t)} - a_{12}e^{u_2(t)} - a_{13}e^{u_3(t)}\right]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}u_2(t) = \left[-r_2 - \frac{1}{2}\sigma_2^2 + a_{21}e^{u_1(t)} - a_{22}e^{u_2(t)} - a_{23}e^{u_3(t)}\right]\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \\ \mathrm{d}u_3(t) = \left[-r_3 - \frac{1}{2}\sigma_3^2 + a_{31}e^{u_1(t)} + a_{32}e^{u_2(t)} - a_{33}e^{u_3(t)}\right]\mathrm{d}t + \sigma_3 \mathrm{d}B_3(t), \end{cases} \tag{3}$$

with $u_i(0) = \ln x_i(0) (i = 1,2,3)$. Let $u(t) = (u_1(t), u_2(t), u_3(t))^T$, $B(t) = (B_1(t), B_2(t), B_3(t))^T$, $f(u) = (f_1(u), f_2(u), f_3(u))^T$, $g(u) = (g_{ij}(u))_{3\times 3}$. Then we can rewrite the above equation in the following form:

$$\mathrm{d}u(t) = f(u)\mathrm{d}t + g(u)\mathrm{d}B(t) \tag{4}$$

with

$$\begin{cases} f_1(u) = r_1 - \frac{1}{2}\sigma_1^2 - a_{11}e^{u_1(t)} - a_{12}e^{u_2(t)} - a_{13}e^{u_3(t)}, \\ f_2(u) = -r_2 - \frac{1}{2}\sigma_2^2 + a_{21}e^{u_1(t)} - a_{22}e^{u_2(t)} - a_{23}e^{u_3(t)}, \\ f_3(u) = -r_3 - \frac{1}{2}\sigma_3^2 + a_{31}e^{u_1(t)} + a_{32}e^{u_2(t)} - a_{33}e^{u_3(t)}, \end{cases}$$

and

$$g(u) = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}.$$

Then for every integer $n \geq 1$ and $u, \tilde{u} \in \mathbb{R}^3_+$ with $|u| \vee |\tilde{u}| \leq n$, we have

$$\begin{aligned} |f(u) - f(\tilde{u})|^2 &= \sum_{i=1}^{3} (f_i(u) - f_i(\tilde{u}))^2 \leq 3 \sum_{i,j=1}^{3} a_{i,j}^2 \left(e^{u_j} - e^{\tilde{u}_j}\right)^2 \\ &\leq 3 \sum_{i,j=1}^{3} a_{i,j}^2 e^{2n} \left(u_j - \tilde{u}_j\right)^2 \leq L_n |u - \tilde{u}|^2 \end{aligned}$$

with $L_n = 3e^{2n}\left(\max_{1\leq i,j\leq 3}\{a_{i,j}^2\}\right)$. By the above inequality and the local boundedness of $f(u)$, we can conclude that (for some constant $\tilde{L}_n$)

$$|f(u)| \leq \tilde{L}_n(1 + |u|), \quad |u| \leq n.$$

Since the corresponding coefficients of equation (3) satisfy the local Lipschitz condition, equation (3) has a unique local solution (Theorem 3.15 on p. 91 of [32]). Then the Itô's formula shows that $x_i(t) = e^{u_i(t)} > 0 (i = 1,2,3)$ is the unique local solution to equation (2) with initial value $x_i(0) > 0, (i = 1,2,3)$.

Next, we claim that the solution of system (2) is global. Consider the following equation:

$$\begin{cases} \mathrm{d}y_1(t) = y_1(t)[r_1 - a_{11}y_1(t)]\mathrm{d}t + \sigma_1 y_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}y_2(t) = y_2(t)[-r_2 + a_{21}y_1(t) - a_{22}y_2(t)]\mathrm{d}t + \sigma_2 y_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}y_3(t) = y_3(t)[-r_3 + a_{31}y_1(t) + a_{32}y_2(t) - a_{33}y_3(t)]\mathrm{d}t + \sigma_3 y_3(t)\mathrm{d}B_3(t), \end{cases} \tag{5}$$

with $y_i(0) = x_i(0)(i = 1,2,3)$. According to Theorem 2.2 in Jiang et al. [16], it is easy to show that the above equation (5) has a unique global solution on $[0, +\infty)$ and the explicit solution is

$$\begin{aligned} y_1(t) &= \frac{\exp\{(r_1 - \sigma_1^2/2)t + \sigma_1 B_1(t)\}}{[x_1(0)]^{-1} + a_{11}\int_0^t \exp\{(r_1 - \sigma_1^2/2)s + \sigma_1 B_1(s)\}\mathrm{d}s}, \\ y_2(t) &= \frac{\exp\{\int_0^t(-r_2 - \sigma_2^2/2 + a_{21}y_1(s))\mathrm{d}s + \sigma_2 B_2(t)\}}{[x_2(0)]^{-1} + a_{22}\int_0^t \exp\{\int_0^s(-r_2 - \sigma_2^2/2 + a_{21}y_1(\tau))\mathrm{d}\tau + \sigma_2 B_2(s)\}\mathrm{d}s}, \\ y_3(t) &= \frac{\exp\{\int_0^t(-r_3 - \sigma_3^2/2 + a_{21}y_1(s) + a_{32}y_2(s))\mathrm{d}s + \sigma_3 B_3(t)\}}{[x_3(0)]^{-1} + a_{33}\int_0^t \exp\{\int_0^s(-r_3 - \sigma_3^2/2 + a_{21}y_1(\tau) + a_{32}y_2(\tau))\mathrm{d}\tau + \sigma_3 B_3(s)\}\mathrm{d}s}. \end{aligned} \tag{6}$$

In view of the stochastic comparison theorem (by the proof of Theorem 1.1 in [15], it is enough that the condition (1.4) in [15] is satisfied locally for x_i), we can see that

$$x_i(t) \le y_i(t), \quad i = 1,2,3.$$

Then $x(t)$ is a global solution of our system (2). □

Remark 2 *There are some relevant results on the above problem (see [10, 20]), but for the completeness of the paper, we give a detailed and much simpler proof on it that is different from the standard methods.*

The next lemma will be used later to prove lemma 4.2, which is necessary in the proof of globally asymptotically stable.

