Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2017) 41: 1308 – 1312
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doi:10.3906/mat-1604-109

On tetravalent normal edge-transitive Cayley graphs on the modular group

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Received: 27.04.2016 • **Accepted/Published Online:** 19.12.2016 • **Final Version:** 28.09.2017

Abstract: A Cayley graph $\Gamma = Cay(G,S)$ on a group G with respective to a subset $S \subseteq G$, $S = S^{-1}, 1 \notin S$, is said to be normal edge-transitive if $N_{\mathbb{A}ut(\Gamma)}(\rho(G))$ is transitive on edges of Γ, where $\rho(G)$ is a subgroup of $\mathbb{A}ut(\Gamma)$ isomorphic to G. We determine all connected tetravalent normal edge-transitive Cayley graphs on the modular group of order $8n$ in the case that every element of S is of order $4n$.

Key words: Cayley graph, edge-transitive, modular group

1. Introduction

Let G be a group and S a subset of G such that $1 \notin S$. The Cayley graph $Cay(G,S)$ is the graph with vertex set $\mathrm{V}(Cay(G,S)) = G$ and edge set $\mathrm{E}(Cay(G,S)) = \{(u,v) | vu^{-1} \in S\}$. The edge set can be identified with set of ordered pairs $\{(g, sg) | g \in G, s \in S\}$. If $S = S^{-1}$, that is, closed under taking the inverse, then $Cay(G,S)$ is an undirected graph. The degree of each vertex is $|S|$ and it is obvious that $Cay(G,S)$ is connected if and only if $G = \langle S \rangle$.

A graph Γ is called vertex-transitive or edge-transitive if the automorphism group $\mathbb{A}ut(\Gamma)$ acts transitively on the vertex-set or edge-set of Γ, respectively. Now let $\Gamma = Cay(G,S)$.

For $g \in G$, let $\rho_g : G \to G$ given by $\rho_g(x) = xg$. The set of all ρ_g, $g \in G$, forms the subgroup $\rho(G)$ (isomorphic to G) of $\mathbb{A}ut(\Gamma)$. Since $\rho(G) \leqslant \mathbb{A}ut(\Gamma)$ acts right regularly on the vertices of Γ, by definition, Γ is vertex-transitive, while Γ is not edge-transitive in general.

In 1999, Praeger [9] introduced the concept of normal edge-transitive Cayley graphs, which plays an important role for understanding Cayley graphs. The graph Γ is called normal edge-transitive if $N_{\mathbb{A}ut(\Gamma)}(\rho(G))$ is transitive on the edges of Γ.

The research on edge-transitive Cayley graphs is an active area of research. One of the standard problems in this respect is the study of normal edge-transitive Cayley graphs of small valencies. Here we mention some references on research about edge-transitive Cayley graphs. In [7] the edge-transitive tetravalent Cayley graphs on groups of square-free order are recognized. In [4] the authors characterized all nonnormal Cayley digraphs of outvalency 2 of all nonabelian groups of order $2p^2$, where p is an odd prime. In [1] the author found normal edge-transitive Cayley graphs of abelian groups. In [6] all the tetravalent edge-transitive Cayley graphs on the group $PSL_2(p)$ and in [2] the normal edge-transitive Cayley graphs of Frobenius groups of order pq, where p

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2010 *AMS Mathematics Subject Classification:* 20D60, 05B25

and q are different primes, are determined. In [3] the authors studied normal edge-transitive Cayley graphs of order $4p$ where p is an odd prime.

Our aim in this paper is to study connected tetravalent normal edge-transitive Cayley graphs of a certain group of order $8n$, $n \in \mathbb{N}$. According to [8] up to isomorphism there are four nonabelian groups of order $8n$ with a cyclic subgroup of order $4n$, if n is a power of 2. One of these groups is called the modular group, with the following presentation:

$$M_{8n} = \langle a, b | a^{4n} = b^2 = 1, bab = a^{2n+1} \rangle.$$

In the following we work with the modular group M_{8n} without assuming that n is a power 2.

