Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2017) 41: 1337 – 1343
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doi:10.3906/mat-1609-12

On subdirectly irreducible regular bands

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Received: 06.09.2016 • **Accepted/Published Online:** 28.12.2016 • **Final Version:** 28.09.2017

Abstract: Subdirectly irreducible regular bands whose structural semilattices are finite chains are characterized in terms of a refined semilattice of semigroups.

Key words: Subdirectly irreducible semigroups, regular bands, refined semilattices of semigroups

1. Introduction and preliminaries

Every nontrivial semigroup is a subdirect product of some subdirectly irreducible semigroups. Therefore, it is certainly of great importance to describe kinds of subdirectly irreducible semigroups. It is known that a nontrivial semigroup S is subdirectly irreducible if and only if S contains the least nontrivial congruence. In [1], Gerhard gave a representation of subdirectly irreducible bands (also called idempotent semigroups) in terms of transformations. However, it is not easy to construct an arbitrary subdirectly irreducible band according to Gerhard's representation. In this paper, we give a construction for a special kind of subdirectly irreducible regular bands, whose structural semilattices form finite chains, by using a structure theorem of regular bands. To date, we are not able to give a construction of a general subdirectly irreducible regular band.

First we introduce some notations and concepts. Let X be a nonempty set. Then we write the identity relation on X as ε_X and write the universal relation on X as ω_X. If X is a partially ordered set, then for any $x, y \in X$, x is said to *immediately cover* y if whenever $x \geq z \geq y$, one has $x = z$ or $y = z$ for any $z \in X$, written as $x \succ y$. Let A, B be nonempty sets. Usually, a mapping from A to the power set 2^B of B (the set of all subsets of B) is called *a set-valued mapping from A to B*. Let ρ be an equivalence on B. *A relational mapping ξ from A to B over ρ* (see [5]) is a set-valued mapping from A to B such that

$$(\forall a \in A \;\&\; \forall b \in B) \;\; |a\xi \cap b\rho| = 1,$$

denoted by $\xi : A \stackrel{\rho}{\longrightarrow} 2^B$. Note that a relational mapping $\xi : A \stackrel{\omega_B}{\longrightarrow} 2^B$ is a usual mapping from A to B if one does not differentiate a singleton set $\{x\}$ from the element x itself. For $\xi : A \stackrel{\rho}{\longrightarrow} 2^B$ and $b \in B$, by $\xi^b : A \to b\rho$ (or $A \to B$), we represent the usual mapping that maps a in A to the unique element in the set $a\xi \cap b\rho$. Obviously,

$$(\forall b_1, b_2 \in B) \;\; b_1 \,\rho\, b_2 \Longleftrightarrow \xi^{b_1} = \xi^{b_2} \Longleftrightarrow (\exists a \in A) a\xi^{b_1} = a\xi^{b_2}.$$

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2010 *AMS Mathematics Subject Classification:* 20M17
Partially supported by the National Natural Science Foundation of China (11501467) and the Fundamental Research Funds for the Central Universities (XDJK2016B038).

Let S be a semigroup. If I is an ideal of S, then we write the Rees congruence determined by I, i.e. $\omega_I \cup \varepsilon_S$, as ρ_I. We denote the lattice of all congruences on S by $\mathcal{C}(S)$. For notation and terminology not explained in this paper, the reader is referred to [2, 3].

Now we give some results required in the next section.

Lemma 1.1 *[[4, Theorem 3.6]] Semigroups S and S^1 are simultaneously subdirectly irreducible or reducible.*

Note that all subdirectly irreducible bands with 2 elements can be easily listed. Let B be a band with $|B| > 2$. If B contains an identity 1_B, then it is clear that $C = B - \{1_B\}$ is also a band. Suppose that B is subdirectly irreducible. From the proof of the above lemma given in [4], one can see that σ is the least nontrivial congruence on C if and only if $\sigma \cup \{(1_B, 1_B)\}$ is the least nontrivial congruence on B. Thus, C contains no identity. Otherwise, assume that 1_C is the identity of C. It is clear that $\omega_{\{1_B,1_C\}} \cup \varepsilon_B \in \mathcal{C}(B)$; hence, B has no least nontrivial congruence. Thus, by Theorem 1.1 in [1], we have the following:

