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The Hahn–Banach theorem for A-linear operators

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Abstract: In this short paper we present a generalization of the Hahn–Banach extension theorem for A-linear operators. Some theoretical applications and results are given.

Key words: Riesz space, positive operator, A-linear operator, Hahn–Banach extension theorem

1. Introduction

We assume the reader to be familiar with the elementary theory of Riesz spaces and order bounded operators. In this regard, we use [1, 3, 7] as sources of unexplained terminology and notation. Moreover, all Riesz spaces under consideration are assumed to be Archimedean.

We denote by $L_b(E,F)$ the class of all order bounded operators from the Riesz space E into F and by $L_b(E)$ the order bounded operators from E into itself. Recall that $\pi \in L_b(E)$ is called an orthomorphism of E if $x \perp y$ in E implies that $\pi(x) \perp y$. Orthomorphisms of E will be denoted by $Orth(E)$. The principal order ideal generated by the identity operator I in $Orth(E)$ is called the ideal center of E and is denoted by $Z(E)$. Let A be a Riesz algebra (lattice ordered algebra), i.e. A is a Riesz space that is simultaneously an associative algebra with the additional property that $a, b \in A_+$ implies that $a \cdot b \in A_+$. An f-algebra A is a Riesz algebra that satisfies the extra requirement that $a \wedge b = 0$ implies $a \cdot c \wedge b = c \cdot a \wedge b = 0$ for all $c \in A_+$. If A is an Archimedean f-algebra, then A is necessarily associative and commutative. The collections $Orth(E)$ are, with respect to composition as multiplication, Archimedean f-algebras with the identity mapping I as a unit element. Another well-known example of f-algebras is $C(K)$ of all real continuous functions on a topological space K.

Let A be an f-algebra and E be a Riesz space. E is said to be a left o-module (order module) over A if there exists a map $A \times E \to E : (a, x) \to ax$ satisfying the following:

1. E is a left module over A,
2. for each $a \in A_+$ and $x \in E_+$ we have $ax \in E_+$.

An o-module over A is called f-module if it has the following property:

3. if $x \perp y$ in E, then for each $a \in A$ we have $ax \perp y$.

If A has a unit element e,

4. $ex = x$ for each $x \in E$,

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and then E is said to be a unitary f-module. A right f-module over A is defined similarly. We shall only consider left f-modules from now on and these will simply be referred to as f-modules. If E is an f-module over A, then there exists a map $p : A \to Orth(E)$ defined by $p(a) = \pi_a$ where $\pi_a(x) = ax$ for each $x \in E$. p is a unital algebra and Riesz homomorphisms. Conversely, if there exists such a map $p : A \to Orth(E)$ then the map $(a, x) \to ax = p(a)(x)$ of $A \times E \to E$ defines an f-module structure on E over A.

Example 1 *(a) A Riesz space E is an f-module over $A = \mathbb{R}$, the f-algebra of the real numbers. The embedding $p : A \to Orth(E)$ is the identification of $\mathbb{R}$ with the subalgebra of $Orth(E)$ consisting of the multiples of the identity.*

(b) A unital f-algebra A is an f-module over itself in a natural way. The mapping $p : A \to Orth(A)$ is the well-known identification (surjective algebra and Riesz isomorphism) of $a \in A$ with the orthomorphism $p(a) = \pi_a$ where $\pi_a(b) = a \cdot b$, for each $b \in A$ [1, 7].

(c) Let E be an f-module over A. The mapping $p : A \to Orth(E)$ is a positive algebra homomorphism and hence p is a Riesz homomorphism [4].

Definition 2 *Let E and F be two modules over A. A mapping $\Gamma : E \to F$ will be called A-linear if $\Gamma(x + y) = \Gamma(x) + \Gamma(y)$ and $\Gamma(ax) = a\Gamma(x)$ for each $a \in A$ and $x, y \in E$. A mapping $q : E \to F$ will be called A-sublinear if $q(x + y) \leq q(x) + q(y)$ and $q(ax) = aq(x)$ for each $a \in A_+$ and $x, y \in E$.*

The set of order bounded and A-linear maps will be denoted by $L_b(E, F; A)$. Order properties of the space of A-linear operators were given by Turan in [5].