Lemma 3 *For any initial value $x(0) \in \mathbb{R}_+^3$ and $p > 0$, if $a_{22} > a_{21}$ and $a_{33} > a_{31} + a_{32}$ there is a constant $K = K(p) > 0$ such that the solution $x(t)$ of model (2) satisfies*

$$\limsup_{t \to +\infty} E[x_i^p(t)] \le K, \quad i = 1,2,3. \tag{7}$$

Proof We shall prove (7) for $p > 0$ first. Define a Lyapunov function $V(x) = x^p, x \in \mathbb{R}_+$. By the Itô's formula, one can derive

$$\begin{aligned} \mathrm{d}e^t V(x_1(t)) &= pe^t x_1^p(t)\left[\frac{1}{p} + r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t) + \frac{p-1}{2}\sigma_1^2\right]\mathrm{d}t \\ &+ p\sigma_1 e^t x_1^p(t)\mathrm{d}B_1(t) \end{aligned} \tag{8}$$

Then taking expectations on both sides, we have (noting that the $K_1(p)$ may change from line to line)

$$\begin{aligned} E\left[e^t V(x_1(t))\right] &= x_1^p(0) + pE\int_0^t e^s x_1^p(s)\left[\frac{1}{p} + r_1 - a_{11}x_1(s) - a_{12}x_2(s) - a_{13}x_3(s) + \frac{p-1}{2}\sigma_1^2\right]\mathrm{d}s \\ &\le x_1^p(0) + pE\int_0^t e^s x_1^p(s)\left[\frac{1}{p} + r_1 - a_{11}x_1(s) + \frac{p}{2}\sigma_1^2\right]\mathrm{d}s \\ &\le x_1^p(0) + pE\int_0^t e^s K_1(p)\mathrm{d}s \\ &\le x_1^p(0) + (e^t - 1)K_1(p), \end{aligned} \tag{9}$$

where

$$K_1(p) = \frac{\left(1 + pr_1 + \frac{1}{2}p^2\sigma_1^2\right)^{(p+1)}}{(p+1)^{(p+1)}a_{11}^p}.$$

Thus we have

$$\limsup_{t\to+\infty} E[x_1^p(t)] \le K_1(p).$$

Similarly,

$$\begin{aligned} \mathrm{d}e^t V(x_2(t)) &= pe^t x_2^p(t)\left[\frac{1}{p} - r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t) + \frac{p-1}{2}\sigma_2^2\right]\mathrm{d}t \\ &\quad + p\sigma_2 e^t x_2^p(t)\mathrm{d}B_2(t). \end{aligned} \tag{10}$$

By taking expectations on both sides, we obtain (noting that $a_{22} > a_{21}$ and the $K_2(p)$ may change from line to line)

$$\begin{aligned} E\left[e^t V(x_2(t))\right] &= x_2^p(0) + pE\int_0^t e^s x_2^p(s)\left[\frac{1}{p} - r_2 + a_{21}x_1(s) - a_{22}x_2(s) - a_{23}x_3(t) + \frac{p-1}{2}\sigma_2^2\right]\mathrm{d}s \\ &\le x_2^p(0) + pE\int_0^t e^s x_2^p(s)\left[\frac{1}{p} + a_{21}x_1(s) - a_{22}x_2(s) + \frac{p}{2}\sigma_2^2\right]\mathrm{d}s \\ &\le x_2^p(0) + p\int_0^t e^s E\left[x_2^p(s)\left(\frac{1}{p} - a_{22}x_2(s) + \frac{p}{2}\sigma_2^2\right)\right]\mathrm{d}s + pa_{21}\int_0^t e^s E\left[x_2^p(s)x_1(s)\right]\mathrm{d}s \\ &\le x_2^p(0) + p\int_0^t e^s E\left[x_2^p(s)\left(\frac{1}{p} + \frac{p}{2}\sigma_2^2 - (a_{22} - a_{21})x_2(s)\right)\right]\mathrm{d}s + \frac{pa_{21}}{p+1}\int_0^t e^s E\left[x_1^{p+1}(s)\right]\mathrm{d}s. \end{aligned} \tag{11}$$

Then for all large t, we have

$$\begin{aligned} E\left[e^t V(x_2(t))\right] &\le x_2^p(0) + p\int_0^t e^s K_2(p)\mathrm{d}s + \frac{pa_{21}}{p+1}\int_0^t e^s K_1(p+1)\mathrm{d}s \\ &\le x_2^p(0) + (e^t - 1)\left(K_2(p) + K_1(p+1)\right), \end{aligned} \tag{12}$$

where

$$K_2(p) = \frac{\left(1 + \frac{1}{2}p^2\sigma_2^2\right)^{(p+1)}}{(p+1)^{(p+1)}(a_{22} - a_{21})^p}.$$

The third inequality is following from the Young's inequality with $p = p, q = 1, \varepsilon = 1$ (see p. 52 [32])

$$|a|^p|b|^q \le |a|^{p+q} + \frac{q}{p+q}\left(\frac{p}{(p+q)}\right)^{p/q}|b|^{p+q},$$

which holds for $\forall a, b \in \mathbb{R}$ and $\forall p, q, \varepsilon > 0$. Then

$$\limsup_{t\to+\infty} E[x_2^p(t)] \le K_2(p).$$

Furthermore, we can prove $\limsup_{t\to+\infty} E[x_3^p(t)] \le K_3(p)$ similarly under the assumption $a_{33} > a_{31} + a_{32}$. Then (7) can be obtained by taking $K(p) = \max\{K_1(p), K_2(p), K_3(p)\}$. □