Remark 1.2 *To characterize all subdirectly irreducible bands, it suffices to characterize those subdirectly irreducible bands with neither identity nor zero.*

Lemma 1.3 *Let $B = (Y, B_\alpha)$ be a subdirectly irreducible band. Then Y contains a zero. Moreover, let B contain no zero, 0 be the zero of Y, and ρ be the least nontrivial congruence on B. Then $\rho \subseteq \rho_{B_0}$.*

Proof Let ρ be the least nontrivial congruence on B. Then there exist $a \in B_\alpha$ and $b \in B_\beta$ with $a \neq b$ such that $(a, b) \in \rho$. Suppose that Y does not contain a zero. Then there exists $\gamma \in Y$ satisfying that $\gamma \not\leq \alpha\beta$. Thus, $I = \bigcup_{\gamma \not\geq \alpha\beta} B_\gamma \neq \phi$. It is obvious that I is an ideal of B and $\rho_I \neq \varepsilon_B$. However, $\rho \nsubseteq \rho_I$, contradicting the fact that ρ is the least nontrivial congruence on B. Therefore, Y contains a zero. Now the second statement is clear since B_0 is a nonzero ideal of B. □

One can see from the following lemma that, to describe all subdirectly irreducible regular bands, it suffices to investigate subdirectly irreducible left (or right) regular ones.

Lemma 1.4 *[[3, Proposition V.1.3]] A band is regular if and only if it is a subdirect product of a left regular band and a right regular band.*

Combining the definition of a relational mapping and the revised definition of a refined semilattice of semigroups in [5] or [6] with Theorem 3.6 in [7], we have the following:

Lemma 1.5 *Let Y be a semilattice and $\{L_\alpha : \alpha \in Y\}$ be a family of pairwise disjoint left zero semigroups. For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\rho_{\alpha,\beta}$ be an equivalence on L_β and $\phi_{\alpha,\beta} : L_\alpha \xrightarrow{\rho_{\alpha,\beta}} 2^{L_\beta}$ be a relational mapping. Suppose also that:*

(1) *for any $\alpha \in Y$, $\rho_{\alpha,\alpha} = \omega_{L_\alpha}$ and $\phi_{\alpha,\alpha}$ is the identity mapping on L_α;*

(2) *for any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$ and for any $a \in L_\alpha$, $b \in L_\beta$ and $c \in L_\gamma$,*

$$\rho_{\alpha,\gamma} \subseteq \rho_{\beta,\gamma}, \tag{1.1}$$

$$a\phi_{\alpha,\beta}\phi_{\beta,\gamma} \subseteq a\phi_{\alpha,\gamma}, \tag{1.2}$$

$$(\exists c' \in L_\gamma)\ \ (b\rho_{\alpha,\beta})\phi^c_{\beta,\gamma} \subseteq c'\rho_{\alpha,\gamma}; \tag{1.3}$$

(3) *for any* $\alpha, \beta, \gamma \in Y$ *with* $\alpha\beta \geq \gamma$ *and for any* $a \in L_\alpha$ *and* $c \in L_\gamma$,

$$(\exists c' \in L_\gamma)\ \ a\phi_{\alpha,\gamma} \cap c\rho_{\alpha\beta,\gamma} \subseteq c'\rho_{\beta,\gamma}.$$

Define an operation $\circ$ *on* $L = \bigcup_{\alpha\in Y} L_\alpha$ *by*

$$a \circ b = (a\phi^x_{\alpha,\alpha\beta})(b\phi^y_{\beta,\alpha\beta}) \quad (a \in L_\alpha, b \in L_\beta), \tag{1.4}$$

where $b\phi_{\beta,\alpha\beta} \subseteq x\rho_{\alpha,\alpha\beta}$ *and* $a\phi_{\alpha,\alpha\beta} \subseteq y\rho_{\beta,\alpha\beta}$. *Then* $(L, \circ)$ *is a left regular band, denoted by* $L = [Y; L_\alpha, \rho_{\alpha,\beta}, \phi_{\alpha,\beta}]$. *Conversely, every left regular band can be so constructed.*

Corollary 1.6 *Let* $L = [Y; L_\alpha, \rho_{\alpha,\beta}, \phi_{\alpha,\beta}]$ *be a left regular band. Then for any* $\alpha, \beta \in Y$ *with* $\alpha \geq \beta$ *and for any* $a \in L_\alpha$ *and* $b \in L_\beta$, $ab = a\phi^b_{\alpha,\beta}$.