We employ the following notation and terminology. The notation $G = K \rtimes H$ is used to indicate that G is a semidirect product of K by H. We denote by $\mathbb{A}ut(G,S)$ the subgroup of $\mathbb{A}ut(G)$ consisting of all $\sigma \in \mathbb{A}ut(G)$ such that $\sigma(S) = S$. It is easy to see that $\mathbb{A}ut(G,S)$ is a subgroup of the automorphisms group of $Cay(G,S)$. $\mathbb{Z}_n$ denotes a cyclic group of order n, and $\mathbb{S}_4$ denotes for a the symmetric group on four letters. D_8 is employed to denote the dihedral group of order 8.

The following theorem is the main result of this paper.

Main Theorem *Let $G = M_{8n}$ and S be a symmetric subset of M_{8n} with cardinality 4 such that each element of S has order $4n$ and $G = \langle S \rangle$. If $\Gamma = Cay(G,S)$ is a normal edge-transitive Cayley graph, then $N_{\mathbb{A}ut(\Gamma)}(\rho(G)) \cong \rho(G) \rtimes \mathbb{Z}_2$.*

2. Preliminaries

We start with a famous lemma.

Lemma 2.1 *([5, Lemma 2.1] or [9]) For a Cayley graph $\Gamma = Cay(G,S)$, we have $N_{\mathbb{A}ut(\Gamma)}(\rho(G)) = \rho(G) \rtimes \mathbb{A}ut(G,S)$.*

Therefore, Γ is normal edge-transitive when $\rho(G) \rtimes \mathbb{A}ut(G,S)$ is transitive on the edge-set of Γ.

Xu in [10] defined a Cayley graph $\Gamma = Cay(G,S)$ to be normal if $\rho(G)$ is a normal subgroup of $\mathbb{A}ut(\Gamma)$, i.e. $N_{\mathbb{A}ut(\Gamma)}(\rho(G)) = \mathbb{A}ut(\Gamma)$.

The following lemma is very useful in this paper.

Lemma 2.2 *([9, Proposition 1(c)]) Consider the Cayley graph $\Gamma = Cay(G,S)$. Then the following are equivalent:*

(i) *Γ is normal edge-transitive;*

(ii) *$S = T \cup T^{-1}$, where T is an $\mathbb{A}ut(G,S)$-orbit in G;*

(iii) *There exists $H \leq \mathbb{A}ut(G)$ and $g \in G$ such that $S = g^H \cup g^{-H}$, where $g^H = \{g^h | h \in H\}$.*

Moreover, $\rho(G) \rtimes \mathbb{A}ut(G,S)$ is transitive on the arcs of Γ if and only if $\mathbb{A}ut(G,S)$ is transitive on S.

3. Proof of the main theorem

First we are going to specify the automorphism group of M_{8n}.

Elements of M_{8n} are of the form a^k or a^kb, $0 \le k < 4n$. Using the defining relations of M_{8n} we can find the orders of elements in M_{8n} as follows: $o(a^k) = \frac{4n}{(k,4n)}$ and

$$o(a^kb) = \begin{cases} \frac{4n}{(k,2n)}, & \text{if k is even,} \\ \frac{4n}{(n+k,2n)}, & \text{if k is odd,} \end{cases}$$

where $0 \le k < 4n$.

Elements of order 2 in M_{8n} are of the form $a^{2n}, a^{2n}b, b$ and if n is odd in addition to the above elements, a^nb and $a^{3n}b$ are also of order 2.

Elements of order $4n$ in M_{8n} are of the form a^k, $(k, 4n) = 1$, and a^kb, k odd, $(n + k, 2n) = 1$, $0 \le k < 4n$. Of course in the latter case n must be even.

Lemma 3.1 $|\mathbb{A}ut(M_{8n})| = 4\varphi(4n)$, *where* φ *refers to the Euler phi function.*

Proof $f \in \mathbb{A}ut(M_{8n})$ is completely ascertained by $f(a)$ and $f(b)$. The elements $f(a)$ and $f(b)$ have orders $4n$ and 2, respectively.

Case(1). n is odd. By what we mentioned earlier we must have $f(a) = a^k, (k, 4n) = 1, 1 \le k < 4n$ and $f(b) \in \{a^{2n}, a^{2n}b, b, a^nb, a^{3n}b\}$. The case $f(b) = a^{2n}$ is impossible and it verified that all other possibilities can happen. Therefore, $|\mathbb{A}ut(M_{8n})| = 4\varphi(4n)$.

