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Tetravalent normal edge-transitive Cayley graphs on a certain group of order $6n$

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Abstract: Let $U_{6n} = \langle a, b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$ be a group of order 6n. In this paper tetravalent normal edge-transitive Cayley graphs on U_{6n} are considered. In this way several nonequivalent normal edge-transitive Cayley graphs on U_{6n} are obtained whose automorphism groups are given exactly.

Key words: Cayley graph, automorphism group, normal edge-transitive Cayley graph, tetravalent graph

1. Introduction

Let $\Gamma = (V, E)$ be a simple graph with vertex set V and edge set E, where no loops or multiple edges are allowed in Γ. If u and v are two distinct vertices of Γ, then the edge joining u to v is denoted by $e = \{u, v\}$. If σ is a permutation on V preserving edges of Γ, then σ is called an automorphism of Γ. The set of all automorphisms of Γ forms a group under composition of mappings, which is denoted by $\mathbb{A}ut\,(\Gamma)$. The group of automorphisms of Γ is denoted by $A = \mathbb{A}ut\,(\Gamma)$, and Γ is called vertex- or edge-transitive if A acts transitively on the set of vertices or edges of Γ, respectively. An arc of Γ is an ordered pair (u, v) of vertices of Γ and Γ is called arc-transitive if A acts transitively on the set of all arcs of Γ.

Let G be a finite group and S be an inverse closed subset of G, i.e. $S = S^{-1}$, such that $1 \notin S$. The Cayley graph of G on S, denoted by $Cay(G, S)$, is a graph with vertex set G where distinct vertices $x, y \in G$ are joined by an edge iff there is $s \in S$ such that $y = sx$. It is easy to verify that $\Gamma = Cay(G, S)$ is a regular graph of valency $|S|$, and it is connected iff G is generated by S.

Let $g \in G$. We define $\rho_g : G \to G$ by $\rho_g(x) = xg$, $x \in G$. It can be verified that ρ_g is a permutation of G that preserves edges of Γ. Therefore, ρ_g is an automorphism of the Cayley graph Γ. The right regular representations of G, denoted by $R(G) = \{\rho_g | g \in G\}$, are a subgroup of $\mathbb{A}ut\,(\Gamma)$ isomorphic to G, which acts regularly on the vertices of Γ, forcing Γ to be a vertex-transitive graph.

For the Cayley graph $\Gamma = Cay(G, S)$ we define $\mathbb{A}ut(G, S) = \{\alpha \in \mathbb{A}ut(G) | \alpha(S) = S\}$, and it can be verified that it is a subgroup of $\mathbb{A}ut(\Gamma)$, which acts on $R(G)$ by $\rho_g^\sigma := \rho_{\sigma^{-1}(g)}$, where $\sigma \in \mathbb{A}ut(G, S)$ and $\rho_g \in R(G)$. Therefore, with this action the semidirect product $R(G) \rtimes \mathbb{A}ut(G, S)$ can be constructed, which is a subgroup of $A = \mathbb{A}ut(\Gamma)$. In [6] it was proved that $N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$, where $N_A(R(G))$ denotes the normalizer of $R(G)$ in A. In [12] the concept of normality of a graph was defined when $R(G)$ is a normal subgroup of A, and in this case we have $A = R(G) \rtimes \mathbb{A}ut(G, S)$.

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The normality of Cayley graphs has been the subject of research by various authors from different points of view. To study the normality of Cayley graphs it suffices to consider connected normal Cayley graphs, because in [11] all disconnected normal Cayley graphs were determined.

Another area of research is the edge-transitivity of Cayley graphs of small valency. In this respect, one of the standard problems is to study normal edge-transitive Cayley graphs. The aim is to determine the Cayley graphs of this type that have specified order and valency. It was suggested in [10] to study Cayley graphs that are not normal edge-transitive or are normal edge-transitive but are not normal. In [8] the edge-transitive tetravalent Cayley graphs on groups of square-free order were determined.

There has been much interest in studying normal edge-transitive Cayley graphs of small valencies. We will mention a few such papers. In [5] the authors determined all nonnormal Cayley digraphs of outvalency 2 of all nonabelian groups of order $2p^2$, where p is an odd prime and consequently the normal ones can be obtained.