Proof Noticing that L_β is a left zero semigroup, we can directly obtain the lemma from the definition of a relational mapping and (1.4). □

2. Main result and proof

The aim of this section is to describe a kind of subdirectly irreducible left regular bands whose structural semilattices are finite chains, but most lemmas in this section are suitable for more general subdirectly irreducible bands. Noticing Remark 1.2 and Lemmas 1.3 and 1.5, we suppose that, *in the following lemmas,* $L = [Y; L_\alpha, \rho_{\alpha,\beta}, \phi_{\alpha,\beta}]$ *always represents a subdirectly irreducible left regular band without zero where* 0 *is the zero of* Y.

Lemma 2.1 *[[4, Theorem 4.7]] For any* $u, v \in L$, $ux = vx$ *for all* $x \in L_0$ *implies that* $u = v$.

Lemma 2.2 *For any* $\alpha \in Y - \{0\}$, $\rho_{\alpha,0} \neq \omega_{L_0}$.

Proof Suppose that there exists $\alpha_1 \in Y - \{0\}$ such that $\rho_{\alpha_1,0} = \omega_{L_0}$. Then $\phi_{\alpha_1,0}$ is a usual mapping from L_{α_1} to L_0. Thus, for any $a \in L_{\alpha_1}$ and $u \in L_0$, we see from Corollary 1.6 that

$$au = a\phi_{\alpha_1,0} = (a\phi_{\alpha_1,0})u.$$

It follows from Lemma 2.1 that $a = a\phi_{\alpha_1,0}$, a contradiction. □

Lemma 2.3 *For any* $\alpha \in Y$, $\rho_{\alpha,0} = \varepsilon_{L_0}$ *implies that* $|\bigcup_{\delta\geq\alpha} L_\delta| = 1$.

Proof Arbitrarily take $\alpha \in Y$ and suppose that $\rho_{\alpha,0} = \varepsilon_{L_0}$. It follows from (1.1) that for any $\delta \geq \alpha$, $\rho_{\delta,0} = \varepsilon_{L_0}$. Thus, for any $x \in L_{\delta_1}, y \in L_{\delta_2}$ with $\delta_1, \delta_2 \geq \alpha$, we see from Corollary 1.6 that

$$xu = x\phi^u_{\delta_1,0} = u = y\phi^u_{\delta_2,0} = yu$$

for all $u \in L_0$. It follows from Lemma 2.1 that $x = y$. Therefore, $|\bigcup_{\delta\geq\alpha} L_\delta| = 1$. □

Lemma 2.4 *Let Y be a chain. Then $\rho_{\alpha,0} = \varepsilon_{L_0}$ for some $\alpha \in Y$ implies that L_α contains the identity of L.*

Proof By Lemma 2.3 we may set $L_\alpha = \{a\}$. Now we prove that a is the identity of L. Note that Y is a chain. For any $u \in L_0$ and $b, c \in S_\beta$ with $(b, c) \in \rho_{\alpha,\beta}$, we see from (1.3) that $(b\phi^u_{\beta,0}, c\phi^u_{\beta,0}) \in \rho_{\alpha,0}$. Hence, $b\phi^u_{\beta,0} = c\phi^u_{\beta,0}$ since $\rho_{\alpha,0} = \varepsilon_{L_0}$. Thus, we see from Corollary 1.6 that

$$bu = b\phi^u_{\beta,0} = c\phi^u_{\beta,0} = cu.$$

It follows from Lemma 2.1 that $b = c$, which means that $\rho_{\alpha,\beta} = \varepsilon_{L_\beta}$. Thus, for any $x \in L_\beta$, again we see from Corollary 1.6 that

$$ax = a\phi^x_{\alpha,\beta} = x,$$

as required. □

Lemma 2.5 *Let Y be a chain. Then for any $\alpha \in Y$ and $a \in L_0$, $\rho = \omega_{a\rho_{\alpha,0}} \cup \varepsilon_L \in \mathcal{C}(L)$.*

