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Schanuel's lemma, the snake lemma, and product and direct sum in H_v-modules

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Abstract: In this paper we find a generalization of the snake lemma and Schanuel's lemma in H_v-modules. We define the isomorph sequences and determine the conditions to split the exact sequences in H_v-modules. Some interesting results on these concepts are given.

Key words: H_v-module, snake lemma, Schanuel's lemma, product and direct sum, star homomorphism, exact sequence

1. Introduction

A couple $(H, *)$ of a nonempty set H and a mapping on $H \times H$ into the family of nonempty subsets of H is called a hyperstructure (or hypergroupoid). A hypergroup is a hyperstructure $(H, *)$ with associative law $(x * y) * z = x * (y * z)$ for every $x, y, z \in H$ and the reproduction axiom is valid: $x * H = H * x = H$ for every $x \in H$; i.e. for every $x, y \in H$ there exist $u, v \in H$ such that $y \in x * u$ and $y \in v * x$. This concept was introduced by Marty in 1934 [11]. If A and B are nonempty subsets of H then $A * B$ is given by $A * B = \bigcup_{a \in A, b \in B} a * b$. Also, $x * A$ is used for $\{x\} * A$ and $A * x$ for $A * \{x\}$. Hyperrings, hypermodules, and other hyperstructures are defined and several books have been written to date [1, 2, 8, 16]. The concept of H_v-structures as a larger class than the well-known hyperstructures was introduced by Vougiouklis at the Fourth Congress on Algebraic Hyperstructures and Applications [14] where the axioms are replaced by the weak ones; that is, instead of the equality on sets, one has nonempty intersections. The basic definitions and results of H_v-structures can be found in [3–6, 9, 10, 12, 15, 16].

The weak-equality and exact sequences in H_v-modules are defined and some results in this respect have been proved [7]. Accordingly, the present authors in [13] proved the five short lemma in H_v-modules. They also introduced $M[-]$ and $-[M]$ functors and then investigated the exactness of them and other problems. The notion of exact sequences is a fundamental concept and it has been widely used in many areas such as ring and module theory. Our aim in this paper is to introduce a generalization of some notions in homological algebra to prove the snake lemma (in H_v-modules) and Schanuel's lemma (in H_v-modules) and also determine the conditions to split a sequence (in H_v-modules); finally, some interesting results are given. We define the concepts of star homomorphism, product and direct sum, isomorph sequences, split sequence, and projective H_v-modules.

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2. Basic concepts and snake lemma

The hyperstructure $(H,*)$ is called an H_v-group if " $*$ " is weak associative: $x*(y*z)\cap(x*y)*z\neq\emptyset$ and the reproduction axiom holds: $x*H=H*x=H$ for every $x\in H$. The H_v-group H is weak commutative if for every $x,y\in H$, $x*y\cap y*x\neq\emptyset$.

A multivalued system $(R,+,\cdot)$ is an H_v-ring if $(R,+)$ is a weak commutative H_v-group, $(R,\cdot)$ is a weak associative hyperstructure where the " $\cdot$ " hyperoperation is weak distributive with respect to " $+$ "; i.e. for every $x,y,z\in R$ we have $x\cdot(y+z)\cap(x\cdot y+x\cdot z)\neq\emptyset$ and $(x+y)\cdot z\cap(x\cdot z+y\cdot z)\neq\emptyset$.

A nonempty set M is a (left) H_v-module over an H_v-ring R if $(M,+)$ is a weak commutative H_v-group and there exists a map $\cdot: R\times M\to\mathcal{P}^*(M)$ denoted by $(r,m)\mapsto rm$ such that for every $r_1,r_2\in R$ and every $m_1,m_2\in M$ we have $r_1(m_1+m_2)\cap(r_1m_1+r_1m_2)\neq\emptyset$, $(r_1+r_2)m_1\cap(r_1m_1+r_2m_1)\neq\emptyset$ and $(r_1r_2)m_1\cap r_1(r_2m_1)\neq\emptyset$. A mapping $f:M_1\longrightarrow M_2$ of H_v-modules M_1 and M_2 over an H_v-ring R is a strong homomorphism if for every $x,y\in M_1$ and every $r\in R$ we have $f(x+y)=f(x)+f(y)$ and $f(rx)=rf(x)$.

By using a certain type of equivalence relations we can connect hyperstructures to ordinary structures. The smallest of these relations are called fundamental relations and denoted by $\beta^*,\gamma^*,\varepsilon^*$. If H is an H_v-group (H_v-ring, H_v-module over an H_v-ring R) then H/β^* is a group (H/γ^* is a ring, H/ε^* is a R/γ^*-module, respectively). According to [16] the fundamental relation ε^* on an H_v-module can be defined as follows:

Consider the left H_v-module M over an H_v-ring R. If ϑ denotes the set of all expressions consisting of finite hyperoperations of either on R and M or of the external hyperoperations applying on finite sets of elements of R and M, a relation ε can be defined on M whose transitive closure is the fundamental relation ε^* so that for every $x,y\in M$; $x\ \varepsilon\ y$ if and only if $\{x,y\}\subseteq u$ for some $u\in\vartheta$; i.e.:

$$x\varepsilon y\Leftrightarrow x,y\in\sum_{i=1}^{n}m_i',\ m_i'=m_i \text{ or } m_i'=\sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}}r_{ijk})m_i,$$

where $m_i\in M,\ r_{ijk}\in R$.