Normal edge-transitive Cayley graphs are a rather special case of edge-transitive ones in which the full automorphism group is known. In fact, in the general case of determining vertex-transitive Cayley graphs, the special case reduces to the determination of the normal edge-transitive Cayley graphs, in which case the full automorphism group of this graph is completely known. In [1] the author found normal edge-transitive Cayley graphs of abelian groups and in [2] the same group G of order $6n$ was considered and it was shown that in general if S is a subset of G with $1 \notin S$, $S = S^{-1}$ and if $Cay(G,S)$ is normal edge-transitive, then $|S|$ is even. In [9] the author obtained interesting results concerning 2-arc transitive Cayley graphs. In [7] all the tetravalent edge-transitive Cayley graphs on the group $PSL_2(p)$ and in [3] the normal edge-transitive Cayley graphs of Frobenius group of order pq where p and q are different prime numbers were determined. In [13] tetravalent nonnormal Cayley graphs of order $4p$, p a prime number, were determined. In [4] the authors studied normal edge-transitive Cayley graphs of order $4p$ where p is an odd prime. To obtain a broader picture of normal edge-transitive graphs we initiated [4] and obtained normal edge-transitive graphs on different subsets. We were motivated by [4] to investigate normal edge-transitive Cayley graphs on a certain group of order $6n$. This family of groups of order $6n$ was chosen arbitrarily, as it presents a minor but interesting advance in the state of our understanding. For feasibility, we chose to restrict our attention to tetravalent graphs. In particular we prove the following:

Main result. Let $U_{6n} = \langle a, b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$. Then, up to isomorphism, the connected tetravalent normal edge-transitive Cayley graphs on U_{6n} are precisely those $\mathrm{Cay}(U_{6n}, S_j)$ with $S_j = \{a, a^{-1}, a^j b, a^{-j} b\}$, where $j^2 \overset{2n}{\equiv} \pm 1$.

2. Preliminary results

Let G be a group and S be a subset of G such that $1 \notin S$. The Cayley digraph $Cay(G,S)$ of G relative to S is a directed graph having G as the set of its vertices and (x, sx) as its edges, where $x \in G$ and $s \in S$. If $S = S^{-1}$, then (x, sx) is an edge if and only if (sx, x) is an edge; therefore, the edge (x, sx) is denoted by $\{x, sx\}$ and $Cay(G,S)$ is an undirected graph that is simply called a Cayley graph. In [10], the following result was proved, which gives a criterion for the normality of a Cayley graph.

Lemma 2.1 *Let $\Gamma = Cay(G,S)$ be the Cayley graph of G with respect to S and let $A = \mathbb{A}ut(\Gamma)$. Then the following hold:*

(i) $N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G,S)$.

(ii) $R(G) \trianglelefteq A$ *if and only if* $A = R(G) \rtimes \mathbb{A}ut(G,S)$.

(iii) Γ *is normal if and only if* $A_1 = \mathbb{A}ut(G,S)$, *where* A_1 *denotes the stabilizer of the vertex* 1 *under* A.

Next we set $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G,S)$ and remark that for normal edge-transitivity of $Cay(G,S)$ the group N need only be transitive on undirected edges, and may or may not be transitive on ordered pairs of adjacent vertices. From [10] we have the following result, which is useful in our further investigation.

Lemma 2.2 *Let* $\Gamma = Cay(G,S)$ *be an undirected Cayley graph of the group* G *on* S *and let* $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G,S)$. *Then the following are equivalent:*

(i) Γ *is normal edge-transitive.*

(ii) $S = T \cup T^{-1}$ *where* T *is an orbit of* $\mathbb{A}ut(G,S)$ *on* S.

(iii) There exist a subgroup H *of* $\mathbb{A}ut(G)$ *and* $g \in G$ *such that* $S = g^H \cup (g^{-1})^H$, *where* $g^H = \{gh | h \in H\}$.

3. Main result

We are going to study the Cayley graphs, and in particular tetravalent normal edge-transitive Cayley graphs, of a certain group of order $6n$ whose presentation is given as follows:

$$U_{6n} = \langle a, b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle.$$

The elements of U_{6n} can be written uniquely in the form $a^i b^j$, where $0 \leq i < 2n$, $j = 0, 1, -1$. Using the defining relations of U_{6n} it can be proved that:

$$ba^i = \begin{cases} a^i b & \text{if i is even} \\ a^i b^{-1} & \text{if i is odd,} \end{cases}$$

and hence $Z(U_{6n}) = \langle a^2 \rangle$. For the inverse of $a^i b^j$ we have:

$$\left(a^i b^j\right)^{-1} = \begin{cases} a^{-i} b^{-j} & \text{if i is even} \\ a^{-i} b^j & \text{if i is odd.} \end{cases}$$

In the Table below we give the orders of elements of U_{6n}.