Proof Obviously, ρ is an equivalence on L. Note that L_0 is a left zero semigroup. To show that $\rho \in \mathcal{C}(L)$, it suffices to verify that ρ is left compatible. Now suppose that $u, v \in a\rho_{\alpha,0}$ and $c \in L$ with $c \in L_\delta$ and $\delta \in Y$. It follows from Corollary 1.6 that

$$cu = c\phi^u_{\delta,0}, \quad cv = c\phi^v_{\delta,0}$$

and from the definition of a relational mapping that

$$(c\phi^u_{\delta,0}, u), \quad (c\phi^v_{\delta,0}, v) \in \rho_{\delta,0}.$$

If $\delta \geq \alpha$, then we obtain from (1.1) that $c\phi^u_{\delta,0}, c\phi^v_{\delta,0} \in a\rho_{\alpha,0}$. If $\delta < \alpha$, then it follows from (1.1) that $(u, v) \in \rho_{\delta,0}$. Thus, we see from the definition of a relational mapping that $cu = cv$. Therefore, $\rho \in \mathcal{C}(L)$. □

Lemma 2.6 *Let Y be a chain. Then for any $\alpha \in Y$, there exists $N_\alpha \subseteq L_0$ such that $\rho_{\alpha,0} = \omega_{N_\alpha} \cup \varepsilon_{L_0}$.*

Proof Assume that there exists $a, b \in L_0$ such that both $|a\rho_{\alpha,0}| > 1$ and $|b\rho_{\alpha,0}| > 1$. Then it follows from Lemma 2.5 that both $\eta_1 = \omega_{a\rho_{\alpha,0}} \cup \varepsilon_L$ and $\eta_2 = \omega_{b\rho_{\alpha,0}} \cup \varepsilon_L$ are nontrivial congruence on L while $\eta_1 \cap \eta_2 = \varepsilon_L$, a contradiction. □

Lemma 2.7 *Let Y be a chain. Then for any $\alpha, \beta \in Y$ with $\alpha \gneq \beta$, $N_\alpha \subsetneq N_\beta$.*

Proof First it follows from Lemma 2.6 and (1.1) that $N_\alpha \subseteq N_\beta$. Suppose that $N_\alpha = N_\beta$. Then we have $\rho_{\alpha,0} = \rho_{\beta,0}$. Pick $a \in L_\alpha$ and $b \in L_\beta$. For any $u \in N_\alpha = N_\beta$, noticing from (1.2) that $a\phi_{\alpha,\beta}\phi_{\beta,0} \subseteq a\phi_{\alpha,0}$, we see from Corollary 1.6 and the definition of a relational mapping that

$$(a\phi^b_{\alpha,\beta})u = a\phi^b_{\alpha,\beta}\phi^u_{\beta,0} = a\phi^u_{\alpha,0} = au.$$

For any $u \in L_0 - N_\alpha$, $au = u = (a\phi^b_{\alpha,\beta})u$, so we have $au = (a\phi^b_{\alpha,\beta})u$ for all $u \in L_0$. It follows from Lemma 2.1 that $a = a\phi^b_{\alpha,\beta}$, which leads to $\alpha = \beta$, a contradiction. □

Lemma 2.8 *Let Y be a chain. Then for any $\alpha \in Y$ and $a \in N_\alpha$, $\phi^a_{\alpha,0}$ is injective.*

Proof Arbitrarily take $x, y \in L_\alpha$. Suppose that $x\phi^a_{\alpha,0} = y\phi^a_{\alpha,0}$. Then for any $u \in L_0$, if $u \in N_\alpha$, then we obtain from Corollary 1.6 and the definition of a relational mapping that

$$xu = x\phi^u_{\alpha,0} = x\phi^a_{\alpha,0} = y\phi^a_{\alpha,0} = y\phi^u_{\alpha,0} = yu;$$

if $u \notin N_\alpha$, then we see from Corollary 1.6 and Lemma 2.6 that $xu = yu = u$. It follows from Lemma 2.1 that $x = y$. Hence, $\phi^a_{\alpha,0}$ is injective. □

Lemma 2.9 *Let Y be a chain. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, there exists $N_{\alpha,\beta} \subseteq L_\beta$ such that $\rho_{\alpha,\beta} = \omega_{N_{\alpha,\beta}} \cup \varepsilon_{L_\beta}$.*