Suppose that $\gamma^*(r)$ is the equivalence class containing $r\in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x\in M$. On M/ε^* the $\oplus$ and the external product $\odot$ using the γ^* classes in R are defined as follows:

For every $x,y\in M$ and for every $r\in R$,

$$\varepsilon^*(x)\oplus\varepsilon^*(y)=\varepsilon^*(c),\ \text{for every } c\in\varepsilon^*(x)+\varepsilon^*(y),$$

$$\gamma^*(r)\odot\varepsilon^*(x)=\varepsilon^*(d),\ \text{for every } d\in\gamma^*(r)\cdot\varepsilon^*(x).$$

The heart of an H_v-module M over an H_v-ring R is denoted by ω_M and defined by $\omega_M=\{x\in M\mid\varepsilon^*_M(x)=0\}$ where 0 is the unit element of the group $(M/\varepsilon^*,\oplus)$. One can prove that the unit element of the group $(M/\varepsilon^*,\oplus)$ is equal to ω_M. By the definition of ω_M we have

$$\omega_{\omega_M}=Ker(\phi:\omega_M\longrightarrow\omega_M/\varepsilon^*_{\omega_M}=0)=\omega_M.$$

Let M_1 and M_2 be two H_v-modules over an H_v-ring R and let ε_1^*, ε_2^*, and ε^* be the fundamental relations on M_1, M_2, and $M_1\times M_2$, respectively; then $(x_1,x_2)\varepsilon^*(y_1,y_2)$ if and only if $x_1\varepsilon_1^*y_1$ and $x_2\varepsilon_2^*y_2$ for all $(x_1,x_2),(y_1,y_2)\in M_1\times M_2$ [15, 16].

Weak equality (monic, epic), exact sequences, and relative results in H_v-modules are defined as follows [7]: let M be an H_v-module. The nonempty subsets X and Y of M are weakly equal if for every $x\in X$ there exists $y \in Y$ such that $\varepsilon_M^*(x) = \varepsilon_M^*(y)$ and for every $y \in Y$ there exists $x \in X$ such that $\varepsilon_M^*(x) = \varepsilon_M^*(y)$ and it is denoted by $X \overset{w}{=} Y$. The sequence $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n$ of H_v-modules and strong homomorphisms is exact if, for every $2 \leq i \leq n$, $Im(f_{i-1}) \overset{w}{=} Ker(f_i)$ where $Ker(f_i) = \{a \in M_{i-1} \mid f_i(a) \in \omega_{M_i}\}$ (that is, an H_v-submodule of M_{i-1}).

The strong homomorphism $f : M_1 \longrightarrow M_2$ is called weak-monic if for every $m_1, m_1' \in M_1$ the equality $f(m_1) = f(m_1')$ implies $\varepsilon_{M_1}^*(m_1) = \varepsilon_{M_1}^*(m_1')$ and f is called weak-epic if for every $m_2 \in M_2$ there exists $m_1 \in M_1$ such that $\varepsilon_{M_2}^*(m_2) = \varepsilon_{M_2}^*(f(m_1))$. Finally, f is called a weak-isomorphism if f is weak-monic and weak-epic.

It is easy to see that every one to one (onto) strong homomorphism is weak-monic (weak-epic), but the converse is not necessarily true. In fact, the concept of weak-monic (weak-epic) is a generalization of the concept of one to one (onto) [see the mapping f in Example 1].

Let $f : A \longrightarrow B$ be a strong homomorphism of H_v-modules over an H_v-ring R. Then we have $f(\omega_A) \subseteq \omega_B$ and so $\omega_A \subseteq Ker(f)$. Moreover, $Ker(f) = \omega_A$ if and only if f is weak-monic.

Lemma 2.1 *[13] Let A and B be H_v-modules. If $\omega_A \xrightarrow{i} A \xrightarrow{f} B$ is exact, then f is weak-monic.*

Proof It is enough to show that $Ker(f) = \omega_A$. We always have $\omega_A \subseteq Ker(f)$. On the other hand, if $a \in Ker(f)$ then there exists $a_1 \in Im(i) = \omega_A$ such that $\varepsilon_A^*(a) = \varepsilon_A^*(a_1) = \omega_A$ and so $a \in \omega_A$. Therefore, $Ker(f) = \omega_A$ and f is weak-monic. □

Now we prove the snake lemma and close this section.

Theorem 2.2 *(**Snake lemma in H_v-modules**) Let*

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & \omega_C \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \\ \omega_{A_1} & \longrightarrow & A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C_1 & & \end{array}$$

be a commutative diagram of H_v-modules and strong homomorphisms over an H_v-ring R with both exact rows. If l is weak-monic, then there exists an exact sequence as follows:

$$Ker(h) \xrightarrow{\alpha} Ker(k) \xrightarrow{\beta} Ker(l).$$

Proof First we want to define α and β. We have

$$\begin{aligned} Ker(h) &= \{a \in A \mid h(a) \in \omega_{A_1}\}, \\ Ker(k) &= \{b \in B \mid k(b) \in \omega_{B_1}\}, \\ Ker(l) &= \{c \in C \mid l(c) \in \omega_{C_1}\}. \end{aligned}$$

Now, for $a \in Ker(h)$, $f_1 \circ h(a) \in f_1(\omega_{A_1}) \subseteq \omega_{B_1}$. Since $f_1 \circ h(a) = k \circ f(a)$, we obtain $f(a) \in Ker(k)$. Also, for $b \in Ker(k)$, $g_1 \circ k(b) \in g_1(\omega_{B_1}) \subseteq \omega_{C_1}$. Since $g_1 \circ k(b) = l \circ g(b)$, we obtain $g(b) \in Ker(l)$.