Example 3 *(a) Let E be a Riesz space. The bilinear map $Z(E) \times E \to E : (\pi, x) \to \pi x = \pi(x)$ shows that E is an f-module over its center. As E is assumed to be Archimedean, $Orth(E)$ is also Archimedean and therefore commutative. Thus, we have:*

$$Orth(E) \subseteq L_b(E, E; Z(E)) \subseteq L_b(E).$$

(b) Let E be a Riesz space of all continuous piecewise linear functions on $[0, 1]$. Then $Z(E) = \{\lambda I : \lambda \in \mathbb{R}\}$ and so $L_b(E, E; Z(E)) = L_b(E)$.

(c) Let E be a Riesz space. If E has the principal projection property, then $L_b(E, E; Z(E)) = Orth(E)$ [1, Theorem 8.3].

(d) Let $A = l_\infty$, $E = l_p; 1 \leq p < \infty$; then $L_b(E, E; A) = Orth(E)$.

Many versions of the Hahn–Banach theorem have been given. In this work, we shall gather an extension theorem for A-linear operators and we shall give some applications of the extension theorem.

2. Generalization of the Hahn–Banach theorem

The following theorem is a general version of what is known as the Hahn–Banach extension theorem.

Theorem 4 *Let E be a module over unital Dedekind complete f-algebra A, F an Archimedean Dedekind complete o-module over A, and $q : E \to F$ an A-sublinear map. If N is a submodule of E and $\Phi : N \to F$*

is an A-linear map satisfying $\Phi(x) \leq q(x)$ for all $x \in N$, then there exists an A-linear map Γ from E to F such that $\Gamma(x) = \Phi(x)$ for all $x \in N$ and $\Gamma(x) \leq q(x)$ for all $x \in E$.

Proof The proof of the one-step extension is different from the other proof of Hahn–Banach theorems. If this is done, then an application of Zorn's lemma guarantees an extension of Φ to all of E with the desired properties. Let N be a submodule of E. If $N = E$ holds, then the proof is trivial. Assume that $N \neq E$ and take $z \in E \backslash N$. Let N_1 be the submodule of E generated by N and z, i.e. $N_1 = \{x + az : x \in N, a \in A\}$. This representation is not unique, since it is possible that $az \in N$ for some nonzero a in A. This leads to difficulties in showing the "one-step extension". It is clear that for each $y, u \in N$ we have

$$\Phi(y) - q(y - z) \leq q(u + z) - \Phi(u).$$

This inequality couplet with the Dedekind completeness of F guarantees that both

$$k = \sup\{\Phi(y) - q(y - z) : y \in N\}$$

and

$$t = \inf\{q(u + z) - \Phi(u) : u \in N\}$$

exist in F, and they satisfy $k \leq t$. Now, for any $w \in F$ satisfying $k \leq w \leq t$ (for instance, $w = k$), let us define

$$\Gamma_1 : N_1 \to F; (x + az) \to \Gamma_1(x + az) = \Phi(x) + aw.$$

We shall now show that Γ_1 is well defined. As A is Dedekind complete, A is e-uniform complete [3, Theorem 42.6]. If $0 \leq a$ in A then $(a + \frac{1}{n}e)^{-1}$ is in A for all n ($n = 1, 2, 3, ...$) by [2, Theorem 11.1]. For all n, we have

$$w \leq t \leq q\left[(a + \frac{1}{n}e)^{-1}x + z\right] - \Phi\left[(a + \frac{1}{n}e)^{-1}x\right]$$

and it follows that

$$\Phi(x) + (a + \frac{1}{n}e)w \leq q\left[x + (a + \frac{1}{n}e)z\right] \leq q(x + az) + \frac{1}{n}eq(z).$$

As F is Archimedean, we have

$$\Phi(x) + aw \leq q(x + az).$$

For general $a \in A$, write $a = a^+ - a^-$. Let $u : A \to B_{a^+}$ be the order projection, where B_{a^+} is the band generated by a^+. We have $u \in Orth(A)$, $u^2 = u$, $u(a) = a^+$. Since A is Riesz and algebra isomorphic to $Orth(A)$ [7, Theorem 141.1], there exists $0 \leq b \in A$ such that $b^2 = b$, $ba = a^+$, $ba^+ = a^+$, $ba^- = 0$, and $(b - e)a = a^-$. From the first case we have

$$\Phi(x) + a^+w \leq q(x + a^+z)$$

and hence

$$\begin{aligned} b(\Phi(x) + a^+w) &= b\Phi(x) + b(a^+w) = b\Phi(x) + (ba^+)w = b\Phi(x) + a^+w \\ &\leq bq(x + a^+z) = q(bx + b(a^+z)) = q(bx + b^2(az)) = bq(x + az). \end{aligned}$$