Proof Suppose that there exist $x, y \in L_\beta$ such that $|x\rho_{\alpha,\beta}| > 1$, $|y\rho_{\alpha,\beta}| > 1$ but $(x, y) \notin \rho_{\alpha,\beta}$. Pick $a \in N_\alpha$. It follows from (1.3) and Lemmas 2.6 and 2.8 that

$$x\rho_{\alpha,\beta}\phi^a_{\beta,0}, y\rho_{\alpha,\beta}\phi^a_{\beta,0} \subseteq N_\alpha.$$

Hence, taking $u \in L_\alpha$, we have $u\phi^x_{\alpha,\beta}\phi^a_{\beta,0}, u\phi^y_{\alpha,\beta}\phi^a_{\beta,0} \in N_\alpha$. By (1.2) and the definition of a relational mapping, we see that

$$u\phi^x_{\alpha,\beta}\phi^a_{\beta,0} = u\phi^y_{\alpha,\beta}\phi^a_{\beta,0},$$

contradicting Lemma 2.8 since $u\phi^x_{\alpha,\beta} \neq u\phi^y_{\alpha,\beta}$. □

Lemma 2.10 *Let Y be a chain. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $|N_{\alpha,\beta}| > 1$, and for any $x \in N_{\alpha,\beta}$, $\phi^x_{\alpha,\beta}$ is injective.*

Proof Suppose that there exist distinct $u, v \in L_\alpha$ such that $u\phi^x_{\alpha,\beta} = v\phi^x_{\alpha,\beta}$. Noticing $|N_{\alpha,\beta}| > 1$, we see from (1.3) and Lemmas 2.6 and 2.8 that $N_{\alpha,\beta}\phi^a_{\beta,0} \subseteq N_\alpha$ so that $u\phi^x_{\alpha,\beta}\phi^a_{\beta,0} = v\phi^x_{\alpha,\beta}\phi^a_{\beta,0} \in N_\alpha$, where $a \in N_\alpha$. It follows from (1.2) and the definition of a relational mapping that $u\phi^a_{\alpha,0} = v\phi^a_{\alpha,0}$, contradicting Lemma 2.8. □

Lemma 2.11 *Let L contain no identity and Y be a chain. Then for any $\alpha, \beta \in Y$ with $\alpha \succ \beta$, $|N_\beta - N_\alpha| = 1$.*

Proof Suppose that there exist distinct $x, y \in N_\beta - N_\alpha$. We claim that $\eta = \omega_{\{x,y\}} \cup \varepsilon_L \in \mathcal{C}(L)$. In fact, it suffices to verify that η is left compatible. Arbitrarily take $u \in L$ with $u \in L_\delta$ and $\delta \in Y$. If $\delta \geq \alpha$, then we see from Corollary 1.6, (1.1), and Lemma 2.6 that $(ux, uy) = (x, y) \in \eta$. If $\delta < \alpha$, then again we see from Lemma 1.6, (1.1), and Lemma 2.6 that $ux = uy$ since $x, y \in N_\delta$. Therefore, $\eta \in \mathcal{C}(L) - \{\varepsilon_L\}$. Note that L contains no identity. It follows from Lemmas 2.6 and 2.4 that $|N_\alpha| > 1$, so we obtain from Lemma 2.5 that $\rho = \omega_{N_\alpha} \cup \varepsilon_L \in \mathcal{C}(L) - \{\varepsilon_L\}$. However, $\rho \cap \eta = \varepsilon_L$, contradicting the fact that L is subdirectly irreducible. □

Lemma 2.12 *Let L contain no identity and Y be a chain. Then there exist $a_1, b_1 \in L_0$ such that $\bigcap_{\alpha \in Y} N_\alpha = \{a_1, b_1\}$. Moreover, $\omega_{\{a_1,b_1\}} \cup \varepsilon_L$ is the least nontrivial congruence on L.*

Proof Note that L contains no identity. We see from Lemmas 2.4 and 2.6 that for any $\alpha \in Y$, $|N_\alpha| > 1$. By Lemma 2.5, we have $|\bigcap_{\alpha \in Y} N_\alpha| > 1$ since

$$\bigcap_{\alpha \in Y} (\omega_{N_\alpha} \cup \varepsilon_L) = \omega_{\bigcap_{\alpha \in Y} N_\alpha} \cup \varepsilon_L \neq \varepsilon_L.$$

If $N = \bigcap_{\alpha \in Y} N_\alpha$ contains at least three elements, say, $a, b, c \in N$, then we now can easily verify that both $\eta_1 = \omega_{\{a,b\}} \cup \varepsilon_L$ and $\eta_2 = \omega_{\{b,c\}} \cup \varepsilon_L$ are congruences on L. However, $\eta_1 \cap \eta_2 = \varepsilon_L$, contradicting the fact that L is subdirectly irreducible.