We define α by $\alpha(a) = f(a)$ for every $a \in Ker(h)$ and β by $\beta(b) = g(b)$ for every $b \in Ker(k)$. Since $Ker(h)$, $Ker(k)$, and $Ker(l)$ are H_v-submodules of A, B, and C, respectively, and f, g are strong homomorphisms, it follows that α and β are strong homomorphisms.

We show that $Im(\alpha) \overset{w}{=} Ker(\beta)$. Letting $x \in Im(\alpha)$, then $x = f(a)$ for some $a \in Ker(h)(\subseteq A)$. The first row is exact, so there exists $b \in Ker(g)$ such that $\varepsilon_B^*(f(a)) = \varepsilon_B^*(b)$, where $g(b) \in \omega_C$. Since l is weak-monic we have $ker(l) = \omega_C$, but $\omega_{ker(l)} = \omega_{\omega_C} = \omega_C$ and so $\beta(b) = g(b) \in \omega_{Ker(l)}$. It is enough to show $b \in Ker(k)$. Since $\varepsilon_B^*(f(a)) = \varepsilon_B^*(b)$ and $f(a) \in Im(\alpha)(\subseteq Ker(k))$, we obtain $b \in Ker(k)$.

Conversely, let $b \in Ker(\beta)$, and then $\beta(b) = g(b) \in \omega_{Ker(l)} = \omega_C$ and $b \in Ker(g)$. Since the first row is exact, there exists $f(a) \in Im(f)$ for some $a \in A$ such that $\varepsilon_B^*(b) = \varepsilon_B^*(f(a))$. It is enough to show $a \in Ker(h)$. Since k is strong and the diagram is commutative, we obtain $\varepsilon_{B_1}^*(k(b)) = \varepsilon_{B_1}^*(k(f(a))) = \varepsilon_{B_1}^*(f_1(h(a)))$. Since $b \in Ker(\beta)(\subseteq Ker(k))$, it follows that $f_1(h(a)) \in \omega_{B_1}$ and $h(a) \in Ker(f_1)$. Since f_1 is weak-monic (by exactness and Lemma 2.1), we have $Ker(f_1) = \omega_{A_1}$. Therefore, $a \in Ker(h)$. □

3. Schanuel's lemma in H_v-modules

In this section we define the concepts of star homomorphism, (star) isomorph sequences, and star projective H_v-modules (we also build and present some examples for these concepts) in order to find a generalization of Schanuel's lemma. We also prove a problem on commutative diagrams.

Definition 3.1 *A mapping $f : M_1 \longrightarrow M_2$ of H_v-modules M_1 and M_2 over an H_v-ring R is called a star homomorphism if for every $x, y \in M_1$ and every $r \in R$: $\varepsilon_{M_2}^*(f(x+y)) = \varepsilon_{M_2}^*(f(x)+f(y))$ and $\varepsilon_{M_2}^*(f(rx)) = \varepsilon_{M_2}^*(rf(x))$; i.e. $f(x+y) \overset{w}{=} f(x)+f(y)$ and $f(rx) \overset{w}{=} rf(x)$.*

Every strong homomorphism is a star homomorphism but the converse is not true necessarily by the following example.

Example 1 *Let R be an H_v-ring. Consider the following H_v-modules on R:*

(1) $M_1 = \{a, b\}$ together with the following hyperoperations:

$*_{M_1}$	a	b
a	a	b
b	b	a

and $\underset{(r,m_1)\mapsto\{a\}}{\cdot_{M_1} : R \times M_1 \to \mathcal{P}^*(M_1)},$

(2) $M_2 = \{0, 1, 2\}$ together with the following hyperoperations:

$*_{M_2}$	0	1	2
0	0	1	2
1	1	0,2	1
2	2	1	0

and $\underset{(r,m_2)\mapsto\{0\}}{\cdot_{M_2} : R \times M_2 \to \mathcal{P}^*(M_2)}.$

We obtain $M_2/\varepsilon_{M_2}^* = \{\varepsilon_{M_2}^*(0) = \{0,2\}, \varepsilon_{M_2}^*(1) = \{1\}\}$. If $f : M_1 \longrightarrow M_2$ defined by $f(a) = 0$ and $f(b) = 1$ then f is a star homomorphism but not a strong homomorphism because $f(b *_{M_1} b) \neq f(b) *_{M_2} f(b)$.

Definition 3.2 *Two mappings $f, g : M \longrightarrow N$ on H_v-modules are called weak equal if for every $m \in M$; $\varepsilon_N^*(f(m)) = \varepsilon_N^*(g(m))$ and denoted by $f \overset{w}{=} g$. The following diagram of H_v-modules and strong homomorphisms* *is called star commutative if* $g \circ f \overset{w}{=} h$.

$$\begin{array}{ccc} & & A \\ & \swarrow^{f} & \downarrow h \\ B & \xrightarrow[g]{} & C \end{array}$$

Also, it is said to be commutative if for every $a \in A$, $g \circ f(a) = h(a)$.