Remark 4 *The idea of this proof is inspired from Cheng [5]. Their result on the ultimate boundedness holds only for $p = 1$. Li et al. [20] just present a similar result for a competitive system without a proof. Here, for the three-species stochastic food-chain system we present a detailed proof for $p > 0$. The stochastic ultimate boundedness of the solutions can be obtained by the above lemma and Chebyshev's inequality.*

From Lemma 3 there exists a $T > 0$ such that $E[x_i^p(t)] \le 2K$ for $t \ge T$. At the same time note that $E[x_i^p(t)]$ is continuous; then there is a constant $\tilde{K} > 0$ such that $E[x_i^p(t)] < \tilde{K}$ for $0 \le t < T$. Define $\widehat{K} = \max\{2K, \tilde{K}\}$. Then

$$E\left[x_i^p(t)\right] \le \widehat{K} = \widehat{K}(p), \quad t \ge 0, p > 0, i = 1, 2, 3. \tag{13}$$

3. Stationary distribution

In this section, we firstly introduce the following notations for simplicity:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{vmatrix}, \qquad D_1 = \begin{vmatrix} r_1 & a_{12} & a_{13} \\ -r_2 & a_{22} & a_{23} \\ -r_3 & -a_{32} & a_{33} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & r_1 & a_{13} \\ -a_{21} & -r_2 & a_{23} \\ -a_{31} & -r_3 & a_{33} \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & r_1 \\ -a_{21} & a_{22} & -r_2 \\ -a_{31} & -a_{32} & -r_3 \end{vmatrix},$$

where all the elements of the above determinants are the parameters of system (2).

Before we go any further, let us introduce some lemmas given by Gard and detailed proofs are available in [11]. Let $X(t)$ be a homogeneous Markov process satisfying the following stochastic differential equation:

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{m=1}^{k} \beta_m(X)\mathrm{d}B_m(t), \tag{14}$$

where $b(\cdot)$ and $\beta_m(\cdot)$ are continuous n-vector valued functions for $t > 0$, $X \in \mathbb{R}^n$ and $B_m(t)(m = 1, 2, \cdots, k)$ are independent scalar Wiener processes. The diffusion matrix is $A(x) = (\alpha_{ij}(x)), \alpha_{ij} = \sum_{m=1}^{k} \beta_m^{(i)}(x)\beta_m^{(j)}(x)$.

Lemma 5 *(p. 132 Theorem 5.3 in [11]) Let U and U_n (for each positive integer n) be open sets in $\mathbb{R}^n$ with*

$$U_n \subseteq U_{n+1}, \quad \overline{U}_n \subseteq U, \quad and \quad U = \bigcup_n U_n,$$

and suppose $b(.)$; $\beta_m(\cdot)$ satisfy the existence and uniqueness conditions for solutions of (14) on each set $t > 0$, $X(t) \in U_n$. Suppose further there is a nonnegative continuous function $V(t, X(t))$ with continuous partial derivatives $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial X_i}$, and $\frac{\partial^2 V}{\partial X_i \partial X_j}$ satisfying $LV \leq cV$ for $t > 0, X \in U$, where c is a positive constant and

$$LV(t, X(t)) = \left[\frac{\partial V}{\partial t} + \nabla_x V \cdot b + \frac{1}{2}\text{tr }(AV_{xx})\right](t, X(t)). \tag{15}$$

If also

$$\inf_{t>0, X \in U \setminus U_n} V(t, X(t)) \to \infty, \quad as \quad n \to \infty,$$

then for any X_0 independent of $B(t)$ such that

$$P(X_0 \in U) = 1,$$

there is a unique solution $X(t)$ of (14) with $X(0) = X_0$, and $X(t) \in U$ for all $t > 0$, that is, $P(\tau_U = \infty) = 1$, where τ_U is the first exit time from U and which is given by $\tau_U = \inf\{t : X(t) \notin U\}$.

Lemma 6 *(p. 144 Theorem 5.9 in [11]) Suppose the conditions of Lemma 5 hold together with the following two conditions: for some positive integer n_0, there are positive constants M and c_1 such that*

$$(i) \quad \sum_{i,j=1}^{k} \alpha_{ij}(X(t))\theta_i\theta_j > M|\theta|^2, \quad X(t) \in \overline{U}_{n_0}, \quad \theta \in \mathbb{R}^k,$$

$$(ii) \quad LV(X(t)) \leq -c_1, \quad for\ all \quad X(t) \in U \setminus \overline{U}_{n_0}.$$

Then there exists an invariant distribution (stationary distribution) $\tilde{P}$ with nowhere zero density in U such that for any Borel set $B \subseteq U$

$$P(t, x, B) \to \tilde{P}(B), \quad as \quad t \to \infty,$$

where $P(t, x, B)$ is the transition probability $P(X(t) \in B | X(0) = x)$ for the solution $X(t)$ of the SDE (14).

Here we define $-\delta = \dfrac{1}{2}\lambda_{\max}\left(B + B^T\right)$, where $\lambda_{\max}\left(B + B^T\right)$ stands for the maximal eigenvalue of the matrix $B + B^T$ and

$$B = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ a_{21} & -a_{22} & -a_{23} \\ a_{31} & a_{32} & -a_{33} \end{pmatrix}. \tag{16}$$

Next we will give the main result of this section.

Theorem 7 *Suppose that $D>0, D_i>0 (i=1,2,3)$. If $\delta>0$ and the noises are sufficiently small such that*

$$\frac{1}{2}\sum_{i=1}^{3}\sigma_i^2 x_i^* < \delta \min_{i=1,2,3}\left\{(x_i^*)^2\right\}, \tag{17}$$

where (x_1^*, x_2^*, x_3^*) is a solution of the following equation:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1, \\ -a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = -r_2, \\ -a_{31}x_1 - a_{32}x_2 + a_{33}x_3 = -r_3. \end{cases} \tag{18}$$

Then there is a stationary distribution $\mu(\cdot)$ for system (2).

