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Chaos-related properties on the product of semiflows

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Abstract: In this paper we generalize some results about the chaos-related properties on the product of two semiflows, which appeared in the literature in the last few years, to the case of the most general possible acting monoids. In order to do that we introduce some new notions, namely the notions of a directional, psp and sip monoid, and the notion of a strongly transitive semiflow. In particular, we obtain a sufficient condition for the Devaney chaoticity of a product, which works for the (very large) class of the psp acting monoids.

Key words: Product of semiflows, directional monoid, psp monoid, sip monoid, Devaney chaos

1. Introduction

In this paper we generalize the continuous versions of the following statements from the papers [3] by Değirmenci and Koçak (published in 2010) and [9] by Li and Zhou (published in 2013) to the case of the most general possible acting monoids: Lemmas 1–4 and Theorems 1–3 from [3], Lemmas 3.5 (1), 3.9 (1)–(3), 3.10 (1)–(3), 3.11 (1)–(3), 3.12 (1)–(4) and Theorems 3.13, 3.14, 3.17 from [9]. However, we have not explicitly stated all these statements as corollaries of our statements, especially when they are immediate corollaries.

The papers in which the semiflows do not necessarily have continuous actions are relatively rare in the literature. In our own research we exclusively deal with continuous actions and so we assume the continuity of all the actions. In the papers [3] and [9] some actions are not assumed to be continuous. However, the papers [3] and [9] deal, respectively, with discrete (i.e. $\mathbb{N}_0$) and continuous (i.e. $\mathbb{R}_+$) semiflows only, while our goal is to investigate the semiflows with as general as possible acting monoids. In that respect we can say that, comparing to [3] and [9], we emphasize a different point of view. In our references [5, 8, 10, 14] the monoid actions are required to be continuous. Let us mention that there are other approaches to the requirement of the continuity of the action. For example, in the book [12] the semiflow is defined with the acting monoid $\mathbb{R}_+$, but, in general, the continuity of the action is required at all points excepts 0. The continuity requirement is discussed in this book on the page 13 and in Commentary 2.7(4) on pages 53–54. In the case of flows (i.e. when the acting monoid is a topological group), the action is almost always required to be continuous (see for example the classical book [6]).

The generality of the statements is the main novelty of our paper. Of course, the statements could not hold for all acting monoids, but we have found large classes of monoids for which they do. They include even monoids for which we cannot say that they are neither discrete nor continuous, but a mix of both (for example

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$T = \{0, 2, 3\} \cup [4, \infty)$ with the operation and topology induced from $\mathbb{R}$). In that way we can, for example, analyze the dynamical systems whose states are first recorded at some discrete moments and which are then observed continuously, after the initial discrete observation qualifies them as "interesting." That partly explains the motivation for the level of generality that is pursued in this paper. The psp monoids we introduce here include all but "pathological" monoids and many statements hold for them, which is quite amazing. Example 8.6 shows a very simple situation (but, nevertheless, nontrivial) in which Devaney's chaos on the product can be proved using our general statements for an acting monoid that is none of the "standard" acting monoids $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}_+$, $\mathbb{R}$.

The paper is self-contained, i.e. all the notions used in the paper are defined in it. The reader can find them in the references [4, 7, 8, 14]. The reader can also consult our paper [10], which is somewhat related to this paper.

We start with some standard definitions. In this paper T will denote a *noncompact* abelian topological monoid whose identity element is 0. For example: $(\mathbb{N}_0, +)$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $(\mathbb{N}_0^2, +)$, $(\mathbb{Z}, +)$, $(\mathbb{R}_+, +)$, where $\mathbb{R}_+ = [0, \infty)$, $(\mathbb{R}_+^2, +)$, $(\mathbb{R}, +)$, etc. A subset A of T is called *syndetic* if there is a compact subset K of T (a *corresponding compact* of A) such that for every $t \in T$ the translate $t + K$ intersects A. A subset A of T is called *thick* if for every compact subset K of T there is a $t \in T$ such that $t + K \subset A$. A subset $A \subset T$ is syndetic if and only if $T \setminus A$ is not thick. A subset $A \subset T$ is thick if and only if $T \setminus A$ is not syndetic. A subset A of T is called *piecewise syndetic* if there is a compact set $K \subset T$ and a thick set $B \subset T$ such that $(b + K) \cap A \neq \emptyset$ for every $b \in B$. If $A \subset T$ and $t \in T$ we denote $-t + A = \{s \in T \mid t + s \in A\}$.

If (X, d) is a metric space, $x \in X$ and $r > 0$, the open ball with center x and radius r is denoted by $B(x, r)$.

A jointly continuous monoid action $\pi : T \times X \to X$ of T on a metric space (X, d) is called a *semiflow* and denoted by (T, X, π) or by (T, X). The element $\pi(t, x)$ will be denoted by $t.x$ or tx, so that the defining conditions for a semiflow have the form

$$s.(t.x) = (s + t).x,$$
$$0.x = x,$$

for any $s, t \in T$ and $x \in X$. The maps $\pi_t : X \to X$, $x \mapsto tx$, are called the *transition maps*. For any $x \in X$ the set $Tx = \{tx \mid t \in T\}$ is called the *orbit* of x. If $Y \subset X$ and $t \in T$ we denote $t^{-1}Y = \{x \in X \mid tx \in Y\}$. When the acting topological monoid is a topological group, a semiflow is called a *flow*.

An $\mathbb{N}_0$-semiflow $(\mathbb{N}_0, X, \pi)$ is called a *cascade*. It is completely determined by the transition map $f := \pi_1$ since $\pi_n = f^n$ for every $n \in \mathbb{N}_0$. It is also denoted by (X, f). For a similar reason the $\mathbb{Z}$-flows are called *cascades*.

A point $x \in X$ in a semiflow (T, X) is called *periodic* if its *fixer* $\mathrm{Fix}(x) = \{t \in T \mid tx = x\}$ is a syndetic submonoid of T.