The second part is now trivial. □

Lemma 2.13 *Let L contain no identity and Y be a finite chain. Then for any $\alpha \in Y$, $|L_\alpha| > 1$.*

Proof Suppose that $L_\alpha = \{u\}$. Obviously, $\alpha \neq 0$, since otherwise u is the zero element of L. Picking $\beta \in Y$ such that $\alpha \succ \beta$, set $N_\beta - N_\alpha = \{c\}$ according to Lemma 2.11 and let $u\phi^{a_1}_{\alpha,0} = d$. Clearly, $c \neq d$. We claim that $\rho = \omega_{\{c,d\}} \cup \varepsilon_L \in \mathcal{C}(L)$. To see this, we only need to verify that for any $x \in L$, $(xc, xd) \in \rho$. We may suppose that $x \in L_\delta$ with $\delta \in Y$. If $\delta \geq \alpha$, then no matter whether d belongs to N_δ or not, from (1.1), (1.2), Corollary 1.6, and the definition of a relational mapping, we always have $xd = x\phi^d_{\delta,0} = d$. Moreover, $xc = c$. If $\delta < \alpha$, we see from Lemma 2.6, (1.1), and the definition of a relational mapping that $xc = xd$. Hence, ρ is a congruence on L. However, this contradicts Lemma 2.12. □

Lemma 2.14 *Let L contain no identity and Y be a finite chain. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, $\rho_{\alpha,\beta} \neq \varepsilon_{L_\beta}$.*

Proof Assume that $\rho_{\alpha,\beta} = \varepsilon_{L_\beta}$ and note Lemma 2.13. Since Y is finite, we may suppose that for any $\delta, \eta \in Y$ with $\alpha \geq \delta \gneq \eta \geq \beta$,

$$\{\delta, \eta\} \neq \{\alpha, \beta\} \Rightarrow \rho_{\delta,\eta} \neq \varepsilon_{L_\eta}.$$

Given that there exists $\zeta \in Y$ such that $\alpha \gneq \zeta \gneq \beta$, we have $\rho_{\alpha,\zeta} \neq \varepsilon_{L_\zeta}$ and $\rho_{\zeta,\beta} \neq \varepsilon_{L_\beta}$. If $x, y \in L_\zeta$, $x \neq y$, $(x, y) \in \rho_{\alpha,\zeta}$ and $u \in N_{\zeta,\beta}$, then we obtain $(x\phi^u_{\zeta,\beta}, y\phi^u_{\zeta,\beta}) \in \rho_{\alpha,\beta}$ from (1.3) and $x\phi^u_{\zeta,\beta} \neq y\phi^u_{\zeta,\beta}$ from Lemma 2.10. However, it follows from $\rho_{\alpha,\beta} = \varepsilon_{L_\beta}$ that $x\phi^u_{\zeta,\beta} = y\phi^u_{\zeta,\beta}$, a contradiction. Hence, we conclude that $\alpha \succ \beta$. Now we obtain from Lemmas 2.6 and 2.12 that $L_\beta\phi^{a_1}_{\beta,0} \subseteq N_\beta$. Noticing that $\rho_{\alpha,\beta} = \varepsilon_{L_\beta}$, for any $c \in L_\beta$, we have $L_\alpha\phi^c_{\alpha,\beta}\phi^{a_1}_{\beta,0} = \{c\phi^{a_1}_{\beta,0}\}$. Thus, we know from (1.2) and Lemmas 2.8 and 2.13 that $c\phi^{a_1}_{\beta,0} \notin N_\alpha$. Therefore, we obtain that $L_\beta\phi^{a_1}_{\beta,0} \cap N_\alpha = \emptyset$. Moreover, note from Lemmas 2.8 and 2.13 that $|L_\beta\phi^{a_1}_{\beta,0}| > 1$. Hence, we have $|N_\beta - N_\alpha| > 1$, contradicting Lemma 2.7. □

Lemma 2.15 *Let L contain no identity and Y be a finite chain. If there exists $u \in L_\alpha$ such that $u\phi^{a_1}_{\alpha,0} = a_1$, then for any $\beta \in Y$ with $\beta \leq \alpha$, there exists $v \in L_\beta$ such that $v\phi^{a_1}_{\beta,0} = a_1$.*

Proof Noticing (1.2), (1.3), and Lemmas 2.8, 2.9, and 2.14, we only need to choose $v = u\phi^x_{\alpha,\beta}$, where $x \in N_{\alpha,\beta}$. □

Now we present the main result.