Proof From equation (18), we know that $x_i^* = \frac{D_i}{D} > 0 (i=1,2,3)$. Then let

$$V(x) = \sum_{i=1}^{3}\left[x_i - \frac{1}{2}x_i^* - x_i^* \ln\frac{x_i}{x_i^*}\right].$$

By the Itô's formula

$$\mathrm{d}V(x) = LV(x)\mathrm{d}t + \sum_{i=1}^{3}\left(x_i - x_i^*\right)\sigma_i \mathrm{d}B_i(t),$$

where

$$\begin{aligned} LV(x) &= (x_1-x_1^*)[r_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3] + \frac{1}{2}\sigma_1^2 x_1^* \\ &\quad +(x_2-x_2^*)[-r_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3] + \frac{1}{2}\sigma_2^2 x_2^* \\ &\quad +(x_3-x_3^*)[-r_3 + a_{31}x_1 + a_{32}x_2 - a_{33}x_3] + \frac{1}{2}\sigma_3^2 x_3^* \\ &= -(x_1-x_1^*)[a_{11}(x_1-x_1^*) + a_{12}(x_2-x_2^*) + a_{13}(x_3-x_3^*)] + \frac{1}{2}\sigma_1^2 x_1^* \\ &\quad -(x_2-x_2^*)[-a_{21}(x_1-x_1^*) + a_{22}(x_2-x_2^*) + a_{23}(x_3-x_3^*)] + \frac{1}{2}\sigma_2^2 x_2^* \\ &\quad -(x_3-x_3^*)[-a_{31}(x_1-x_1^*) - a_{32}(x_2-x_2^*) + a_{33}(x_3-x_3^*)] + \frac{1}{2}\sigma_3^2 x_3^* \\ &\le (x_1-x_1^*, x_2-x_2^*, x_3-x_3^*)\frac{1}{2}\lambda_{\max}\left(B+B^T\right)(x_1-x_1^*, x_2-x_2^*, x_3-x_3^*)^T \\ &\quad +\frac{1}{2}\sum_{i=1}^{3}\sigma_i^2 x_i^* \\ &= \sum_{i=1}^{3} -\delta(x_i-x_i^*)^2 + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2 x_i^*. \end{aligned} \tag{19}$$

From the conditions $\delta>0$ and (17) we know that the sphere (noting that $x_i^*>0$ for $i=1,2,3$)

$$\sum_{i=1}^{3} -\delta(x_i-x_i^*)^2 + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2 x_i^* = 0, \tag{20}$$

is entirely included in $\mathbb{R}^3_+$. Let U_{n_0} be a neighborhood of the sphere with $\overline{U}_{n_0} \subseteq \mathbb{R}^3_+$, and then $LV(x) < -c_1$ for some constant $c_1 > 0$ and any $x \in \mathbb{R}^3_+ \backslash \overline{U}_{n_0}$. This makes the Assumption (ii) of Lemma 6 hold with $U = \mathbb{R}^3_+$. Moreover, for all $x \in \mathbb{R}^3_+$, we have $V(x) \geq \sum_{i=1}^{3} \frac{1}{2} x_i^* > 0$. Thus we can easily get $LV(x) \leq cV(x)$ for all $x \in \mathbb{R}^3_+$. On the other hand, since the diffusion matrix of our system (2) is

$$A(x) = \begin{pmatrix} \sigma_1^2 x_1^2 & 0 & 0 \\ 0 & \sigma_2^2 x_2^2 & 0 \\ 0 & 0 & \sigma_3^2 x_3^2 \end{pmatrix}, \tag{21}$$

then there exists a constant $c > 0$ such that

$$\sum_{i=1}^{3} \sigma_i^2 (x_i)^2 \theta_i^2 > c|\theta|^2,$$

for $x \in \overline{U}_{n_0}$ and $\theta \in \mathbb{R}^3$ ($|\cdot|$ means the Euclidean norm of a vector), which verifies the Assumption (i) of Lemma 6. Consequently, from Lemma 6 we obtain that system (2) has a stationary distribution $\mu(\cdot)$. □

Remark 8 *We know that assumption (i) of Lemma 3.2 shows that the operator L is uniformly elliptical in $\overline{U}_{n_0}$, which can guarantee that the smallest eigenvalue of the diffusion matrix A is bounded away from zero (see p. 103 of Gard [11]and p. 349 of Strang [37]). Assumption (ii) of Lemma 3.2 shows that the mean time τ at which a path issuing from x reaches the set $\overline{U}_{n_0}$ is finite for any $x \in U \backslash \overline{U}_{n_0}$, and $\sup_{x \in U_0} E_x \tau < +\infty$ for every compact subset $U_0 \subset U$ (see p. 1163 of Zhu and Yin [38]). Then by adopting the conclusion of Theorem 5.1 on p. 121 of Khasminskii [12], we can conclude that under the assumption of Theorem 3.1 system (2) has the ergodic property with $U = \mathbb{R}^3_+$*

$$\mathcal{P}\left\{ \lim_{t \to +\infty} \frac{1}{t} \int_0^t x_i(s) \mathrm{d}s = \int_{\mathbb{R}^3_+} z_i \mu(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3) \right\} = 1, \quad \text{for all } z_i \in \mathbb{R}+, \ i = 1, 2, 3.$$

4. Global asymptotic stability

In this section, let us first introduce the definition of global stability and some useful lemmas.