Table. Order of elements in the group U_{6n}.

Type	Element	Order
I	a^i, $1 \leq i \leq 2n$	$\frac{2n}{(i,2n)}$
II	$a^i b^{\pm 1}$, $1 \leq i \leq 2n$, i odd	$2\frac{n}{(i,n)}$
III	$a^i b^{\pm 1}$, $1 \leq i \leq 2n$, i even	$3\frac{2n}{(3i,2n)}$

Lemma 3.1 $\mathbb{A}ut(U_{6n})$ *is a group of order* $6\varphi(n)$, *where* φ *denotes the Euler totient function.*

Proof Any automorphism f of U_{6n} is completely determined by its effect on a and b. Since $O(a) = 2n$ and $O(b) = 3$, the order of $f(a)$ and $f(b)$ must be $2n$ and 3, respectively.

However, using the Table, we see that elements of order $2n$ in U_{6n} are either a^i, $(i, 2n) = 1$, or $a^i b^{\pm 1}$, $(i, n) = 1$, i odd, $1 \leq i \leq 2n$, and the elements of order 3 in U_{6n} are $b^{\pm 1}$ or $a^{\pm \frac{2n}{3}}$ if n is a multiple of 3.

Since $a^{\pm\frac{2n}{3}} \in Z(U_{6n})$ and $b \notin Z(U_{6n})$, we have $f(b) \neq a^{\pm\frac{2n}{3}}$. Therefore, all automorphisms of U_{6n} have the following definition:

$$\begin{cases} f(a) = a^k \\ f(b) = b^{\pm 1} \end{cases} \quad \text{or} \quad \begin{cases} f(a) = a^k b^{\pm 1} \\ f(b) = b^{\pm 1} \end{cases}$$

where $(k, 2n) = 1$. Hence, all together we have $6\varphi(2n)$ choices for automorphisms of U_{6n}, i.e. $|\mathbb{A}ut(U_{6n})| = 6\varphi(2n)$. □

From Lemma 3.1 we observe that the general form of an automorphism of U_{6n}, which is defined on the generators a and b, should be as follows:

$$\begin{cases} f_{k,l,r}(a) = a^k b^l \\ f_{k,l,r}(b) = b^r \end{cases}$$

where $(k, 2n) = 1$, $l = 0, \pm 1$, and $r = \pm 1$.

It can be verified that $H = \{f_{k,0,1} | (k, 2n) = 1,\ 1 \leq k < 2n\}$ and $K = \{f_{1,l,r} | l = 0, \pm 1, r = \pm 1\}$ are normal subgroups of $\mathbb{A}ut(U_{6n})$ of order $\varphi(2n)$ and 6 respectively and $H \cap K = 1$. We have $H \cong \Phi_{2n}$ as the group of units of the ring $\mathbb{Z}_{2n}$, and $K \cong \mathbb{S}_3$. Therefore, $\mathbb{A}ut(U_{6n}) \cong \Phi_{2n} \times \mathbb{S}_3$.

By Lemma 2.2, if $Cay(G, S)$ is a connected normal edge-transitive graph, then $S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(G, S)$ on S. Therefore, all elements of S have the same order. From the Table it is easy to see that the only elements of order 2 in U_{6n} are a^n, $a^n b$, and $a^n b^{-1}$ when n is odd, and if n is even then U_{6n} has a unique element of order 2, namely a^n. Therefore, if $\langle S \rangle = U_{6n}$ and all elements of S have the same order, then S does not contain elements of order 2 and $|S| \geq 4$ is an even number.