Definition 3.3 *The sequences*

$$\omega_A \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \omega_C$$

and

$$\omega_{A'} \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow \omega_{C'}$$

are called isomorph (star isomorph) if there exist weak-isomorphisms (star homomorphisms) $\alpha : A \longrightarrow A'$, $\beta : B \longrightarrow B'$, *and* $\gamma : C \longrightarrow C'$ *such that the following diagram is commutative (star commutative):*

$$\begin{array}{ccccccccc} \omega_A & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & \omega_C \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ \omega_{A'} & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & \omega_{C'}. \end{array}$$

Definition 3.4 *An* H_v*-module* P *is called star projective if for every diagram of strong homomorphisms and* H_v*-modules as follows*

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ M & \xrightarrow{g} & N & \longrightarrow & \omega_N \end{array} \qquad ,$$

such that its row is exact, there exists a strong homomorphism $\phi : P \longrightarrow M$ *such that* $g \circ \phi \overset{w}{=} f$.

According to [7], for every strong homomorphism $f : M \longrightarrow N$ there is the R/γ^*-homomorphism $F : M/\varepsilon^*_M \longrightarrow N/\varepsilon^*_N$ of R/γ^*-modules defined by $F(\varepsilon^*_M(m)) = \varepsilon^*_N(f(m))$.

Lemma 3.5 *[13] Let* $f : A \longrightarrow B$ *be a strong homomorphism of* H_v*-modules. Then* f *is weak-epic (weak-monic) if and only if* F *is onto (one to one). Thus,* f *is a weak-isomorphism if and only if* F *is an isomorphism.*

Proof Suppose that f is weak-epic and $\varepsilon^*_B(b) \in B/\varepsilon^*_B$. Since f is weak-epic, there exists $a \in A$ such that $\varepsilon^*_B(f(a)) = \varepsilon^*_B(b)$, but $\varepsilon^*_B(f(a)) = F(\varepsilon^*_A(a))$. Thus, $F(\varepsilon^*_A(a)) = \varepsilon^*_B(b)$ and consequently F is onto.

Conversely, let F be onto. Then, for every $b \in B$, there exists $\varepsilon^*_A(a) \in A/\varepsilon^*_A$ such that $F(\varepsilon^*_A(a)) = \varepsilon^*_B(b)$, but $F(\varepsilon^*_A(a)) = \varepsilon^*_B(f(a))$. Thus, there exists $a \in A$ such that $\varepsilon^*_B(f(a)) = \varepsilon^*_B(b)$ and consequently f is weak-epic. The second part is proved in [7]. The third part is an obvious result. □

Theorem 3.6 ***(Schanuel's lemma in*** H_v ***-modules)*** *Let* P_1 *and* P_2 *be two star projective* H_v *-modules. Then the following exact sequences are star isomorph:*

$$\omega_K \longrightarrow K \xrightarrow{f} P_1 \xrightarrow{g} M \longrightarrow \omega_M \ , \tag{1}$$

$$\omega_L \longrightarrow L \xrightarrow[f_1]{} P_2 \xrightarrow[g_1]{} M \longrightarrow \omega_M \ . \tag{2}$$

Proof Let $\gamma : M \longrightarrow M$ be identity on M. Since P_1 is a star projective H_v-module, there exists a strong homomorphism $\beta : P_1 \longrightarrow P_2$ such that for every $p \in P_1$; $\varepsilon^*_M(g_1 \circ \beta(p)) = \varepsilon^*_M(g(p))$. Now, for every $k \in K$; $f(k) \in P_1$ and then by exactness of sequence (1) we have $\beta \circ f(k) \in Ker(g_1)$ and so by exactness of sequence (2) there exists $l_k \in L$ such that $\varepsilon^*_{P_2}(\beta(f(k))) = \varepsilon^*_{P_2}(f_1(l_k))$. We define $\alpha : K \longrightarrow L$ by $\alpha(k) = l_k$. Supposing $k_1, k_2 \in K$ and $r \in R$, we have:

$$\begin{aligned}
\varepsilon^*_{P_2}(\beta \circ f(k_1 + k_2)) &= \varepsilon^*_{P_2}(\beta(f(k_1)) + \beta(f(k_2))) \\
&= \varepsilon^*_{P_2}(\beta f(k_1)) \oplus \varepsilon^*_{P_2}(\beta(f(k_2))) \\
&= \varepsilon^*_{P_2}(f_1(l_{k_1})) \oplus \varepsilon^*_{P_2}(f_1(l_{k_2})) \\
&= \varepsilon^*_{P_2}(f_1(l_{k_1}) + f_1(l_{k_2})) \\
&= \varepsilon^*_{P_2}(f_1(l_{k_1} + l_{k_2})) \\
&= \varepsilon^*_{P_2}(f_1(\alpha(k_1) + \alpha(k_2)) \\
&= F_1(\varepsilon^*_L(\alpha(k_1) + \alpha(k_2))),
\end{aligned}$$

while on the other hand

$$\begin{aligned}
\varepsilon^*_{P_2}(\beta \circ f(k_1 + k_2)) &= \{\varepsilon^*_{P_2}(\beta(f(t)))|\ t \in k_1 + k_2\} \\
&= \{\varepsilon^*_{P_2}(f_1(l_t))|\ t \in k_1 + k_2;\ \varepsilon^*_{P_2}(\beta(f(t))) = \varepsilon^*_{P_2}(f_1(l_t))\} \\
&= \{\varepsilon^*_{P_2}(f_1(\alpha(t)))|\ t \in k_1 + k_2\} \\
&= \varepsilon^*_{P_2}(f_1(\alpha(k_1 + k_2))) \\
&= F_1(\varepsilon^*_L(\alpha(k_1 + k_2))).
\end{aligned}$$