Definition 9 *Let $x(t)$ and $\widetilde{x}(t)$ be two arbitrary solutions of system (2) with initial values $x(0) \in \mathbb{R}^3_+$ and $\widetilde{x}(0) \in \mathbb{R}^3_+$, respectively. If for every $x(0), \widetilde{x}(0) \in \mathbb{R}^3_+$, $\lim_{t \to \infty} |x_i(t) - \widetilde{x}_i(t)| = 0, (i = 1, 2, 3)$ a.s., then we say (2) is globally asymptotically stable (or globally attractive).*

Lemma 10 *(See [18, 31]) Suppose that an n-dimensional stochastic process $X(t)$ on $t \geq 0$ satisfies the condition*

$$E|X(t) - X(s)|^{\lambda_1} \leq c|t - s|^{1+\lambda_2}, \quad 0 \leq s, t < \infty,$$

for some positive constants λ_1, λ_2*, and* c*. Then there exists a continuous modification* $\tilde{X}(t)$ *of* $X(t)$ *which has the property that for every* $\gamma \in (0, \lambda_2/\lambda_1)$ *there is an almost surely positive random variable* $h(\omega)$ *such that*

$$\mathcal{P}\left\{\omega : \sup_{0<|t-s|<h(\omega), 0\le s,t<\infty} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t-s|^{\gamma}} \le \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ *is locally but uniformly Hölder continuous with exponent* γ*.*

Lemma 4.1 is the well-known Kolmogorov–Čentsov theorem on the continuity of a stochastic process. Karatzas and Shreve [18] give a proof of this result in the case when the stochastic process $X(t)$ is on the finite interval $[0,T]$. Mao [31] points out that a little bit of modification of the proof works for the case when $X(t)$ is on the entire $\mathbb{R}_+$.

Lemma 11 *Let* $x(t)$ *be a positive solution of system (2); if* $a_{22} > a_{21}$ *and* $a_{33} > a_{31} + a_{32}$*, then almost every sample path of* $x_i(t) (i = 1, 2, 3)$ *is uniformly continuous.*

Proof The equivalent integral equation of system (2) is

$$\begin{aligned}
x_1(t) = \quad & x_1(0) + \int_0^t x_1(s)[r_1 - a_{11}x_1(s) - a_{12}x_2(s) - a_{13}x_3(s)]\mathrm{d}s \\
& + \int_0^t \sigma_1 x_1(s)\mathrm{d}B_1(s), \\
x_2(t) = \quad & x_2(0) + \int_0^t x_2(s)[-r_2 + a_{21}x_1(s) - a_{22}x_2(s) - a_{23}x_3(s)]\mathrm{d}s \\
& + \int_0^t \sigma_2 x_2(s)\mathrm{d}B_2(s), \\
x_3(t) = \quad & x_3(0) + \int_0^t x_3(s)[-r_3 + a_{31}x_1(s) + a_{32}x_2(s) - a_{33}x_3(s)]\mathrm{d}s \\
& + \int_0^t \sigma_3 x_3(s)\mathrm{d}B_3(s).
\end{aligned} \tag{22}$$

For $0 < \tau < t < \infty, t - \tau \le 1$, $p > 2$, and $i = 1, 2, 3$, it is obvious

$$\begin{aligned}
& E\left|x_i(t) - x_i(\tau)\right|^p \\
& \le E\left|\int_\tau^t x_i(s)\left[r_i + \sum_{j=1}^3 a_{ij}x_j(s)\right]\mathrm{d}s + \int_\tau^t \sigma_i x_i(s)\mathrm{d}B_i(s)\right|^p \\
& \le 2^{p-1}E\left|\int_\tau^t x_i(s)\left[r_i + \sum_{j=1}^3 a_{ij}x_j(s)\right]\mathrm{d}s\right|^p + 2^{p-1}E\left|\int_\tau^t \sigma_i x_i(s)\mathrm{d}B_i(s)\right|^p \\
& \le [2(t-\tau)]^{p-1}\int_\tau^t E\left|x_i(s)\left[r_i + \sum_{j=1}^3 a_{ij}x_j(s)\right]\right|^p \mathrm{d}s \\
& \quad + \frac{1}{2}[2p(p-1)]^{\frac{p}{2}}\left|\sigma_i\right|^p (t-\tau)^{\frac{p-2}{2}}\int_\tau^t E\left|x_i(s)\right|^p \mathrm{d}s.
\end{aligned} \tag{23}$$

Here the last inequality is followed from the continuous Hölder inequality and Theorem 7.1 from p. 39 [30]. By using the discrete Hölder inequality, one can deduce the following useful inequality:

$$\left|\sum_{i=1}^{n} a_i\right|^p \leq n^{(p-1)} \sum_{i=1}^{n} |a_i|^p,$$

where $p \geq 1, a_i \in \mathbb{R}, n \geq 2$. Then we note that (from the above inequality and Lemma 3)

$$\begin{aligned}
&E\left|x_i\left[r_i + \sum_{j=1}^{3} a_{ij}x_j\right]\right|^p \\
&\leq \frac{1}{2}E\left|x_i\right|^{2p} + \frac{1}{2}E\left|r_i + \sum_{j=1}^{3} a_{ij}x_j\right|^{2p} \\
&\leq \frac{1}{2}\left\{E\left|x_i\right|^{2p} + 4^{2p-1}\left[\left|r_i\right|^{2p} + a_{ii}^{2p}E\left|x_i\right|^{2p} + \sum_{j=1,j\neq i}^{3} a_{ij}^{2p}E\left|x_j\right|^{2p}\right]\right\} \\
&\leq \frac{1}{2}\left\{\widehat{K}(2p) + 4^{2p-1}\left[\left|r_i\right|^{2p} + a_{ii}^{2p}\widehat{K}(2p) + \widehat{K}(2p)\sum_{j=1,j\neq i}^{3} a_{ij}^{2p}\right]\right\} =: \overline{K}(p).
\end{aligned} \tag{24}$$