A semiflow (T, X) is called *minimal* (MIN) if the orbit Tx of every point x is dense, i.e. $\overline{Tx} = X$ for every $x \in X$. Otherwise, (T, X) is called *nonminimal* (NMIN). A semiflow (T, X) is called *topologically transitive* (TT) if for any nonempty open subsets U, V of X there is a $t \in T$ such that $tU \cap V \neq \emptyset$. A semiflow (T, X) is called *syndetically transitive* (SyndT) if for any nonempty open subsets U, V of X the set $\{t \in T \mid tU \cap V \neq \emptyset\}$ is syndetic. A semiflow (T, X) is called *weakly mixing* (WM) if for any four nonempty open subsets U, V, U', V' of X there is a $t \in T$ such that $tU \cap V \neq \emptyset$ and $tU' \cap V' \neq \emptyset$. A semiflow (T, X) is called *strongly mixing* (SM) if for any two nonempty open subsets U, V of X there is a compact subset K of T such that for every $t \in T \setminus K$, $tU \cap V \neq \emptyset$. A semiflow (T, X) is called *sensitive* (S) if there is a number $c > 0$ (a *sensitivity constant*) such that for any nonempty open set $U \subset X$ there are two points $x, y \in U$ and $t \in T$ such that $d(tx, ty) > c$.

If (T, X) and (T, Y) are two semiflows, their *product* is the T-semiflow on $X \times Y$, defined by $t.(x, y) = (tx, ty)$. We assume that the metric on $X \times Y$ is given by $d((x, y), (x', y')) = d(x, x') + d(y, y')$, where the same letter d denotes the metrics on X, Y and $X \times Y$.

2. Comparing (SM), (SyndT), (ST), and (TT)

Here is the first new notion that we introduce.

Definition 2.1 *A semiflow (T, X) is called strongly transitive (ST) if for any nonempty open subsets U, V of X and any compact $K \subset T$ there is a $t \in T \setminus K$ such that $tU \cap V \neq \emptyset$.*

Proposition 2.2 *For any semiflow (T, X) we have*

$$(SM) \Rightarrow (ST) \Rightarrow (TT).$$

Proof Suppose (T, X) is (SM). Let U, V be two nonempty open subsets of X. Let K be a compact subset of T. Let K' be a compact subset of T such that for every $t \in T \setminus K'$, $tU \cap V \neq \emptyset$. Then $K \cup K'$ is a compact subset of T and for every $t \in T \setminus (K \cup K') \subset T \setminus K'$ we have $tU \cap V \neq \emptyset$. Also $T \setminus (K \cup K') \subset T \setminus K$. Thus (T, X) is (ST).

Clearly (ST) implies (TT). □

Example 2.3 *An example of a (ST) semiflow that is not (SM). Let $X = \{0, 1\}$ with a discrete metric and $T = \mathbb{Z}$. Let the action of T on X be defined by $t.x = (t + x)(\mathrm{mod}\ 2)$. The only nonempty open sets in X are $\{0\}$, $\{1\}$, and $\{0, 1\}$. Let $U = \{0\}$ and $V = \{1\}$. It is clear that there is no compact subset K of $T = \mathbb{Z}$ such that for every $t \in T \setminus K$, $t.\{0\} \cap \{1\} \neq \emptyset$. Thus (T, X) is not (SM). However, for these U and V, outside of every compact $K \subset T$ there is a t (an odd number) such that $t.\{0\} \cap \{1\} \neq \emptyset$. The situation is also clear for all other choices of U and V. Thus (T, X) is (ST).*

Example 2.4 *An example of a (TT) semiflow that is not (ST). The $\mathbb{Z}$-semiflow $\mathbb{Z}$, defined by $t.x = t + x$ for every $t, x \in \mathbb{Z}$, is (TT), but not (ST).*

Here is the second new notion that we introduce.

Definition 2.5 *A topological monoid T is called directional if for every compact subset K of T there is a $t_0 \in T$ such that $(t_0 + T) \cap K = \emptyset$.*

Examples 2.6 *(i)* $\mathbb{N}_0$, $\mathbb{R}_+$.

(ii) Any nonzero submonoid of $\mathbb{N}_0$ or $\mathbb{R}_+$.

(iii) $T = \{0, 2, 3\} \cup [4, \infty)$ with the addition and the topology induced from $\mathbb{R}$.

(iv) $T = [0, 1)$ with the topology induced from $\mathbb{R}$ and the addition $s + t = \max\{s, t\}$.

(v) Any product $T = T_1 \times T_2$ of two monoids, at least one of which is directional.

Indeed, suppose that T_1 is directional (similarly if we assume that T_2 is directional). Let K be a compact subset of T. We need to show that there is $t = (t_1, t_2) \in T$ such that $(t + T) \cap K = \emptyset$. We can assume that $K = K_1 \times K_2$, where K_1 (resp. K_2) is a compact subset of T_1 (resp. T_2). There is $t_1 \in T_1$ such that $(t_1 + T_1) \cap K_1 = \emptyset$. Let t_2 be any element of T_2 and put $t = (t_1, t_2)$. We have $t + T = (t_1 + T_1) \times (t_2 + T_2)$. Any point $(k_1, k_2) \in K$ has $k_1 \in T_1 \setminus (t_1 + T_1)$, and so $(k_1, k_2) \notin (t_1 + T_1) \times (t_2 + T_2)$. Thus $(t + T) \cap K = \emptyset$.

Since one factor can be any monoid (and the product is directional if the other factor is directional), we can intuitively say that the class of directional monoids is very large.

Note that no topological group is directional.

Proposition 2.7 *Let T be a directional monoid. Then for any semiflow (T, X) in which all transition maps have dense images we have*

$$(ST) \Leftrightarrow (TT).$$

Proof Clearly (ST) implies (TT). We need to prove the converse. Suppose (T, X) is (TT). Let U, V be two nonempty open subsets of X and let K be a compact subset of T. Since T is directional, there is a $t_0 \in T$ such that $(t_0 + T) \cap K = \emptyset$. Since all transition maps have dense images, $V' = t_0^{-1}V$ is a nonempty open set. Let $t' \in T$ be such that $t'U \cap V' \neq \emptyset$. Then $(t_0 + t')U \cap V \neq \emptyset$ and $t_0 + t' \notin K$ since $t_0 + t' \in t_0 + T$. Hence (T, X) is (ST). □

Proposition 2.8 *Let $(\mathbb{N}_0, X) = (X, f)$ be a (TT) cascade. Then all transition maps $f^n : X \to X$ have dense images.*

Proof If X is a singleton, clear. Suppose that X has at least two elements. Let V be a nonempty open subset of X.

Claim. $f^{-1}(V) \neq \emptyset$.