Similarly, since $k \leq w$ we get

$$(e - b)(\Phi(x) - a^-w) \leq (e - b)q(x + az).$$

Adding these inequalities, we obtain

$$\Phi(x) + aw \leq q(x + az) \tag{1}$$

for all $a \in A$. We suppose $x, y \in N$ and $a, c \in A$ are such that $x + az = y + cz$. Then, from (1), we have

$$\Phi(x - y) + (a - c)w \leq q((x - y) + (a - c)z) = q(0) = 0$$

and

$$\Phi(y - x) + (c - a)w \leq q((y - x) + (c - a)z) = q(0) = 0,$$

and therefore $\Gamma_1(x+az) = \Gamma_1(y+cz)$. Evidently, Γ_1 is dominated by q and is an A-linear map. An application of Zorn's lemma guarantees an extension of Φ to all of E with the desired properties. □

Let K be an extremally disconnected compact Hausdorff space (sometimes called a Stonian space). Then $C(K)$ is a Dedekind complete f-algebra. Taking $A = C(K)$, we can get the following result that was proved by Vincent-Smith as Theorem 1 in [6].

Corollary 5 *Let E be a module over Dedekind complete $A = C(K)$ and $q : E \to A$ be an A-sublinear map. If N is a submodule of E and $\Phi : N \to A$ is an A-linear map satisfying $\Phi(x) \leq q(x)$ for all $x \in N$, then there exists some A-linear map Γ on E that extends Φ and is such that $\Gamma(x) \leq q(x)$ for all $x \in E$.*

If we take $A = \mathbb{R}$, then we obtain the following corallary that was given as Theorem 83.13 [7] or Theorem 2.1 [1].

Corollary 6 *Let E be a (real) vector space, F a Dedekind complete Riesz space, and $q : E \to F$ a sublinear map. If N is a vector subspace of E and $\Phi : N \to F$ is an operator satisfying $\Phi(x) \leq q(x)$ for all $x \in N$, then there exists some linear map Γ on E that extends Φ and is such that $\Gamma(x) \leq q(x)$ for all $x \in E$.*

It is known that there are many applications of the Hahn–Banach extension theorem. Some of these applications were given for linear positive operators in [1]. Similar applications of Theorem 1 can be given for A-linear positive operators. Two of them will be given below. Let E be a unitary module over A. Then every submodule of E is a vector subspace of E. A submodule N of a unitary module E is called a Riesz submodule whenever N is closed under the lattice operations of E (i.e. whenever N is a Riesz subspace of E).

Corollary 7 *Let E be a unitary f-module over unital Dedekind complete f-algebra A, F an Archimedean Dedekind complete o-module over A, and $\Gamma : E \to F$ a positive A-linear operator. If N is a Riesz submodule of E and $\Phi : N \to F$ is a positive A-linear map satisfying $\Phi(x) \leq \Gamma(x)$ for all $x \in N_+$, then there exists some positive A-linear map H on E that extends Φ and is such that $H(x) \leq \Gamma(x)$ for all $x \in E_+$.*

Proof It is similar to the proof of Theorem 2.2, which is given in [1]. □

Corollary 8 *Let E be a unitary f-module over unital Dedekind complete f-algebra A and F an Archimedean Dedekind complete o-module over A. If N is a Riesz submodule of E and $\Phi : N \to F$ is a positive A-linear map, then the following statements are equivalent:*

1. *Φ extends to a positive A-linear map.*
2. *Φ extends to an order bounded A-linear map.*
3. *There exists a monotone A-sublinear mapping $q : E \to F$ satisfying $\Phi(x) \leq q(x)$ for all $x \in N$.*

Proof The proof is the same as the proof of Theorem 2.3 [1] with a slight difference. □

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