Thus, $F_1(\varepsilon^*_L(\alpha(k_1 + k_2))) = F_1(\varepsilon^*_L(\alpha(k_1) + \alpha(k_2)))$. Now by Lemma 2.1 and Lemma 3.5, F_1 is one to one and $\varepsilon^*_L(\alpha(k_1 + k_2)) = \varepsilon^*_L(\alpha(k_1) + \alpha(k_2))$.

Also,

$$\begin{aligned}
\varepsilon^*_{P_2}(\beta \circ f(rk_1)) &= \varepsilon^*_{P_2}(r\beta(f(k_1)) \\
&= \gamma^*(r) \odot \varepsilon^*_{P_2}(\beta(f(k_1)) \\
&= \gamma^*(r) \odot \varepsilon^*_{P_2}(f_1(l_{k_1})) \\
&= \varepsilon^*_{P_2}(rf_1(l_{k_1})) \\
&= \varepsilon^*_{P_2}(rf_1(\alpha(k_1))) \\
&= \varepsilon^*_{P_2}(f_1(r\alpha(k_1))) \\
&= F_1(\varepsilon^*_L(r\alpha(k_1)),
\end{aligned}$$

while on the other hand

$$\begin{aligned}
\varepsilon^*_{P_2}(\beta(f(rk_1))) &= \{\varepsilon^*_{P_2}(\beta(f(t)))|\ t\in rk_1\}\\
&= \{\varepsilon^*_{P_2}(f_1(l_t))|\ t\in rk_1;\ \varepsilon^*_{P_2}(\beta(f(t)))=\varepsilon^*_{P_2}(f_1(l_t))\}\\
&= \{\varepsilon^*_{P_2}(f_1(\alpha(t)))|\ t\in rk_1\}\\
&= \varepsilon^*_{P_2}(f_1(\alpha(rk_1)))\\
&= F_1(\varepsilon^*_L(\alpha(rk_1))).
\end{aligned}$$

Thus, $F_1(\varepsilon^*_L(\alpha(rk_1))) = F_1(\varepsilon^*_L(r\alpha(k_1))$. Now by Lemma 2.1 and Lemma 3.5, F_1 is one to one and $\varepsilon^*_L(\alpha(rk_1)) = \varepsilon^*_L(r\alpha(k_1)$, and α is a star homomorphism.

One can check the star commutativity on these star homomorphisms. □

Theorem 3.7 *(i) Let*

$$\begin{array}{ccccccc}
\omega_A & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C\\
 & & & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma}\\
[1ex]\omega_{A_1} & \longrightarrow & A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C_1
\end{array}$$

be a star commutative diagram of H_v-modules and strong H_v-homomorphisms with both exact rows. Then there exists a star homomorphism $\alpha : A \longrightarrow A_1$ such that it star-commutes the diagram.

(ii) Let

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & \omega_C\\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & & & \\
A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C_1 & \longrightarrow & \omega_{C_1}
\end{array}$$

be a star commutative diagram of H_v-modules and strong homomorphisms with both exact rows. Then there exists a star homomorphism $\gamma : C \longrightarrow C_1$ such that it star-commutes the diagram.

Proof (i) For every $a\in A$ we have $\varepsilon^*_{C_1}(g_1\circ\beta\circ f(a)) = \varepsilon^*_{C_1}(\gamma\circ g\circ f(a))$. The first row is exact and γ is strong homomorphism. Then $g\circ f(a)\in\omega_C$ and $\gamma\circ g\circ f(a)\in\omega_{C_1}$. Thus, $\beta\circ f(a)\in Ker(g_1)$ and there exists $a_1\in A_1$ such that $\varepsilon^*_{B_1}(\beta\circ f(a)) = \varepsilon^*_{B_1}(f_1(a_1))$. Now we define $\alpha : A\longrightarrow A_1$ by $\alpha(a) = a_1$.

Similar to the proof of Theorem 3.6 one can show that α is a star homomorphism. Also, for every $a\in A$, we have

$$\varepsilon^*_{B_1}(f_1\circ\alpha(a)) = \varepsilon^*_{B_1}(f_1(a_1)) = \varepsilon^*_{B_1}(\beta\circ f(a)).$$

(ii) Since g is weak-epic for every $c\in C$ there exists $b_c\in B$ such that $\varepsilon^*_C(c) = \varepsilon^*_C(g(b_c))$. We define $\gamma : C\longrightarrow C_1$ by $\gamma(c) = g_1\circ\beta(b_c)$. The remainder of the proof is straightforward and similar to the proof of (i). □

4. Product and direct sum in H_v-modules

In this section we define the concepts of the product and direct sum of H_v-modules (we also build and present some examples for these concepts), and we determine the conditions to split an exact sequence.