Proof of the claim. If $V = X$, clear. Suppose $V \neq X$. Suppose to the contrary, i.e. $f^{-1}(V) = \emptyset$. Then for any $n \geq 1$ we have $f^{-n}(V) = f^{-(n-1)}(f^{-1}(V)) = f^{-(n-1)}(\emptyset) = \emptyset$. Let $x \in V$ and $y \in X \setminus \{x\}$. Let $\varepsilon > 0$ be such that $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$ and $B(x, \varepsilon) \subset V$. By (TT) there is an $n \geq 1$ such that $f^n(B(y, \varepsilon)) \cap B(x, \varepsilon) \neq \emptyset$. Hence $f^{-n}(B(x, \varepsilon)) \neq \emptyset$, in particular $f^{-n}(V) \neq \emptyset$, a contradiction. The claim is proved.

Since $f^{-1}(V)$ is a nonempty open subset of X, applying the claim to $f^{-1}(V)$ we get $f^{-2}(V) \neq \emptyset$, etc., $f^{-n}(V) \neq \emptyset$ for every $n \geq 0$. Thus all f^n, $n \geq 0$, have dense images. □

Proposition 2.9 *For any cascade $(\mathbb{N}_0, X)$, (ST) $\Leftrightarrow$ (TT).*

Proof Follows from Propositions 2.7 and 2.8. □

Remark 2.10 *If $(\mathbb{R}_+, X)$ is a semiflow on a Polish space X such that there exists $x \in X$ and $t_0 > 0$ with $\overline{[t_0, \infty).x} = \mathbb{R}$ (i.e. which has a "strictly positive dense orbit"), then by a theorem of Birkhoff, for any two nonempty open subsets U, V of X there exists $t \geq 1$ such that $tU \cap V \neq \emptyset$. This, in turn, implies that all transition maps of $(\mathbb{R}_+, X)$ have dense images.*

Proposition 2.11 *For any semiflow that has a dense set of points whose orbits have empty interior, (ST) $\Leftrightarrow$ (TT).*

Proof Clearly (ST) implies (TT). We need to prove the converse. Suppose (T, X) is (TT). Let U, V be two nonempty open subsets of X and let K be a compact subset of T. We need to show that there is a $t \in T \setminus K$ such that $tU \cap V \neq \emptyset$. Suppose to the contrary, i.e. that for every $t \in T \setminus K$ we have $tU \cap V = \emptyset$. Let $x \in U$ be a point such that Tx has an empty interior. Hence there is a point $y \in V$ that does not belong to Tx. Let $\alpha = d(y, Kx)$. Let $\varepsilon \in (0, \alpha/3)$ be such that $B(y, \varepsilon) \subset V$. Every point of X is a point of equicontinuity of the compact set K; in particular x is such a point. Hence (for ε) there is a $\delta > 0$ such that $B(x, \delta) \subset U$ and for any $x_1 \in X$, $d(x_1, x) < \delta$ implies $d(tx_1, tx) < \varepsilon$ for all $t \in K$. Hence no element of K maps anything from $B(x, \delta)$ to $B(y, \varepsilon)$. Also, by the assumption, since $B(x, \delta) \subset U$ and $B(y, \varepsilon) \subset V$, there is no element of $T \setminus K$ that maps any element of $B(x, \delta)$ to $B(y, \varepsilon)$. This contradicts (TT) of (T, X). □

Here is the third new notion that we introduce.

Definition 2.12 *A topological monoid T is called a psp monoid if it satisfies the following piecewise syndetic property: no piecewise syndetic subset of T is relatively compact.*

Proposition 2.13 *If T is a directional monoid, then it is psp.*

Proof Suppose to the contrary. Let $A \subset T$ be a piecewise syndetic set contained in a compact $K \subset T$. Let $t_0 \in T$ be such that $(t_0 + T) \cap K = \emptyset$. Let $K' \subset T$ be a compact and $B \subset T$ a thick set such that $(t + K') \cap A \neq \emptyset$ for every $t \in B$. If $t \in t_0 + T$, then $t = t_0 + t'$ for some $t' \in T$; hence $t + K' = t_0 + t' + K' \subset t_0 + T$, and so $(t + K') \cap A = \emptyset$. Hence $(t + K') \cap A \neq \emptyset$ implies $t \in T \setminus (t_0 + T)$. Hence $B \subset T \setminus (t_0 + T)$. Hence $T \setminus (t_0 + T)$ is thick. So $t_0 + T$ is not syndetic. However, $t_0 + T$ is syndetic (with a corresponding compact $\{t_0\}$), a contradiction. □

We can intuitively say that the class of psp monoids is very large since it includes the class of directional monoids (and this one is already very large, as we observed earlier).

Examples 2.14 *Here are some examples of psp monoids.*

(i) All directional monoids (by Proposition 2.13).

(ii) $\mathbb{Z}^n$, $\mathbb{R}^n$ $(n \geq 1)$.

(iii) Let $T = [0, 1)$ with the metric induced from $\mathbb{R}$ and the operation $s + t = \max\{s, t\}$. Then $A \subset T$ is thick if and only if A contains a sequence $t_1 < t_2 < t_3 < \ldots$ of elements of T that converges to 1. Also $A \subset T$ is piecewise syndetic if there is a compact $K \subset T$ and a thick subset $B \subset T$ such that every translate $t + K$, $t \in B$, intersects A. It follows that A is piecewise syndetic if and only if $A \supset B \cap [t, 1)$ for some thick subset B of T and some $t \in T$. Hence A is piecewise syndetic if and only if A is thick. Hence T is psp.

However, not all monoids are psp. For example, let $T = [0, 1] \cap \mathbb{Q}$ with the metric induced from $\mathbb{R}$ and the operation $s + t = \max\{s, t\}$. Then $A = \{1\}$ is a piecewise syndetic subset of T, which is compact.

In the next lemma we show that in the definition of a thick subset A of T the compacts can be translated by the elements of A. We follow [7, Theorem 4.47].

Lemma 2.15 *A subset A of a monoid T is thick if and only if for every compact $K \subset T$ there is an element $a \in A$ such that $a + K \subset A$.*

Proof Let s be an element of T. By definition there is a $t \in T$ such that $t + ((s+K) \cup \{s\}) \subset A$. Hence $t + s + K \subset A$ and $t + s \in A$. We now put $a = t + s$. □

Lemma 2.16 *A subset A of a monoid T is piecewise syndetic if and only if there is a compact $K \subset T$ such that $\cup_{k\in K}(-k+A)$ is thick.*

Proof Let A be piecewise syndetic. Then there is a compact set $K \subset T$ and a thick set $B \subset T$ such that for every $b \in B$ there is a $k \in K$ with $k + b \in A$. Hence for every $b \in B$ there is a $k \in K$ such that $b \in -k + A$. Thus $\cup_{k\in K}(-k+A) \supset B$ and so $\cup_{k\in K}(-k+A)$ is thick.