Let us assume that $|S| = 4$ and each element of S has the same order. We may assume $S = \{x, y, x^{-1}, y^{-1}\}$. Obviously elements of type I can not generate U_{6n}. Suppose we have two elements of type II: $a^i b^{\pm 1}$ and $a^j b^{\pm 1}$. By the defining relations for U_{6n}, we have:

$$a^i b a^j b = a^{i+j},$$

$$a^i b a^j b^{-1} = a^{i+j} b,$$

$$a^i b^{-1} a^j b^{-1} = a^{i+j},$$

and since $i + j$ is even and $S = S^{-1}$, two elements of type II can not generate U_{6n}. Similarly, two elements of type III can not generate U_{6n}, so elements must be of two distinct types. It can proved that there are no elements of types III and II with the same order. If they come from types III and I, then it can proved that $\langle S \rangle = \langle x, y \rangle \leqslant \langle a^2, b \rangle < U_{6n}$. If x and y come from I and II with order $k < 2n$, then $\langle S \rangle = \langle x, y \rangle \leqslant \langle a^{\frac{2n}{k}}, b \rangle < U_{6n}$. Hence, elements of S are of types I and II with order $2n$. Therefore, we assume that S contains a^i and $a^j b$ (or $a^j b^{-1}$) and their inverses, where we have $(i, 2n) = (j, 2n) = 1$, $1 \leq i, j \leq 2n$, and $S = \{a^i, a^{-i}, a^j b, a^{-j} b\}$ or $S = \{a^i, a^{-i}, a^j b^{-1}, a^{-j} b^{-1}\}$. It is clear that in these cases $\langle S \rangle = U_{6n}$.

Next we define a concept that is useful in our further investigation. If S and S' are two inverse closed subsets of a group G such that $1 \notin S \cup S'$, and if there is an automorphism f of G such that $f(S) = S'$, then $Cay(G, S)$ and $Cay(G, S')$ are isomorphic graphs. In this case we call S and S' equivalent.

Proposition 3.2 *If $\langle S \rangle = U_{6n}$, $|S| = 4$, $S = S^{-1}$, and each element of S has order $2n$, then S is equivalent to $\{a, a^{-1}, a^j b, a^{-j} b\}$ where $1 \leq j \leq 2n$, $(j, 2n) = 1$.*

Proof By what we proved above S has one of the following shapes: $S_1 = \{a^i, a^{-i}, a^j b, a^{-j} b\}$ or $S_2 = \{a^i, a^{-i}, a^j b^{-1}, a^{-j} b^{-1}\}$. Now if we take $f \in \mathbb{A}ut(U_{6n})$ with the definition $f(a) = a^{-1}$, $f(b) = b^{-1}$, then $f(S_1) = S_2$, showing that S_1 is equivalent to S_2. Next let $S = \{a^i, a^{-i}, a^j b, a^{-j} b\}$. Since $(i, 2n) = 1$, i has a multiplicative inverse k in $\mathbb{Z}_{2n}$, and hence if k is its inverse, then we define φ by $\varphi(a) = a^k$, $\varphi(b) = b$, and then $\varphi(S) = \{a^{ik}, a^{-ik}, a^{jk} b, a^{-jk} b\} = \{a, a^{-1}, a^{jk} b, a^{-jk} b\}$. Since jk runs through invertible elements of $\mathbb{Z}_{2n}$ as j varies we have S equivalent to $\{a, a^{-1}, a^j b, a^{-j} b\}$ where $1 \leq j \leq 2n$, $(j, 2n) = 1$. □

Lemma 3.3 *Let $S = \{a, a^{-1}, a^j b, a^{-j} b\}$ and $T = \{a, a^{-1}, a^k b, a^{-k} b\}$ be two distinct subsets of U_{6n} consisting of elements of order $2n$. If S and T are equivalent, then $jk \overset{2n}{\equiv} \pm 1$.*

Proof Suppose that there is an $f \in \mathbb{A}ut(U_{6n})$ such that $f(S) = T$. We consider two cases:

Case (i): $f(a) = a^{\pm 1}$. In this case $f(a^j b) = a^{\pm j} f(b)$ and by Lemma 3.1, $f(b) = b^{\pm 1}$; hence, $f(a^j b) = a^{\pm j} b^{\pm 1}$. This implies that $f(S) \neq T$, which is not the case.