Definition 4.1 *Let M be an H_v-module; H and K are H_v-submodules of M. M is said to be the direct sum of H and K if $H \cap K \subseteq \omega_M$ and $\varepsilon^*(H+K) = \varepsilon^*(M)$. We denote it by $H \oplus K = M$.*

Example 2 *For every H_v-module M we have $M = \omega_M \oplus M$.*

Example 3 *Consider the following weak commutative H_v-group:*

$*_M$	0	1	2	3	4	5	6
0	0,1	0,1	2	3	4	5	6
1	0,1	0,1	2	3	4	5	6
2	2	2	0,1	5,6	5,6	2,3,4	2,3,4
3	3	3	5,6	0,1	5,6	2,3,4	2,3,4
4	4	4	5,6	5,6	0,1	2,3,4	2,3,4
5	5	5	2,3,4	2,3,4	2,3,4	6	0,1
6	6	6	2,3,4	2,3,4	2,3,4	0,1	5

One can check that $R = (M, *_M, .)$ is an H_v-ring where $r_1.r_2 = \{0,1\}$ for every r_1, $r_2 \in R$ and M is an H_v-module over the H_v-ring R. Also,

$$M/\varepsilon^*_M = \{\varepsilon^*_M(0), \varepsilon^*_M(2)\},$$

where

$$\varepsilon^*_M(0) = \omega_M = \{0,1,5,6\}, \qquad \varepsilon^*_M(2) = \{2,3,4\}.$$

Now $H = \{0,1,2\}$ and $K = \{0,1,5,6\}$ are H_v-submodules of M and $H \oplus K = M$.

Proposition 4.2 *Let $f : M \longrightarrow M$ be a strong homomorphism of H_v-modules such that $f^2 = f$. Then M is the direct sum of $Im(f)$ and $Ker(f)$. Moreover, f is identity on $Im(f) \cap Ker(f)$.*

Proof Let $m \in Im(f) \cap Ker(f)$, and then

$$m = f(m_1) \text{ for some } m_1 \text{ in } M \tag{3}$$

and

$$f(m) \in \omega_M. \tag{4}$$

By applying f on Eq. (3) we obtain $f(m) = f^2(m_1) = f(m_1) = m$ as a member of ω_M by Eq. (4), so $Im(f) \cap Ker(f) \subseteq \omega_M$ and f is identity on $Im(f) \cap Ker(f)$. Now, for every $m \in M$, we have:

$$F(F(\varepsilon^*(m))) = F(\varepsilon^*(f(m))) = \varepsilon^*(f^2(m)) = \varepsilon^*(f(m)) = F(\varepsilon^*(m)).$$

Thus, $Im(F) + Ker(F) = M/\varepsilon^*_M$, since F is a R/γ^*-module such that $F^2 = F$. Therefore, $\varepsilon^*(Im(f) + Ker(f)) = \varepsilon^*(M)$. □

Let $\{M_i\}_{i\in I}$ be a nonempty collection of H_v-modules. The product of this collection,

$$\sqcap_{i\in I}\{M_i\} = \{(x_i) | \; x_i \in M; \; \forall\, i \in I\},$$

with the following hyperoperations is an H_v-module:

$$(x_i) + (y_i) = \{(z_i) | \; z_i \in x_i + y_i\},$$

$$r(x_i) = \{(w_i) | \; w_i \in rx_i\}.$$

Lemma 4.3 *Let $\sqcap_{i\in I} M_i$ be the product of the nonempty collection of H_v-modules. Then:*
(i) $P_k : \sqcap M_i \longrightarrow M_k$ defined by $P_k((x_i)) = x_k$ is a strong homomorphism.
(ii) For every exact sequence $M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2$ the mapping
$\lambda_1 : M_1 \longrightarrow M_1 \sqcap M_2$ defined by $\lambda_1(x) = (x, \psi\phi(x))$ is a strong homomorphism. Also, $\lambda_2 : M_2 \longrightarrow M_1 \sqcap M_2$, defined by $\lambda_2(x) = (a, x)$, where a is an arbitrary member of ω_{M_1}, is a star homomorphism. In particular, if there exists a $t \in \omega_{M_1}$ such that $t + t = t$, then λ_2 is a strong homomorphism.
(iii) $P_k\lambda_k = I_{M_k}$.

Proof (i)

$$P_k((x_i) + (y_i)) = P_k(\{(z_i)|\ z_i \in x_i + y_i\}) \quad = \{z_k|\ z_k \in x_k + y_k\}.$$

On the other hand,

$$P_k((x_i)) + P_k((y_i)) = x_k + y_k.$$

Similarly, we obtain $P_k(r(x_i)) = rP_k((x_i))$.

(ii) We have

$$\begin{aligned} \lambda_1(x+y) &= \textstyle\bigcup_{a\in x+y, b\in\psi\phi(x+y)}(a,b) \\ &= (x, \psi\phi(x)) + (y, \psi\phi(y)) \\ &= \lambda_1(x) + \lambda_1(y). \end{aligned}$$

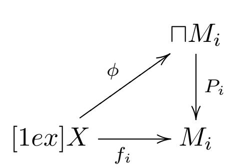
obtain $\lambda_1(rx) = r\lambda_1(x)$. Also,

$$\begin{aligned} \varepsilon^*(\lambda_2(x+y)) &= \varepsilon^*(\textstyle\bigcup_{a\in\omega_{M_1}, b\in x+y}(a,b)) \\ &= \varepsilon^*((a_1, x) + (a_1, y)) \quad \text{where } \mathrm{a}_1 \in \omega_{\mathrm{M}_1} \\ &= \varepsilon^*((a_1, x)) \oplus \varepsilon^*((a_1, y)) \\ &= \varepsilon^*(\lambda_2(x)) \oplus \varepsilon^*(\lambda_2(y)) \\ &= \varepsilon^*(\lambda_2(x) + \lambda_2(y)). \end{aligned}$$

Similarly, $\varepsilon^*(\lambda_2(rx)) = \varepsilon^*(r\lambda_2(x))$.