Conversely, suppose that there is a compact K such that $\cup_{k\in K}(-k+A)$ is thick. Denote this set by B. Hence for every $b \in B$ there is a $k \in K$ such that $b+k \in A$, or, in other words, for every $b \in B$, $(b+K) \cap A \neq \emptyset$. □

In the next lemma we characterize piecewise syndetic subsets of a monoid. We follow [7, Theorem 4.49].

Lemma 2.17 *A subset A of a monoid T is piecewise syndetic if and only if $A = B \cap C$ with $B \subset T$ syndetic and $C \subset T$ thick.*

Proof ($\Leftarrow$) Let $A = B \cap C$ with $B \subset T$ syndetic and $C \subset T$ thick. Let K be a corresponding compact for B. Hence $T = \cup_{k\in K}(-k+B)$. We claim that $\cup_{k\in K}(-k+A)$ is thick (so that, by Lemma 2.16, A is piecewise syndetic). Let K' be a compact subset of T. We need to show that some translate of K' is contained in $\cup_{k\in K}(-k+A)$. Let $t \in T$ be such that $t + K + K' \subset C$. We will show that $t + K' \subset \cup_{k\in K}(-k+A)$. To see that, let k' be any element of K'. Let $k \in K$ be such that $t + k' \in (-k+B)$. Then $k + k' + t \in B \cap C = A$.

($\Rightarrow$) Suppose A is piecewise syndetic. Then, by Lemma 2.16, there is a compact subset $K \subset T$ such that $\cup_{k\in K}(-k+A)$ is thick. Let $C = A \cup (\cup_{k\in K}(-k+A))$ and let $B = A \cup (T \setminus C)$. Clearly C is thick and $A = B \cap C$, and so it is sufficient to show that B is syndetic. Suppose to the contrary. Then $T \setminus B$ is thick; hence, by Lemma 2.15, there is a $t \in T \setminus B$ such that $t + K \subset T \setminus B$. Now $T \setminus B = C \setminus A \subset \cup_{k\in K}(-k+A)$, and so we can pick a $k \in K$ such that $t + k \in A$. Then $t + k \in B$, a contradiction. □

Proposition 2.18 *Let T be a psp monoid, $A \subset T$ a syndetic, and $K \subset T$ a compact subset of T. Then $A \setminus K$ is a syndetic subset of T.*

Proof Without loss of generality we can assume that $K \subset A$. We want to show that $A \setminus K$ is syndetic, or, equivalently, that $T \setminus (A \setminus K) = (T \setminus A) \cup K$ is not thick. Suppose to the contrary, i.e. that $(T \setminus A) \cup K$ is thick. Then $K = [(T \setminus A) \cup K] \cap A$ is an intersection of a thick and a syndetic set, and so, by Lemma 2.17, it is a piecewise syndetic set, contradicting to T being psp. □

Proposition 2.19 *Suppose T is a psp monoid. Then for any semiflow (T, X) we have*

$$\text{(SM)} \Rightarrow \text{(SyndT)} \Rightarrow \text{(ST)} \Rightarrow \text{(TT)}.$$

Proof Suppose (T, X) is (SM). Let U, V be two nonempty open subsets of X. Let K be a compact subset of T such that for every $t \in T \setminus K$, $tU \cap V \neq \emptyset$. Since T is psp, by Proposition 2.18 the set $T \setminus K$ is syndetic. Hence (T, X) is (SyndT).

Suppose (T, X) is (SyndT). Let U, V be two nonempty open subsets of X. Let K be a compact subset of T. The set $A = \{t \in T \mid tU \cap V \neq \emptyset\}$ is syndetic. Since T is psp, A is not contained in K. Any $t \in A \cap (T \setminus K)$ satisfies $tU \cap V \neq \emptyset$, and so (T, X) is (ST).

Clearly (ST) implies (TT). □

3. (TT) on the product

Proposition 3.1 *If $(T, X \times Y)$ is (TT), then (T, X) and (T, Y) are both (TT).*

Proof Let U_1, U_2 be arbitrary nonempty open sets in X and V_1, V_2 in Y. Since $(T, X \times Y)$ is (TT), there is a $t \in T$ such that $t.(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$. Hence $tU_1 \cap U_2 \neq \emptyset$ and $tV_1 \cap V_2 \neq \emptyset$. Hence both (T, X) and (T, Y) are (TT). □

Example 3.2 *The converse of the previous proposition is not true. Consider the $\mathbb{Z}$-semiflow $\mathbb{Z}$, defined by $t.n = t + n$ for every $t, n \in \mathbb{Z}$. This semiflow is (TT); however, its product with itself is not (TT) since for $U = \{(0, 1)\}$ and $V = \{(0, 2)\}$ there is no $t \in \mathbb{Z}$ such that $tU \cap V \neq \emptyset$. Thus some stronger conditions should be imposed on the factors in order to imply (TT) of the product.*

Proposition 3.3 *If (T, X) is (SM) and (T, Y) is (ST), then $(T, X \times Y)$ is (TT).*

Proof Let $U_1 \times V_1$ and $U_2 \times V_2$ be two nonempty open sets in $X \times Y$. Since (T, X) is (SM), there is a compact $K \subset T$ such that for every $t \in T \setminus K$, $tU_1 \cap U_2 \neq \emptyset$. Since (T, Y) is (ST), there is a $t_0 \in T \setminus K$ such that $t_0 V_1 \cap V_2 \neq \emptyset$. Hence $t_0(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$ and so $(T, X \times Y)$ is (TT). □

Corollary 3.4 *Let (T, X) be a (SM) and (T, Y) a (TT) semiflow and suppose that at least one of the following two conditions holds:*

(i) T is directed and all transition maps of (T, Y) have dense images;

(ii) Y has a dense set of points whose orbits in (T, Y) have empty interior.

Then $(T, X \times Y)$ is (TT).