Case (ii): $f(a) = a^{\pm k} b$. In this case $f(a^j b) = (a^{\pm k} b)^j f(b)$. Since j is odd we have $(a^{\pm k} b)^j = a^{\pm kj} b$; hence, $f(a^j b) = a^{\pm kj} b f(b)$. Therefore, $f(b) = b^{-1}$ and $\pm jk \overset{2n}{\equiv} 1$ and the lemma is proved. □

Proposition 3.4 *Let $S = \{a, a^{-1}, a^j b, a^{-j} b\}$, $1 \leq j \leq 2n$, $(j, 2n) = 1$. If $j^2 \overset{2n}{\equiv} \pm 1$ then $\mathbb{A}ut(U_{6n}, S)$ acts transitively on S; otherwise, $\mathbb{A}ut(U_{6n}, S)$ has two orbits on S.*

Proof Note that $f_{2n-1,0,1}$ takes each element of S to its inverse, so there are at most 2 orbits on S. By the same proof as in Lemma 3.3, if there is a single orbit on S, then $j^2 \overset{2n}{\equiv} \pm 1$. Conversely, if $j^2 \overset{2n}{\equiv} \pm 1$, then f defined by $f(a) = a^j b$, $f(b) = b^{-1}$ will be an element of $Aut(U_{6n}, S)$ mapping a to $a^j b$, so the action is transitive. □

By Lemma 2.2 if $Cay(U_{6n}, S)$ is normal edge-transitive, then $S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(U_{6n}, S)$. If $j^2 \overset{2n}{\not\equiv} \pm 1$, then by Proposition 3.4 there is no $f \in \mathbb{A}ut(U_{6n}, S)$ taking a to $a^j b$; therefore, there is no T orbit of $\mathbb{A}ut(U_{6n}, S)$ such that $S = T \cup T^{-1}$. Therefore, in this case, $Cay(U_{6n}, S)$ is not normal edge-transitive. The above proof shows that the action of $f_{2n-1,0,1}$ implies that the edge-transitive action is in fact always arc-transitive on these graphs.

Theorem 3.5 *Let $S = \{a, a^{-1}, a^j b, a^{-j} b\}$, $1 \leq j \leq 2n$, $(j, 2n) = 1$.*

(a) If $j^2 \overset{2n}{\not\equiv} \pm 1$, then $Cay(U_{6n}, S)$ is not normal edge-transitive.

(b)If $j \overset{2n}{\equiv} \pm 1$, then $Cay(U_{6n}, S)$ is normal edge-transitive with automorphism group isomorphic to $U_{6n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

(c) If the multiplicative order of j in $\mathbb{Z}_2$ is 4, i.e. $j^2 \overset{2n}{\equiv} -1$, then $Cay(U_{6n}, S)$ is normal edge-transitive with automorphism group isomorphic to $U_{6n} \rtimes D_8$.

Proof Since S generates U_{6n}, the group $\mathbb{A}ut(U_{6n}, S)$ acts faithfully on S and hence $\mathbb{A}ut(U_{6n}, S)$ is isomorphic to a subgroup of $\mathbb{S}_4$. If $\mathbb{A}ut(U_{6n}, S)$ contains a permutation σ of order 3, then σ would fix an element, say α, in S, but in this case $\sigma(\alpha^{-1}) = \alpha^{-1}$ and σ cannot be a 3-cycle. Therefore, $|\mathbb{A}ut(U_{6n}, S)| \big| 8$.

By Proposition 3.4 and Lemma 2.2, case (a) is clear.

Examine the action of $Aut(U_{6n}, S)$ on S. It contains $f_{2n-1,0,1}$, which exchanges the first two elements and the last two elements of S. We have in fact seen that it is generated by this, together with f, where $f(a) = a^j b$ and $f(b) = b^{-1}$. We see that $f^2(a) = f(a^j b) = a^{j^2}$, which is a if $j \overset{2n}{\equiv} \pm 1$ and a^{-1} if $j^2 \overset{2n}{\equiv} -1$. Accordingly, if $j \overset{2n}{\equiv} \pm 1$ then f has order 2 and commutes with $f_{2n-1,0,1}$, so by examining the action on S it is easy to see that $Aut(U_{6n}, S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. On the other hand, if $j \overset{2n}{\equiv} -1$, then f has order 4 and does not commute with $f_{2n-1,0,1}$, and by examining the action on S we see that $Aut(U_{6n}, S) \cong \mathbb{D}_8$.

In the case that $Cay(U_{6n}, S)$ is normal its automorphism group is isomorphic to $U_{6n} \rtimes \mathbb{A}ut(U_{6n}, S)$, from which statements (b) and (c) follow, and the theorem is proved. $\square$

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