(iii) The proof of this part is straightforward. □

Theorem 4.4 *Let $\{M_i\}$ be a nonempty collection of H_v-modules. For every H_v-module X and every collection of strong homomorphisms $\{f_i : X \longrightarrow M_i\}$ there exists a unique strong homomorphism $\phi : X \longrightarrow \sqcap M_i$ defined by $\phi(x) = (f_i(x))$ such that for every $i \in I$ the following diagram is commutative.*



Proof The proof is straightforward. □

We want to define the inverse of a weak-isomorphism to determine the conditions for splitting an exact sequence.

Lemma 4.5 *Let* $f: M \longrightarrow N$ *be a weak-isomorphism. Then* $f^{-1}: N \longrightarrow M$ *defined by* $f^{-1}(n) = m_n$ *for selected* $m_n \in F^{-1}(\varepsilon_N^*(n))$ *is a star homomorphism such that* $f^{-1} \circ f \overset{w}{=} I_M$ *and* $f \circ f^{-1} \overset{w}{=} I_N$ *.*

Proof Since f is a weak-isomorphism by Lemma 3.5, F is an isomorphism and has an inverse. For every $n_1, n_2 \in N$ we have

$$f^{-1}(n_1 + n_2) = \{m_c |\ m_c \in F^{-1}(\varepsilon_N^*(c)),\ c \in n_1 + n_2\}. \tag{5}$$

On the other hand,

$$\begin{aligned} f^{-1}(n_1) + f^{-1}(n_2) &= m_{n_1} + m_{n_2} \\ &\subseteq F^{-1}(\varepsilon_N^*(n_1)) + F^{-1}(\varepsilon_N^*(n_2)) \\ &= F^{-1}(\varepsilon_N^*(n_1 + n_2)). \end{aligned} \tag{6}$$

From Eq. (5) and Eq. (6) we obtain $\varepsilon_M^*(f^{-1}(n_1 + n_2)) = \varepsilon_M^*(f^{-1}(n_1) + f^{-1}(n_2))$ (notice that for every $n_1, n_2 \in N$, $n_1 + n_2 \subseteq \varepsilon_N^*(n)$ for some $n \in n_1 + n_2$).

Similarly, we obtain $\varepsilon_M^*(f^{-1}(rn)) = \varepsilon_M^*(rf^{-1}(n))$.

Finally, for every $m \in M$ we have

$$\begin{aligned} f^{-1} \circ f(m) &\in F^{-1}(\varepsilon_N^*(f(m))) \\ &= F^{-1}(F(\varepsilon_M^*(m))), \\ &= \varepsilon_M^*(m) \end{aligned}$$

and for every $n \in N$,

$$\begin{aligned} f \circ f^{-1}(n) &= f(m_n),\ \text{where } m_n \in F^{-1}(\varepsilon_N^*(n)), \\ &\text{but } f(m_n) \in \varepsilon_N^*(n). \end{aligned}$$

□

Definition 4.6 *Letting* f *be a weak-isomorphism, the* f^{-1} *defined in Lemma 4.5 is called the inverse of* f *. It is clear that this inverse is not necessarily unique.*

Theorem 4.7 *Let* M_1 *,* M_2 *, and* M *be three* H_v *-modules and the sequence*

$$\omega_{M_1} \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow \omega_{M_2} \tag{7}$$

is exact:

(i) If there exists a star homomorphism $\phi^{'}: M \longrightarrow M_1$ *(*$\psi^{'}: M_2 \longrightarrow M$ *) such that* $\phi^{'}\phi \overset{w}{=} I_{M_1}$ *(*$\psi\psi^{'} \overset{w}{=} I_{M_2}$ *), then the sequence (7) is star isomorph with the sequence*

$$\omega_{M_1} \longrightarrow M_1 \xrightarrow{\lambda_1} M_1 \sqcap M_2 \xrightarrow{P_2} M_2 \longrightarrow \omega_{M_2}. \tag{8}$$

(ii) If the sequences (7) and (8) are isomorph, then there exist star homomorphisms $\phi^{'}: M \longrightarrow M_1$ *and* $\psi^{'}: M_2 \longrightarrow M$ *such that* $\phi^{'}\phi \overset{w}{=} I_{M_1}$ *,* $\psi\psi^{'} \overset{w}{=} I_{M_2}$ *.*

Proof (i) We define $\alpha: M \longrightarrow M_1 \sqcap M_2$ by $\alpha(x) = (\phi^{'}(x), \psi(x))$. It is easy to see that α is a star homomorphism. Since for every $m_1 \in M_1$ we have $\phi^{'}\phi(m_1) \in \varepsilon_{M_1}^*(m_1)$ and $\psi\phi(m_1) \in \omega_{M_1}$, the following diagram is star commutative with both exact rows.