Proof Using Proposition 2.7 in the case (i) and Proposition 2.11 in the case (ii) we conclude that (T, Y) is (ST). Now the statement follows from Proposition 3.3. □

Corollary 3.5 ([9]) *If $(\mathbb{N}_0, X)$ is (SM) and $(\mathbb{N}_0, Y)$ is (TT), then $(\mathbb{N}_0, X \times Y)$ is (TT).*

Proof Follows from Corollary 3.4 and Proposition 2.8. □

Example 3.6 *An example where (T, X) is (SM) and (T, Y) is (TT) but $(T, X \times Y)$ is not (TT). Let $(\mathbb{Z}, X)$ be any (SM) semiflow such that there are two nonempty open subsets U_1, U_2 of X and $k \in \mathbb{Z}$ with $kU_1 \cap U_2 = \emptyset$. Let $Y = \mathbb{Z}$ and let a $\mathbb{Z}$-semiflow Y be defined by $t.n = t + n$ for any $t, n \in \mathbb{Z}$. Let $V_1 = \{0\}$ and $V_2 = \{k\}$. Then the only element $t \in T = \mathbb{Z}$ such that $tV_1 \cap V_2 \neq \emptyset$ is $t = k$. Now consider the open subsets $U_1 \times V_1$ and $U_2 \times V_2$. There is no $t \in T$ such that $t(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$. Hence $(T, X \times Y)$ is not (TT).*

Example 3.7 *An example of two (ST) semiflows $(T,X),(T,Y)$ such that $(T,X\times Y)$ is not (TT). Let $(T,X),(T,Y)$ both be the $\mathbb{Z}$-semiflows on the discrete spaces $X=Y=\{0,1\}$, defined by $t.x=t+x\,(\mathrm{mod}\ 2)$, $t\in T=\mathbb{Z}$, $x\in X=Y$. We know from before (Example 2.3) that $(T,X),(T,Y)$ are (ST) (but not (SM)). Consider the T-semiflow $X\times Y$. Let $U=\{(0,0)\}$, $V=\{(0,1)\}$ be two open subsets of $X\times Y$. Then there is no $t\in T$ such that $tU\cap V\neq\emptyset$, as only even t's map 0 to 0 and only odd t's map 0 to 1. Hence $(T,X\times Y)$ is not (TT).*

Definition 3.8 *A semiflow (T,X) is said to have (DPP) if X has a dense subset consisting of periodic points.*

Definition 3.9 *A semiflow (T,X) is said to have the Touhey property (TP) if for any nonempty open subsets U,V of X there is a periodic point $x\in U$ and $t\in T$ such that $tx\in V$.*

This property was introduced in the paper [13] by Touhey.

Note that

$$\text{(TP)} \Leftrightarrow \text{(DPP)} + \text{(TT)}.$$

Proposition 3.10 *Suppose T is a psp monoid. Then if (T,X) is (SM) and (T,Y) has (TP), $(T,X\times Y)$ is (TT).*

Proof Let $U_1\times V_1$ and $U_2\times V_2$ be two nonempty open subsets of $X\times Y$. There is a compact subset K of T such that for every $t\in T\setminus K$, $tU_1\cap U_2\neq\emptyset$. There is a periodic point y_0 in V_1 and $t_0\in T$ such that $t_0y_0\in V_2$. Let $S=\mathrm{Fix}(y_0)=\{t\in T\,|\,ty_0=y_0\}$. S is a syndetic subset of T; hence t_0+S is also syndetic. Hence t_0+S is not contained in K (since T is psp). Let $t_0+s_0\in T\setminus K$. Then $(t_0+s_0)y_0\in V_2$, and so $(t_0+s_0)V_1\cap V_2\neq\emptyset$. Since $t_0+s_0\notin K$, $(t_0+s_0)U_1\cap U_2\neq\emptyset$. Hence $(t_0+s_0)(U_1\times V_1)\cap(U_2\times V_2)\neq\emptyset$. Hence $(T,X\times Y)$ is (TT). □

4. (SM) and (SyndT) on the product

Proposition 4.1 *Let T be a topological monoid and $(T,X),(T,Y)$ two semiflows. The product $(T,X\times Y)$ is (SM) if and only if each of $(T,X),(T,Y)$ is (SM).*

Proof Let $U_1,U_2\subset X$ and $V_1,V_2\subset Y$ be nonempty open sets. Let K be a compact subset of T such that for every $t\in T\setminus K$, $t(U_1\times U_2)\cap(U_2\times V_2)\neq\emptyset$. Hence for every $t\in T\setminus K$ we have $tU_1\cap U_2\neq\emptyset$ and $tV_1\cap V_2\neq\emptyset$. Hence (T,X) and (T,Y) are both (SM).

Let $U_1,U_2\subset X$, $V_1,V_2\subset Y$ be nonempty open sets. There is a compact K_1 such that for every $t\in T\setminus K_1$ we have $tU_1\cap U_2\neq\emptyset$. There is a compact K_2 such that for every $t\in T\setminus K_2$ we have $tV_1\cap V_2\neq\emptyset$. Then $K_1\cup K_2$ is a compact with the property that for every $t\in T\setminus(K_1\cup K_2)$ we have $t(U_1\times V_1)\cap(U_2\times V_2)\neq\emptyset$. Hence $(T,X\times Y)$ is (SM). □

Proposition 4.2 *Suppose T is a psp monoid. Let (T,X) be (SM) and (T,Y) (SyndT). Then $(T,X\times Y)$ is (SyndT).*

Proof Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ be nonempty open sets. Let $K \subset T$ be a compact such that for every $t \in T \setminus K$, $tU_1 \cap U_2 \neq \emptyset$. Let S be a syndetic subset of T such that for every $s \in S$, $sV_1 \cap V_2 \neq \emptyset$. Since T is psp, then by Proposition 2.18, $S \setminus K$ is syndetic and all the elements of $S \setminus K$ are outside of K. Hence for every $s \in S \setminus K$ we have $s(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$. Thus $(T, X \times Y)$ is (SyndT). □

5. (DPP) on the product

Proposition 5.1 *Let T be a monoid and $(T,X),(T,Y)$ two semiflows. If $(T, X \times Y)$ has (DPP), then each of $(T,X),(T,Y)$ has (DPP).*

Proof Let $U \subset X$, $V \subset Y$ be nonempty open sets. Then there is a periodic point $(x,y) \in U \times V$ of $(T, X \times Y)$. Hence $\mathrm{Fix}(x,y)$ is syndetic. However, $\mathrm{Fix}(x) \supset \mathrm{Fix}(x,y)$ and $\mathrm{Fix}(y) \supset \mathrm{Fix}(x,y)$, and so both of these sets are syndetic. Hence $x \in U$ is periodic and $y \in V$ is periodic. Hence each of $(T,X),(T,Y)$ has (DPP). □

Here is the fourth new notion that we introduce.

Definition 5.2 *A monoid T is said to have the syndetic intersection property, or that it is a sip monoid if the intersection of any two syndetic submonoids of T is a syndetic submonoid of T.*

Examples 5.3 *Here are some examples of sip monoids. The monoids in the first three examples are also psp.*

(i) Submonoids of $(\mathbb{N}_0, +)$.