One should take note that $\widehat{K}(2p)$ is from(13). By taking the inequality (24) into (23)

$$\begin{aligned}
&E\left|x_i(t) - x_i(\tau)\right|^p \\
&\leq [2(t-\tau)]^{p-1}\int_{\tau}^{t}\bar{K}(p)\mathrm{d}s + \frac{1}{2}[2p(p-1)]^{\frac{p}{2}}\left|\sigma_i\right|^p(t-\tau)^{\frac{p-2}{2}}\int_{\tau}^{t}\widehat{K}(p)\mathrm{d}s \\
&= 2^{p-1}(t-\tau)^p\bar{K}(p) + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left|\sigma_i\right|^p(t-\tau)^{\frac{p}{2}}\widehat{K}(p) \\
&\leq 2^{p-1}(t-\tau)^{\frac{p}{2}}\left[(t-\tau)^{\frac{p}{2}} + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right]K_0(p) \\
&\leq 2^{p-1}(t-\tau)^{\frac{p}{2}}\left[1 + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right]K_0(p),
\end{aligned} \tag{25}$$

where $K_0(p) = \max\{\overline{K}(p), |\sigma_i|^p\widehat{K}(p)\}$. Then since $p > 2$ we can get that almost every sample path of $x_i(t)(i = 1,2,3)$ is local uniformly Hölder continuous with exponent $\gamma \in (0, \frac{p-2}{2p})$, which is following from Lemma 10. Therefore almost every sample path of $x_i(t)(i = 1,2,3)$ is uniformly continuous on $t > 0$. □

Lemma 12 *(Barbalat's Lemma [2]) Let f be a nonnegative function defined on $[0,+\infty)$ such that f is integrable and is uniformly continuous. Then* $\lim_{t\to\infty} f(t) = 0$.

Then we give the main theorem of this section.

Theorem 13 *System (2) is globally asymptotically stable if* $\widetilde{B}$ *is irreducible and*

$$a_{ii} > \sum_{j=1, j\neq i}^{3} a_{ji}, \quad i = 1,2,3, \tag{26}$$

where

$$\widetilde{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{27}$$

Proof Suppose $x(t), \widetilde{x}(t)$ be two solution of equation (2) with initial values $x(0), \widetilde{x}(0) \in \mathbb{R}_+^3$ and $x(0) \neq \widetilde{x}(0)$, respectively. Let c_i be the cofactor of the i-th diagonal element of $L_{\widetilde{B}}$, where

$$L_{\widetilde{B}} = \begin{pmatrix} a_{12}+a_{13} & -a_{12} & -a_{13} \\ -a_{21} & a_{21}+a_{23} & -a_{23} \\ -a_{31} & -a_{32} & a_{31}+a_{32} \end{pmatrix}. \tag{28}$$

Then $c_i > 0$. Consider the following Lyapunov function:

$$V(t) = \sum_{i=1}^{3} c_i \left| \ln x_i(t) - \ln \widetilde{x}_i(t) \right|.$$

By calculating the right differential $\mathrm{d}^+V(t)$, we obtain

$$\begin{aligned} \mathrm{d}^+V(t) \quad &= \sum_{i=1}^{3} c_i \mathrm{sgn}\,(x_i(t) - \widetilde{x}_i(t))\, \mathrm{d}\left(\ln x_i(t) - \ln \widetilde{x}_i(t)\right) \\ &= c_1 \mathrm{sgn}\,(x_1 - \widetilde{x}_1)\left[-a_{11}(x_1 - \widetilde{x}_1) - a_{12}(x_2 - \widetilde{x}_2) - a_{13}(x_3 - \widetilde{x}_3)\right] \mathrm{d}t \\ &\quad + c_2 \mathrm{sgn}\,(x_2 - \widetilde{x}_2)\left[a_{21}(x_1 - \widetilde{x}_1) - a_{22}(x_2 - \widetilde{x}_2) - a_{23}(x_3 - \widetilde{x}_3)\right] \mathrm{d}t \\ &\quad + c_3 \mathrm{sgn}\,(x_3 - \widetilde{x}_3)\left[a_{31}(x_1 - \widetilde{x}_1) + a_{32}(x_2 - \widetilde{x}_2) - a_{33}(x_3 - \widetilde{x}_3)\right] \mathrm{d}t \\ &\leq \sum_{i=1}^{3} \left[-c_i a_{ii} \left| x_i - \widetilde{x}_i \right| + \sum_{j=1, j\neq i}^{3} c_i a_{ji} \left| x_i - \widetilde{x}_i \right| \right] \mathrm{d}t \\ &= -\sum_{i=1}^{3} c_i \left[a_{ii} - \sum_{j=1, j\neq i}^{3} a_{ji} \right] \left| x_i - \widetilde{x}_i \right| \mathrm{d}t. \end{aligned}$$

The above inequality is equivalent to

$$V(t) \leq V(0) - \int_0^t \sum_{i=1}^{3} c_i \left[a_{ii} - \sum_{j=1, j\neq i}^{3} a_{ji} \right] \left| x_i(s) - \widetilde{x}_i(s) \right| \mathrm{d}s.$$

That is

$$V(t) + \int_0^t \sum_{i=1}^{3} c_i \left[a_{ii} - \sum_{j=1, j\neq i}^{3} a_{ji} \right] \left| x_i(s) - \widetilde{x}_i(s) \right| \mathrm{d}s \leq V(0) < +\infty.$$

Since $V(t) > 0$ is obvious and with assumption (26), we can see that $|x_i(t) - \widetilde{x}_i(t)|$ is integrable.

Then by using Lemma 11 and Lemma 12, the solution of our system (2) is globally asymptotically stable. □

5. Example and numerical simulations

We consider the following example:

$$\begin{cases} dx_1(t) = x_1(t)[1 - 1.6x_1(t) - 1.2x_2(t) - 0.3x_3(t)]dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)[-0.4 + 0.85x_1(t) - 2.5x_2(t) - 0.4x_3(t)]dt + \sigma_2 x_2(t)dB_2(t), \\ dx_3(t) = x_3(t)[-0.05 + 0.4x_1(t) + x_2(t) - 3x_3(t)]dt + \sigma_3 x_3(t)dB_3(t), \end{cases} \tag{29}$$

with $\frac{\sigma_1^2}{2} = \frac{\sigma_2^2}{2} = \frac{\sigma_3^2}{2} = 0.001$. We give the simulations of the solution of Eq. (29) by using Milstein's method (see e.g. [13]). Consider the following discretization equation:

$$\begin{aligned} x_1^{(n+1)} &= x_1^{(n)} + x_1^{(n)}[1 - 1.6x_1^{(n)} - 1.2x_2^{(n)} - 0.3x_3^{(n)}]\Delta t \\ &\quad + \sigma_1 x_1^{(n)}\sqrt{\Delta t}\xi^{(n)} + \frac{\sigma_1^2}{2}x_1^{(n)}\Big[(\xi^{(n)})^2\Delta t - \Delta t\Big], \\ x_2^{(n+1)} &= x_2^{(n)} + x_2^{(n)}[-0.4 + 0.85x_1^{(n)} - 2.5x_2^{(n)} - 0.4x_3^{(n)}]\Delta t \\ &\quad + \sigma_2 x_2^{(n)}\sqrt{\Delta t}\zeta^{(n)} + \frac{\sigma_2^2}{2}x_2^{(n)}\Big[(\zeta^{(n)})^2\Delta t - \Delta t\Big], \\ x_3^{(n+1)} &= x_3^{(n)} + x_3^{(n)}[-0.05 + 0.4x_1^{(n)} + x_2^{(n)} - x_3^{(n)}]\Delta t \\ &\quad + \sigma_3 x_3^{(n)}\sqrt{\Delta t}\eta^{(n)} + \frac{\sigma_3^2}{2}x_3^{(n)}\Big[(\eta^{(n)})^2\Delta t - \Delta t\Big]. \end{aligned} \tag{30}$$

(i) By calculating we get $x_1^* = 0.5898$, $x_2^* = 0.0291$, $x_3^* = 0.0692$, $\delta = 3.1499$ and then $\frac{1}{2}\sum_{i=1}^{3}\sigma_i^2 x_i^* - \delta \min_{i=1,2,3}\{(x_i^*)^2\} = -0.0909 < 0$. Therefore system (29) has a stationary distribution according to Theorem 4 (see Figure 1).

(ii) The matrix $\tilde{B}$ corresponding to system (29) is irreducible and satisfying $a_{ii} > \sum_{j=1, j\neq i}^{3} a_{ji}, (i = 1, 2, 3)$ which satisfy the conditions of Theorem 13; then the system (29) is globally asymptotically stable (see Figure 2).

6. Conclusion

In this paper, we formulate a three-species stochastic food-chain model (2) based on deterministic model (1) by considering the growth rate influenced by random fluctuations. Under our assumption that deterministic model (1) has a globally stable positive equilibrium, if the intensities of the noises are sufficiently small, we conclude that the stochastic system (2) has a stationary distribution. This shows that the population distribution of the stochastic model will be around the positive equilibrium point. Then the sufficient conditions for global asymptotic stability are established. That is to say no matter what the initial population may be each population change will be almost the same after some time.

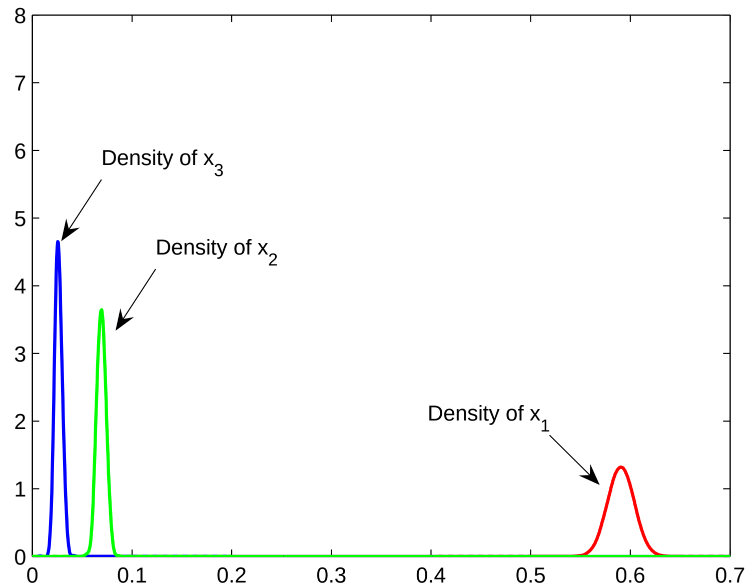

Figure 1. Distribution of Eq. (29) with step size $\Delta t = 0.001$ $\frac{\sigma_1^2}{2} = \frac{\sigma_2^2}{2} = \frac{\sigma_3^2}{2} = 0.001$.

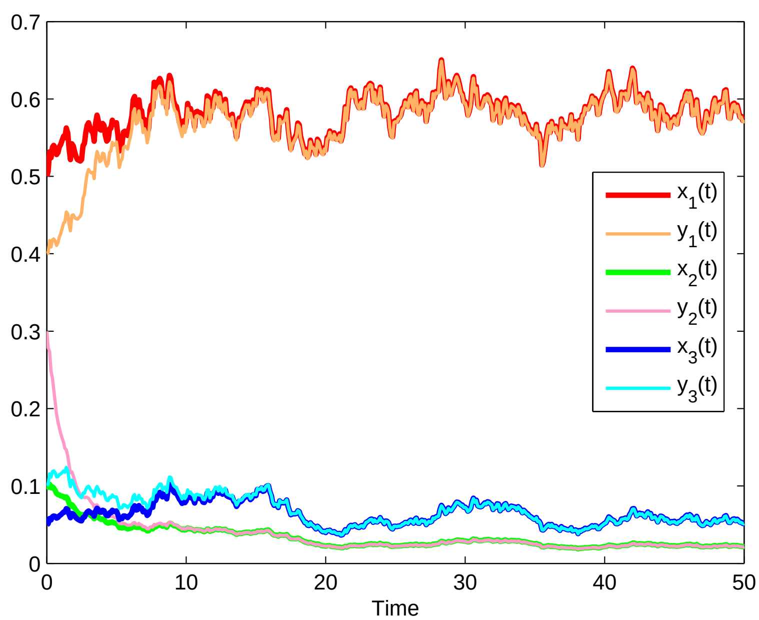

Figure 2. Solution of Eq. (29) for initial data $x_1(0) = 0.5, x_2(0) = 0.1, x_3(0)) = 0.05,$ $y_1(0) = 0.4, y_2(0) = 0.3, y_3(0) = 0.1$, step size $\Delta t = 0.001$.