$$\begin{array}{ccccccccc}
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow{\phi} & M & \xrightarrow{\psi} & M_2 & \longrightarrow & \omega_{M_2} \, . \\
 & & \downarrow{\scriptstyle 1_{M_1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1_{M_2}} & & \\
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow[\lambda_1]{} & M_1 \sqcap M_2 & \xrightarrow[P_2]{} & M_2 & \longrightarrow & \omega_{M_2}
\end{array}$$

Now let there exist the star homomorphism $\psi^{'} : M_2 \longrightarrow M$ such that $\psi\psi^{'} \overset{w}{=} I_{M_2}$. We define the mapping $\beta : M_1 \times M_2 \longrightarrow M$ by $\beta((m_1, m_2)) = m_{m_1,m_2}$ where m_{m_1,m_2} is a member of $\phi(m_1)+\psi^{'}(m_2)$ (according to the choice axiom). We show that β is a star homomorphism. We have: $\varepsilon^*(\beta((a_1, a_2) + (a_1^{'}, a_2^{'}))) = \varepsilon^*(\beta((t_1, t_2))$, where $t_1 \in a_1 + a_1^{'}$ and $t_2 \in a_2 + a_2^{'}$.
and

$$\begin{aligned}
\varepsilon^*(\beta((a_1, a_2))) \oplus \varepsilon^*(\beta((a_1^{'}, a_2^{'}))) &= \varepsilon^*(\phi(a_1) + \psi^{'}(a_2)) \oplus \varepsilon^*(\phi(a_1^{'}) + \psi^{'}(a_2^{'})) \\
&= \varepsilon^*(\phi(a_1) + \psi^{'}(a_1^{'}) + \phi(a_2) + \psi^{'}(a_2^{'})) \\
&= \varepsilon^*(\phi(t_1)) \oplus \varepsilon^*(\psi^{'}(t_2) \\
&= \varepsilon^*(\phi(t_1) + \psi^{'}(t_2)) \\
&= \varepsilon^*(\beta((t_1, t_2)),
\end{aligned}$$

where $t_1 \in a_1 + a_1^{'}$ and $t_2 \in a_2 + a_2^{'}$. Thus, β is a star homomorphism. One can show that the following diagram is star commutative:

$$\begin{array}{ccccccccc}
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow{\phi} & M & \xrightarrow{\psi} & M_2 & \longrightarrow & \omega_{M_2} \, . \\
 & & \uparrow{\scriptstyle 1_{M_1}} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle 1_{M_2}} & & \\
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow[\lambda_1]{} & M_1 \sqcap M_2 & \xrightarrow[P_2]{} & M_2 & \longrightarrow & \omega_{M_2}
\end{array}$$

(ii) By hypothesis there exist weak-isomorphisms
$\alpha : M_1 \longrightarrow M_1$, $\beta : M \longrightarrow M_1 \sqcap M_2$, and $\gamma : M_2 \longrightarrow M_2$ that commute the following diagram:

$$\begin{array}{ccccccccc}
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow{\phi} & M & \xrightarrow{\psi} & M_2 & \longrightarrow & \omega_{M_2} \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow[\lambda_1]{} & M_1 \sqcap M_2 & \xrightarrow[P_2]{} & M_2 & \longrightarrow & \omega_{M_2}
\end{array}$$

By Lemma 4.5, there exists star homomorphism $\alpha^{-1} : M_1 \longrightarrow M_1$ such that $\alpha^{-1} \circ \alpha \overset{w}{=} I_{M_1}$. Now we define $\phi^{'} : M \longrightarrow M_1$ by $\phi^{'} = \alpha^{-1} P_1 \beta$. Consequently, $\phi^{'}$ is a star homomorphism and

$$\phi^{'}\phi = \alpha^{-1} P_1 \beta\phi = \alpha^{-1} P_1 \lambda_1 \alpha = \alpha^{-1} 1_{M_1} \alpha \overset{w}{=} I_{M_1}.$$

Similarly, by hypothesis, there exist weak-isomorphisms
$\alpha : M_1 \longrightarrow M_1$, $\beta : M_1 \sqcap M_2 \longrightarrow M$, and $\gamma : M_2 \longrightarrow M_2$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow{\lambda_1} & M_1 \sqcap M_2 & \xrightarrow{P_2} & M_2 & \longrightarrow & \omega_{M_2} \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
\omega_{M_1} & \longrightarrow & M_1 & \xrightarrow[\phi]{} & M & \xrightarrow[\psi]{} & M_2 & \longrightarrow & \omega_{M_2}
\end{array}$$

By Lemma 4.5, there exists star homomorphism $\gamma^{-1} : M_2 \longrightarrow M_2$ such that $\gamma \circ \gamma^{-1} \overset{w}{=} I_{M_2}$ Now we define $\psi^{'} : M_2 \longrightarrow M$ by $\psi^{'} = \beta\lambda_2\gamma^{-1}$. Obviously $\psi^{'}$ is a star homomorphism and

$$\psi\psi^{'} = \psi\beta\lambda_2\gamma^{-1} = \gamma P_2\lambda_2\gamma^{-1} = \gamma 1_{M_2}\gamma^{-1} \overset{w}{=} I_{M_2}.$$

□

An exact sequence in Theorem 4.7 is called a split sequence.

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