(ii) Submonoids of $(\mathbb{Z}, +)$ (note that a submonoid of $\mathbb{Z}$ that contains at least one positive and at least one negative element must be a subgroup of $\mathbb{Z}$).

(iii) $T = [0,1)$ with the metric induced from $\mathbb{R}$ and the operation $s + t = \max\{s,t\}$.

(iv) $T = [0,1] \cap \mathbb{Q}$ with the metric induced from $\mathbb{R}$ and the operation $s + t = \max\{s,t\}$.

Proposition 5.4 *Let T be a sip monoid. If $(T,X),(T,Y)$ have (DPP), then $(T, X \times Y)$ has (DPP).*

Proof Let $U \times V$ be a nonempty open subset of $X \times Y$. Then there are periodic points $x \in U$ and $y \in V$. Let $S_1 = \mathrm{Fix}(x)$ and $S_2 = \mathrm{Fix}(y)$. These are two syndetic submonoids of T; hence $S_1 \cap S_2$ is a syndetic submonoid of T. Since $\mathrm{Fix}(x,y) = S_1 \cap S_2$, $\mathrm{Fix}(x,y)$ is a syndetic submonoid of T. Hence (x,y) is a periodic point of $(T, X \times Y)$, contained in $U \times V$. Hence $(T, X \times Y)$ has (DPP). □

Corollary 5.5 ([3]) *Let $(\mathbb{N}_0, X)$ and $(\mathbb{N}_0, Y)$ be two cascades.Then $(\mathbb{N}_0, X \times Y)$ has (DPP) if and only if $(\mathbb{N}_0, X)$ and $(\mathbb{N}_0, Y)$ have (DPP).*

Proof Follows from Propositions 5.1 and 5.4. □

6. (WM) on the product

Proposition 6.1 *Let T be a monoid and $(T,X),(T,Y)$ two semiflows. If $(T, X \times Y)$ is (WM), then (T,X) and (T,Y) are (WM).*

Proof Let $U_1, U_2, U_3, U_4 \subset X$ and $V_1, V_2, V_3, V_4 \subset Y$ be nonempty open sets. Since $(T, X \times Y)$ is (WM), there is a $t \in T$ such that $t(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$ and $t(U_3 \times V_3) \cap (U_4 \times V_4) \neq \emptyset$. Hence $tU_1 \cap U_2 \neq \emptyset$ and $tU_3 \cap U_4 \neq \emptyset$, and so (T, X) is (WM). Also $tV_1 \cap V_2 \neq \emptyset$ and $tV_3 \cap V_4 \neq \emptyset$, and so (T, Y) is (WM). □

Definition 6.2 *A monoid T is called a C monoid if for every $t \in T$ the set $T \setminus (t + T)$ is relatively compact in T.*

This type of monoids was introduced in the paper [8] by Kontorovich and Megrelishvili.

Examples 6.3 *(i)* $\mathbb{N}_0$, $\mathbb{R}_+$.

(ii) Topological groups.

Proposition 6.4 *Let T be a C monoid and (T, X) a (NMIN) (WM) semiflow. Then all transition maps of (T, X) have dense images.*

Proof Let V be a nonempty open subset of X. Suppose there is a $t_0 \in T$ such that $t_0 X \cap V = \emptyset$. Then $(t_0 + T)X \cap V = \emptyset$ since $(t_0 + t)^{-1}V = t^{-1}(t_0^{-1}V) = t^{-1}\emptyset = \emptyset$ for any $t \in T$. Let $x \in X$ be a point such that $\overline{Tx} \neq X$ (it exists since (T, X) is (NMIN)) and let $y \in X \setminus \overline{Tx}$. Let $\alpha = d(\overline{Tx}, y)$. The set $R = T \setminus (t_0 + T)$, being relatively compact since T is a C monoid, acts equicontinuously on x, and so there is a $\delta > 0$ such that the set $RB(x, \delta)$ is contained in the $\alpha/3$-neighborhood of $\overline{Tx}$ (which is the set of all points of X whose distance from $\overline{Tx}$ is $< \alpha/3$). This implies that $y \notin V$, otherwise, choosing $\varepsilon \in (0, \alpha/3)$ such that $B(y, \varepsilon) \subset V$, we would have that no element of T maps any element of $B(x, \delta)$ to $B(y, \varepsilon)$, contradicting to (TT) of (T, X) (which is implied by (WM)). Let $z \in V$. Let $\varepsilon \in (0, \alpha/3)$ be such that $B(z, \varepsilon) \cap B(y, \varepsilon) = \emptyset$ and $B(z, \varepsilon) \subset V$. Because of the (WM), there is a $t \in T$ such that

$$tB(x, \delta) \cap B(y, \varepsilon) \neq \emptyset,$$
$$tB(x, \delta) \cap B(z, \varepsilon) \neq \emptyset.$$

Now $t \notin t_0 + T$ since those elements do not map anything to V, in particular to $B(z, \varepsilon)$. Also $t \notin R$ since those elements do not map anything from $B(x, \delta)$ to $B(y, \varepsilon)$. We got a contradiction. Hence all transition maps have dense images. □

Remark 6.5 *An intuitive explanation of this proposition is this: for a (NMIN) semiflow to achieve (WM) it is necessary that all transition maps have dense images. This works for C monoids as acting monoids since, in a certain sense, they are small monoids. Big monoids, however, can have enough transition maps to achieve (WM) even when the semiflow is (NMIN) and not all transition maps have dense images.*

Proposition 6.6 *Let T be a directional monoid. Suppose (T, X) is (SM), (T, Y) is (WM), and all transition maps of (T, Y) have dense images. Then $(T, X \times Y)$ is (WM).*

Proof Let $U_1 \times V_1, U_2 \times V_2, U_3 \times V_3, U_4 \times V_4$ be nonempty open subsets of $X \times Y$. We want to show that there is a $t \in T$ such that $t(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$ and $t(U_3 \times V_3) \cap (U_4 \times V_4) \neq \emptyset$. There is a compact subset K of T such that for every $t \in T \setminus K$ we have $tU_1 \cap U_2 \neq \emptyset$ and $tU_3 \cap U_4 \neq \emptyset$. Let $t_0 \in T$ be such that $(t_0 + T) \cap K = \emptyset$. Since all transition maps of (T, Y) have dense images, the open sets $t_0^{-1}V_3, t_0^{-1}V_4$ are nonempty. Since (T, Y)

is (WM), there is a $t \in T$ such that $tV_1 \cap t_0^{-1}V_3 \neq \emptyset$ and $tV_2 \cap t_0^{-1}V_4 \neq \emptyset$. Consider now the element $t + t_0$ of T. The set $(t + t_0)V_1 = t_0(tV_1)$ has a nonempty intersection with $t_0(t_0^{-1}V_3)$, and hence with V_3. The set $(t + t_0)V_2 = t_0(tV_2)$ has a nonempty intersection with $t_0(t_0^{-1}V_4)$, and hence with V_4. Since $t + t_0$ is not in K, we have $(t+t_0)(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$, $(t+t_0)(U_3 \times V_3) \cap (U_4 \times V_4) \neq \emptyset$. Hence $(T, X \times Y)$ is (WM). □