Case(2). n is even. In this case $f(a) = a^k, (k, 4n) = 1, 1 \le k < 4n$, or $f(a) = a^lb, l$ odd, $(n + l, 2n) = 1, 0 \le l < 4n$, and $f(b) \in \{a^{2n}, a^{2n}b, b\}$. The automorphisms of M_{8n} are of two kinds. One kind is defined by $f(a) = a^k, (k, 4n) = 1, 1 \le k < 4n$ and $f(b) = a^{2n}b$ or b. The number of these automorphisms is $2\varphi(4n)$. The other kind of automorphisms of M_{8n} is defined by $f(a) = a^lb, l$ odd, $(n + l, 2n) = 1, 0 \le l < 4n$, and $f(b) \in \{a^{2n}, a^{2n}b, b\}$. However, $Z(M_{8n}) = \langle a^2 \rangle$ and hence $f(a^2) = a^{2t}$ and $f(b) = a^{2n}$ make a contradiction. Therefore, $f(b) = a^{2n}b$ or b.

However, it is easy to see that $(n + l, 2n) = 1$ if and only if $(l, n) = 1$ (note that n is even and l is odd), and $(l, n) = 1$ if and only if $(l, 4n) = 1$. Therefore, the number of automorphisms f is equal to $2\varphi(4n)$ and altogether we have $4\varphi(4n)$ possibilities for elements of $\mathbb{A}ut(M_{8n})$. This completes the proof.

□

Let us consider the Cayley graph $\Gamma = Cay(M_{8n}, S)$ where $|S| = 4$ and $M_{8n} = \langle S \rangle$. We are interested in the case where Γ is normal edge-transitive. By Lemma 2.2 elements of S have the same order and $\mathbb{A}ut(M_{8n}, S)$ on S is either transitive or has two orbits, T and T^{-1}.

We are interested in the case where each element of S has order $4n$. Therefore, elements of S are of the form $a^k, (k, 4n) = 1, 0 \le k < 4n$ or $a^lb, (n + l, 2n) = 1, l$ odd, $0 \le k < 4n$. It is obvious that n must be even. Therefore, from now on, we will assume that n is even.

Theorem 3.1 *Let* n *be an even number and* $\Gamma = Cay(M_{8n}, S)$ *be a normal connected edge-transitive Cayley graph where* $|S| = 4$ *and each element of* S *has order* $4n$. *Then* S *is of the following form:* $\{a, zab, a^{-1}, z^{-1}b^{-1}a^{-1}\}$, *where* $z \in Z(M_{8n})$.

Proof Elements of order $4n$ in M_{8n}, n even, are of the following types:

Type I: $a^k, 0 \leq k < 4n, (k, 4n) = 1$.

Type II: $a^l b, 0 \leq l < n, l$ odd, $(n+l, 2n) = 1$.

Let S be a generating set for M_{8n} such that $o(x) = 4n, \forall x \in S$, and $|S| = 4, S = S^{-1}$. Since $a^l b a^{l'} b = a^{l+l'(2n+1)}$ is a central element of M_{8n}, two elements of the same type can not generate M_{8n}. Therefore, we have to choose one element from each type. Let $S = \{x, y, x^{-1}, y^{-1}\}, M_{8n} = \langle S \rangle = \langle x, y \rangle$. Let $x = a^k, 0 \leq k < 4n, (k, 4n) = 1$, and $y = a^l b, 0 \leq l < 4n, l$ odd, $(n+l, 2n) = 1$. From $a^k \in S$ it is easy to deduce that $a \in \langle S \rangle$; hence, $b \in \langle S \rangle$. Therefore, for any x and y with the above conditions S is a generating set for M_{8n}.

If we take the automorphism $f \in \mathbb{A}ut(M_{8n})$ with $f(a) = a^{k'}, f(b) = b$, and choose k' in such a way that $kk' \equiv 1 (mod\ 4n)$, then $f(a^k) = a$ and $f(a^l b) = a^{k'l} b$. Since k' and l are odd, we can write $k'l = 1 + 2t$, and hence $a^{k'l} b = a^{1+2t} b = a^{2t} ab$. However, $Z(M_{8n}) = \langle a^2 \rangle$, and we see that $a^{2t} = z \in Z(M_{8n})$ and $f(S) = \{a, zab, a^{-1}, z^{-1} b^{-1} a^{-1}\}$, and the theorem is proved.