Theorem 2.16 *Let $L = [Y; L_\alpha, \rho_{\alpha,\beta}, \phi_{\alpha,\beta}]$ be a left regular band with neither zero nor identity, where Y is a finite chain with zero 0 and identity ι. Then L is subdirectly irreducible if and only if the following statements hold:*

(a) *for all $\alpha \in Y$, there exists $N_\alpha \subseteq L_0$ such that $\rho_{\alpha,0} = \omega_{N_\alpha} \cup \varepsilon_{L_0}$;*

(b) *there exist $a_1, b_1 \in L_0$ such that $N_\iota = \{a_1, b_1\}$;*

(c) *for all $\alpha \in Y$, $\phi_{\alpha,0}^{a_1}$ is injective;*

(d) *for all $\alpha, \beta \in Y$, $\alpha \succ \beta$ implies that $|N_\beta - N_\alpha| = 1$;*

(e) *for all $\alpha \in Y$, there exist $u, v \in L_\alpha$ such that $u\phi_{\alpha,0}^{a_1} = a_1, v\phi_{\alpha,0}^{b_1} = b_1$.*

Proof *Necessity.* We only need to prove (e) since we already have Lemmas 2.6, 2.8, and 2.11 and one can directly deduce (b) from Lemma 2.12. It follows from Lemmas 2.8 and 2.13 that $|L_\iota| = 2$, so $\phi_{\iota,0}^{a_1}$ is a bijection. Thus, there must exist $u, v \in L_\iota$ such that $u\phi_{\iota,0}^{a_1} = a_1$ and $v\phi_{\iota,0}^{b_1} = b_1$. Now we obtain (e) from Lemma 2.15.

Sufficiency. One can easily verify that $\rho_0 = \omega_{\{a_1,b_1\}} \cup \varepsilon_L \in \mathcal{C}(L)$ by noticing Lemma 1.5. Arbitrarily take $\sigma \in \mathcal{C}(L) - \varepsilon_L$. Then there exist distinct $x, y \in L$, say, $x \in L_\alpha$ and $y \in L_\beta$, with $(x, y) \in \sigma$. If $\alpha = \beta$, then we obtain from (c) that

$$xa_1 = x\phi_{\alpha,0}^{a_1} \neq y\phi_{\alpha,0}^{a_1} = ya_1.$$

If $\alpha \neq \beta$, then we may suppose that $\alpha > \beta$. If $y\phi_{\beta,0}^{a_1} \in N_\beta - N_\alpha$, then

$$xa_1 = x\phi_{\alpha,0}^{a_1} \neq y\phi_{\beta,0}^{a_1} = ya_1$$

since $x\phi_{\alpha,0}^{a_1} \in N_\alpha$. If $y\phi_{\beta,0}^{a_1} \in N_\alpha$, then taking $z \in N_\beta - N_\alpha$, we have

$$xz = z \neq y\phi_{\beta,0}^{a_1} = yz.$$

Thus, we see that there always exist $s, t \in L_0$ such that $s \neq t$ and $(s, t) \in \sigma$. If $\{s, t\} = \{a_1, b_1\}$, then we have $\rho_0 \subseteq \sigma$. If $\{s, t\} \neq \{a_1, b_1\}$, then we obtain from (d) that there exists $\delta \in Y$ such that exactly one of s, t, say s, lies in N_δ but the other does not. According to (e), there exist $u, v \in L_\delta$ such that $u\phi_{\delta,0}^{a_1} = a_1, v\phi_{\delta,0}^{b_1} = b_1$. Thus, we have

$$(us, ut) = (a_1, t),\ (vs, vt) = (b_1, t).$$

Then $(a_1, b_1) \in \sigma$, which again leads to $\rho_0 \subseteq \sigma$, so we obtain that ρ_0 is the least nontrivial congruence on L and hence L is subdirectly irreducible. □

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