Corollary 6.7 ([9]) *If* $(\mathbb{N}_0, X)$ *is (SM) and* $(\mathbb{N}_0, Y)$ *is (WM), then* $(\mathbb{N}_0, X \times Y)$ *is (WM).*

Proof Follows from Propositions 6.6 and 2.8. □

Corollary 6.8 *Let* T *be a directional C monoid. Suppose that* (T, X) *is (SM) and* (T, Y) *is (NMIN) (WM). Then* $(T, X \times Y)$ *is (WM).*

Proof Follows from Propositions 6.6 and 6.4. □

7. (S) on the product

Proposition 7.1 *Let* T *be a monoid and* $(T, X), (T, Y)$ *two semiflows. The product* $(T, X \times Y)$ *is (S) if and only if at least one of* $(T, X), (T, Y)$ *is (S).*

Proof Suppose $(T, X \times Y)$ is sensitive with a sensitivity constant c. Let $U \subset X$ and $V \subset Y$ be two nonempty open sets. Then there are points $(x_1, y_1), (x_2, y_2) \in U \times V$ and $t \in T$ such that $d(t(x_1, y_1), t(x_2, y_2)) > c$. Hence $d(tx_1, tx_2) + d(ty_1, ty_2) > c$. Hence either $d(tx_1, tx_2) > c/2$ or $d(ty_1, ty_2) > c/2$. Hence at least one of the semiflows $(T, X), (T, Y)$ is sensitive with a sensitivity constant $c/2$.

Conversely, suppose (T, X) is sensitive with a sensitivity constant c. Let $U \times V$ be a nonempty open set in $X \times Y$. There are points $x_1, x_2 \in U$ and $t \in T$ such that $d(tx_1, tx_2) > c$. Let y be any point in V. Then $d(t(x_1, y), t(x_2, y)) = d((tx_1, ty), (tx_2, y)) = d(tx_1, tx_2) + d(ty, ty) > c$. Hence $(T, X \times Y)$ is sensitive with a sensitivity constant c. □

Example 7.2 *Let* (T, X) *be the semiflow defined in Example 4.3 in our paper [11]. It is sensitive and so* $(T, X \times X)$ *is also sensitive according to Proposition 7.1. This gives an example where our proposition works and the acting monoid is none of the monoids* $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}_+$, $\mathbb{R}$.

8. Devaney chaos on the product

Definition 8.1 *A semiflow* (T, X) *is called Devaney chaotic if it is (NMIN), (TT) and has (DPP).*

Equivalently, a semiflow is Devaney chaotic if it is (TT), has (DPP), and is (S) (see the papers [1, 5, 11], where it is shown that, when (NMIN) holds, (S) follows from (TT)+(DPP)).

Note that (NMIN)+(TT)+(DPP) is the same as (NMIN)+(TP).

The notion of Devaney chaos was introduced in [4] by Devaney.

Proposition 8.2 *Let* T *be a psp monoid. Let* $(T, X), (T, Y)$ *be two semiflows. Suppose that:*

(i) (T, X) *is (SM);*

(ii) (T, Y) *is (TT);*

(iii) $(T, X \times Y)$ *has (DPP);*

(iv) at least one of (T, X), (T, Y) *is (NMIN).*

Then $(T, X \times Y)$ *is Devaney chaotic.*

Proof Because of (iv), $(T, X \times Y)$ is (NMIN). By Proposition 3.10, $(T, X \times Y)$ is (TT). Also $(T, X \times Y)$ is assumed to have (DPP). Hence $(T, X \times Y)$ is Devaney chaotic. □

Corollary 8.3 ([9]) *Let* $(\mathbb{R}_+, X)$ *and* $(\mathbb{R}_+, Y)$ *be two semiflows. Suppose that:*

(i) $(\mathbb{R}_+, X)$ *is (SM);*

(ii) $(\mathbb{R}_+, Y)$ *is Devaney chaotic;*

(iii) $(\mathbb{R}_+, X \times Y)$ *has (DPP).*

Then $(\mathbb{R}_+, X \times Y)$ *is Devaney chaotic.*

Proof Follows immediately from Proposition 8.2. □

Corollary 8.4 *Let* T *be a sip psp monoid. Let* $(T, X), (T, Y)$ *be two semiflows. Suppose that:*

(i) (T, X) *is (SM) and has (DPP);*

(ii) (T, Y) *has (TP);*

(iii) at least one of (T, X), (T, Y) *is (NMIN).*

Then $(T, X \times Y)$ *is Devaney chaotic.*

Proof Since (T, X) and (T, Y) have (DPP) and T is sip; then, by Proposition 5.4, $(T, X \times Y)$ has (DPP). Now the statement follows from Proposition 8.2. □

Corollary 8.5 ([3]) *Let* $(\mathbb{N}_0, X), (\mathbb{N}_0, Y)$ *be two cascades. Suppose that:*

(i) $(\mathbb{N}_0, X)$ *is (SM) and Devaney chaotic;*

(ii) $(\mathbb{N}_0, Y)$ *has (TP).*

Then $(\mathbb{N}_0, X \times Y)$ *is Devaney chaotic.*

Proof Follows from Corollary 8.4. □

Example 8.6 *This is a simple example of two semiflows* (T, X) *and* (T, Y) *in which the acting monoid is neither* $\mathbb{N}_0$ *(as in [3]) nor* $\mathbb{R}_+$ *(as in [9]) and where we can deduce the Devaney chaoticity of the product* $(T, X \times Y)$ *using our Corollary 8.4. Suppose that* $(\mathbb{N}_0, X)$ *is (SM) and has (DPP), and that* $(\mathbb{N}_0, Y)$ *has (TP). Also suppose that at least one of* $(\mathbb{N}_0, X)$, $(\mathbb{N}_0, Y)$ *is (NMIN). Any pair of semiflows* $(\mathbb{N}_0, X)$, $(\mathbb{N}_0, Y)$ *that satisfy [3, Theorem 3] (i.e. the above Corollary 8.5) would work. Let now* $T = \mathbb{N}_0 \setminus \{1\}$. *This is a submonoid of* $\mathbb{N}_0$ *and so we can consider the restricted semiflows* (T, X) *and* (T, Y). *We will show that* $(T, X \times Y)$ *is Devaney chaotic by applying Corollary 8.4, i.e. by checking that all the conditions from Corollary 8.4 are satisfied.*