□

Now we are going to prove the main theorem.

By Theorem 3.1, S is equivalent to $\{a, zab, a^{-1}, (zab)^{-1}\}$, and by Lemma 2.1 we have $N_{\mathbb{A}ut(\Gamma)}(\rho(G)) = \rho(G) \rtimes \mathbb{A}ut(G, S)$. It is enough to find $\mathbb{A}ut(G, S)$. Because of $G = \langle S \rangle$, we have $\mathbb{A}ut(G, S) \leqslant \mathbb{S}_4$. The group $\mathbb{A}ut(G, S)$ does not contain elements of order 3 because if $\sigma \in \mathbb{A}ut(G, S)$ fixes $x \in S$, then it will fix x^{-1} as well. Therefore, $|\mathbb{A}ut(G, S)| \mid 8$, and $\mathbb{A}ut(G, S) \cong \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, D_8$.

We consider the following cases:

Case I. $\mathbb{A}ut(G, S)$ does not contain elements of order 4.

Let $\sigma \in \mathbb{A}ut(G, S)$ be of order 4. Then σ induces a cycle of length 4 on S. If $x \in S$, obviously $\sigma(x) = x^{-1}$ is impossible because then σ would be the product of two cycles. Therefore, we may assume that $\sigma = (a, zab, a^{-1}, (zab)^{-1})$. Since $z \in Z(M_{8n}) = \langle a^2 \rangle$, we set $z = a^{2t}, t \in \mathbb{N}$.

From $\sigma(a) = zab, \sigma(zab) = a^{-1}$ we obtain:

$$a^{-1} = \sigma(zab) = \sigma(z)\sigma(a)\sigma(b) = \sigma(z) zab \sigma(b) \Rightarrow \sigma(z) = z^{-1} a^{-2} \text{ or } z^{-1} a^{-2-2n}.$$

However, $\sigma(a)$ can only be of the form $\sigma(a) = a^l b$ where l is odd and hence $a^l b = zab$, from which it follows that $l = 2t + 1$.

Now:

$$\sigma(z) = \sigma(a^{2t}) = \sigma(a)^{2t} = (a^l b)^{2t} = a^{2(l-1)t} (ab)^{2t} = a^{2(l-1)t} a^{(2n+2)t} = z^{n+l}$$

. If $\sigma(z) = z^{-1} a^{-2} = z^{n+l}$, then $z^{n+l+1} a^2 = 1$, from which we obtain $2t(n+l+1) + 2 = 4mn$ for some $m \in \mathbb{N}$. It follows that $t(n+l+1) = 2mn - 1$, but the left-hand side of the last equality is even whereas its right-hand side is odd, a contradiction.

Similarly, the case $\sigma(z) = z^{-1} a^{-2-2n}$ results in a contradiction. Therefore, $\mathbb{A}ut(G, S)$ cannot be isomorphic to $\mathbb{Z}_4, D_8$.

Case II. $\mathbb{A}ut(G,S)$ does not contain a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is enough to prove that $\mathbb{A}ut(G,S)$ does not contain an element σ with $\sigma(a) = zab$ and $\sigma(zab) = a$. From the form of the automorphism of $\mathbb{A}ut(G)$ we have $\sigma(a) = a^k$ for some k, $(k, 4n) = 1$. If $\sigma(a) = zab$, then $a^k = zab$, from which we obtain $b = a^{-2t+k-1}$, which is not the case because a and b are independent generators of G.

Case III. $\mathbb{A}ut(G,S)$ contains an element of order 2.

If we define $\sigma(a) = a^{-1}$, $\sigma(b) = a^{2n}b$, we see that the cycle structure of $\sigma \in \mathbb{A}ut(G,S)$ on S is $(a, a^{-1})(zab, (zab)^{-1})$.

Therefore, $\mathbb{A}ut(G,S)$ is isomorphic to $\mathbb{Z}_2$. This completes the proof.

Acknowledgment

The authors express their deep gratitude to the referees for their extensive comments leading to a much clearer presentation.

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