There still are some interesting further questions that deserve to be considered. One may study the dynamic properties of the nonautonomous food-chain models. One can also consider the persistence and extinction properties of the three-species Lotka–Volterra model under regime switching.

Acknowledgments

We would like to thank the editor and referees for their important and valuable comments.

References

[1] Bahar A, Mao X. Stochastic delay population dynamics. Int J Pure Appl Math 2004; 11: 377-400.

[2] Barbalat I. Systems d'equations differentielles d'oscillations nonlineaires. Rev Roum Math Pures Appl 1959; 4: 267-270.

[3] Caswell H. Predator mediated coexistence: a non-equilibrium model. Am Nat 1978; 112: 127-154.

[4] Chattopadhyay J, Arino O. A predator-prey model with disease in the prey. Nonlinear Anal 1999; 36: 747-766.

[5] Cheng S. Stochastic population systems. Stoch Anal Appl 2009; 27: 854-874.

[6] Dubey B, Upadhyay RK. Persistence and extinction of one-prey and two-predators system. Nonlinear Anal: Model Control 2004; 9: 307-329.

[7] Farkas M, Freedman H. Stability conditions for two predator one prey systems. Acta Appl Math 1989; 14: 3-10.

[8] Freedman HI, Waltman P. Persistence in models of three interacting predator-prey populations. Math Biosci 1984; 68: 213-231.

[9] Freedman HI, So JW. Global stability and persistence of simple food chains. Math Biosci 1985; 76: 69-86.

[10] Fu J, Li H, Han Q, Li H. Analysis of a stochastic food chain model with finite delay. J Nonlinear Sci Appl 2016; 9: 553-567.

[11] Gard TC. Introduction to Stochastic Differential Equations. New York, NY, USA: Dekker, 1988.

[12] Khasminskii R. Stochastic Stability of Differential Equations. Alphen aan den Rijn, Netherlands: Sijthoff & Noordhoff, 1980.

[13] Higham D. An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev 2001; 43: 525-546.

[14] Hutson V, Vickers GT. Acriterion for permanent co-existence of species with an application to a two-prey one-predator system. Math Bio Sci 1983; 63: 253-269.

[15] Ikeda N, Watanabe S. A comparison theorem for solutions of stochastic differential equations and its applications. Osaka J Math 1977; 14: 619-633.

[16] Jiang D, Shi N. A note on non-autonomous logistic equation with random perturbation. J Math Anal Appl 2005; 303: 164-172.

[17] Jiang D, Ji C, Li X, O'Regan D. Analysis of autonomous Lotka-Volterra competition systems with random perturbation. J Math Anal Appl 2012; 390: 582-595.

[18] Karatzas I, Shreve S. Brownian Motion and Stochastic Calculus. Berlin, Germany: Springer, 1991.

[19] Krikorian N. The Volterra model for three species predator-prey systems: boundedness and stability. J Math Biology 1979; 7: 117-132.

[20] Li X, Mao X. Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation. Discrete Contin Dyn Syst 2009; 24: 523-545.

[21] Li H, Cong F, Jiang D, Hua H. Persistence and nonpersistence of a food chain model with stochastic perturbation. Abstr Appl Analj 2013; 4: 764-787.

[22] Liu M, Bai C. Analysis of a stochastic tri-trophic food-chain model with harvesting. J Math Biol 2016; 73: 597-625.

[23] Liu M, Bai C. Optimal harvesting of a stochastic mutualism model with Lévy jumps. Appl Math Comput 2016; 276: 301-309.

[24] Liu M, Wang K. Survival analysis of a stochastic cooperation system in a polluted environment. J Biol Syst 2011; 19: 183-204.

[25] Liu M, Wang K, Wu Q. Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle. Bull Math Biol 2011; 73: 1969-2012.

[26] Liu M, Wang K. Global stability of a nonlinear stochastic predator-prey system with Beddington-DeAngelis functional response. Commun. Nonlinear Sci Numer Simulat 2011; 16: 1114-1121.

[27] Liu M, Wang K. Dynamics of a two-prey one-predator system in random environments. J Nonlinear Sci 2013; 23: 751-775.

[28] Liu M, Mandal PS. Dynamical behavior of a one-prey two-predator model with random perturbations. Commun. Nonlinear Sci Numer Simulat 2015; 28: 123-137.

[29] Luo Q, Mao X. Stochastic population dynamics under regime switching II. J Math Anal Appl 2009; 355: 577-593.

[30] Mao X. Stochastic Differential Equations and Applications. Chichester, UK: Horwood Publishing, 1997.

[31] Mao X. Stochastic versions of the LaSalle theorem. J Differ Equations 1999; 153: 175-195.

[32] Mao X, Yuan C. Stochastic Differential Equations with Markovian Switching. London, UK: Imperial College Press, 2006.

[33] Murray JD. Mathematical Biology I: an Introduction. New Delhi, India: Springer, 2002.

[34] Pande LK. Ecosystems with three species: one-prey-and-two-predators system in an exactly solvable model. J Theor Biol 1978; 74: 591-598.

[35] Qiu H, Deng WM. Optimal harvesting of a stochastic delay logistic model with Lévy jumps. J Phys A: Math Theor 2016; 49: 1-10.

[36] Rudnicki R, Pichor K. Influence of stochastic perturbation on prey-predator systems. Math Biosci 2007; 206: 108-119.

[37] Strang G. Linear Algebra and Its Applications. New York, NY, USA: Thomson Learning, 1988.

[38] Zhu C, Yin G. Asymptotic properties of hybrid diffusion systems. SIAM J Control Optim 2007; 46: 1155-1179.

[39] Zhu C, Yin G. On hybrid competitive Lotka-Volterra ecosystems. Nonlinear Anal 2009; 71: 1370-1379.