Indeed, since $(\mathbb{N}_0, X)$ *is (SM), then* (T, X) *is clearly (SM). Let* $x \in X$ *be a periodic point in* $(\mathbb{N}_0, X)$. *We show that* x *is then a periodic point in* (T, X) *too. Let* S *be the fixer* $Fix_{\mathbb{N}_0}(x)$ *of* x *in* $(\mathbb{N}_0, X)$. *The set*

$S' = S\setminus\{1\}$ is the fixer $Fix_T(x)$ of x in (T,X). Let us show the syndeticity of S' in T. Let $K = \{0,1,\dots,n_0\}$ be a corresponding compact of S in $\mathbb{N}_0$. We show that $K' = \{0,2,3,4,\dots,n_0,n_0+1,\dots,2n_0+1\}$ is a corresponding compact of S' in T. Indeed, for any $t \in T$ the translate $t+K'$ contains the disjoint union $(t+K)\cup((t+n_0+1)+K)$. Each of the disjoint sets $t+K$ and $(t+n_0+1)+K$ intersects S, and hence one of them has to intersect S'. Thus $(t+K')\cap S' \neq \emptyset$. Hence x is periodic in (T,X). It follows that (DPP) of $(\mathbb{N}_0,X)$ implies (DPP) of (T,X). Let us show that (T,Y) has (TP). Let U,V be two nonempty open subsets of Y. There is a point $y \in U$, periodic in $(\mathbb{N}_0,Y)$, and $t \in \mathbb{N}_0$ such that $ty \in V$. The point y is also periodic in (T,Y) by what we have shown above. Since every element of $t + Fix_{\mathbb{N}_0}(y)$ maps y to V and $Fix_{\mathbb{N}_0}(y)$ is a syndetic submonoid of $\mathbb{N}_0$, there is an element of T (in $t+Fix_{\mathbb{N}_0}(y)$) that maps y to V. Thus (T,Y) has (TP).

Next we check that T is psp. Indeed, let A be a piecewise syndetic subset of T. It is enough to show that A is not bounded. By Lemma 2.17, $A = B\cap C$, where B is a syndetic and C is a thick subset of T. Suppose A is bounded. Let $N \in \mathbb{N}_0$ be such that $a \leq N$ for every $a \in A$. Let $K = \{0,2,3,\dots,n_0\}$ be a corresponding compact for B in T. Let $K' = \{0,2,3,\dots,N+n_0+1\}$. There is a translate $t+K'$ of K' ($t \in T$) contained in C. It contains an element of B in the interval $\{t+N+1, t+N+3, t+N+4, \dots, t+N+n_0+1\} = t+N+1+K \subset t+K'$. Hence it contains an element of A, bigger than N, a contradiction. Hence A is not bounded and so not relatively compact in T. Finally we check that T is sip. Indeed, let S_1 and S_2 be two syndetic submonoids of T. Let $a \in S_1\setminus\{0\}$ and $b \in S_2\setminus\{0\}$. Then $ab \in S_1\cap S_2$ (since they are monoids). Hence $\{0,ab,2ab,3ab,\dots\} \subset S_1\cap S_2$ and so $S_1\cap S_2$ is a syndetic submonoid of T.

All the conditions of Corollary 8.4 are checked and so the product $(T, X\times Y)$ is Devaney chaotic.

9. Conclusion

We investigated some chaos-related properties on the product of two semiflows for the case of very general acting monoids. Among the results that we obtained is a sufficient condition for the Devaney chaoticity of the product for the psp acting monoids (Proposition 8.2), generalizing the results from [3] and [9]. The importance of studying very general acting monoids was discussed in the Introduction. Studying products of any structures (in particular semiflows) is a natural course of action since they are a way of obtaining more complex objects from the basic ones. Also some dynamical properties are defined or characterized in terms of products, like, for example, weak mixing or scattering (see, for example, [2, Proposition 4.1]). Here are some natural topics for further research, related to our manuscript.

Question 9.1 *Is it possible to improve Proposition 3.10 (which is used in the proof of Proposition 8.2)? For example, can one weaken the assumption of (SM) for (T,X), or the assumption (TP) for (T,Y)? One way to weaken (TP) for (T,Y) would be to replace it by (TT)+(DAP), where (DAP) means the density of almost periodic points. (A point $x \in X$ is almost periodic in (T,X) if for every neighborhood U of x the set $D(x,U) = \{t \in T \mid tx \in U\}$ is syndetic.)*

Question 9.2 *The following question is raised in [9] for the case of the acting monoids $\mathbb{N}_0$ and $\mathbb{R}_+$: if $(T, X\times Y)$ is (SyndS) (resp. (CofinS); (MultiS)), is one of the factors necessarily (SyndS) (resp. (CofinS); (MultiS))? Here are the definitions. A semiflow (T,X) is syndetically sensitive (SyndS) if there exists $c>0$ such that for every nonempty open set $U \subset X$ the set $D(U,c) = \{t \in T \mid d(tx,ty) > c \text{ for some } x,y \in U\}$ is syndetic. A subset A of T is cofinal if $A \supset T\setminus K$ for some compact subset K of T. A semiflow (T,X) is* *cofinally sensitive (CofinS)) if there exists* $c > 0$ *such that for every nonempty open set* $U \subset X$ *the set* $D(U, c)$ *is cofinal. A semiflow* (T, X) *is multisensitive (MultiS) if there exists* $c > 0$ *such that for any nonempty open subsets* $U_1, \ldots, U_n$ *of* X *the set* $\cap_{i=1}^{n} D(U_i, c)$ *is nonempty.*

Question 9.3 *In the paper [15] the chaos-related properties are investigated on finite products of cascades with arbitrarily many factors and on countable infinite products of cascades. The sufficient conditions are obtained for the products to be at the same time Devaney chaotic and strongly mixing. It would be of interest to find sufficient conditions for the Devaney chaoticity of the products of semiflows (with an arbitrary finite or an infinite number of factors) for some large class of acting monoids, but without the products being at the same time strongly mixing, or necessarily having any other additional property.*

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