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# A note on locally graded minimal non-metahamiltonian groups

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**Abstract:** We prove that a nonperfect locally graded minimal non-metahamiltonian group $G$ is a soluble group with derived length of at most 4. On the other hand, if $G$ is perfect, then $G/\Phi(G)$ is isomorphic to $A_5$, where $\Phi(G)$ is the Frattini subgroup of $G$ and $A_5$ is the alternating group. Moreover, we show that under some conditions, if G is a $p$-group, then G is metabelian, where $p$ is a prime integer.

**Key words:** Locally graded minimal non-metahamiltonian, soluble, metabelian, $d$-maximal

## 1. Introduction

A group $G$ is called *metahamiltonian* if every nonabelian subgroup of $G$ is normal. Metahamiltonian groups were introduced and investigated in a series of papers by Romalis and Sesekin (see [17–19]); they proved in particular that the commutator subgroup of any group with such property is finite with prime-power order. In [6, Theorem 3.4], De Mari and De Giovanni proved that a locally graded metahamiltonian group is soluble with derived length of at most 3. In [14, Theorem 1], Mahnev proved that for prime integer p, the commutator subgroup of a finite metahamiltonian $p$-group is abelian. For more details and the current knowledge related to these kinds of groups, we recommend the book by Kuzenniy and Semko [12].

A group $G$ is called *minimal non-metahamiltonian* if every proper subgroup of $G$ is a metahamiltonian but $G$ itself is not. In [5, Lemma 4.2], De Falco et al. proved that a locally graded group with such property is finite. In [5], the authors also gave the alternating group $A_5$ as an example that shows that there exist finite insoluble minimal non-metahamiltonian groups. Further, $A_5$ is perfect simple with every proper subgroup metabelian. Corollary 2 shows that a simple locally graded minimal non-metahamiltonian group is isomorphic to $A_5$.

Therefore, every minimal non-metahamiltonian group may not soluble. It is natural to ask: "Is there a soluble minimal non-metahamiltonian group?" The answer to this question is yes. It is well known that the general linear group $GL(2,3)$ is a soluble group with derived length 4. On the other hand, this group is also a minimal non-metahamiltonian group: indeed, its subgroups of order sixteen are nonabelian and not normal. Also, one can see that every proper subgroup of $GL(2,3)$ is metahamiltonian. Theorem 1(ii) shows that a nonperfect locally graded minimal non-metahamiltonian group is soluble with derived length of at most 4.

The second crucial question is "Does there exist an insoluble nonsimple minimal non-metahamiltonian group?" The answer to this question is yes. If we consider the unique nonsplitting central extension of a group

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of order 2, which is the Schur multiplier of $A_5$, by the group $A_5$ itself, this group is perfect, but not simple. It is easy show that it is a minimal non-metahamiltonian group with its only proper nontrivial normal subgroup being its center, and also, the Frattini factor of it is isomorphic to $A_5$. Theorem 1(i) shows that the Frattini factor of a perfect nonsimple locally graded minimal non-metahamiltonian group is isomorphic to $A_5$.

The generalized dihedral group for the direct product of $Z_4$ and $Z_4$ is known as a metabelian 2-group, but it is also a minimal non-metahamiltonian group: indeed, it has nonnormal nonabelian subgroups of order eight. One can also see that every proper subgroup is metahamiltonian.

Another question is "Does there exist a non-metabelian locally graded minimal non-metahamiltonian 2-group?" The technique used by O'Brein in [15] shows that there is no non-metabelian minimal non-metahamiltonian group of order dividing $2^9$, which means that there is no such a group of order dividing $2^9$. Furthermore, we could not find any example that shows that there exists a metabelian (or not) minimal non-metahamiltonian group of order at least $2^{10}$ in the literature. As a consequence, we have no answer to this question, neither yes nor no. In this study, for any prime integer $p$, we show that a locally graded minimal non-metahamiltonian $p$-group for which every proper subgroup is metacyclic is metabelian (see Theorem 3). We also prove for any prime integer $p$ that a $d$-maximal locally graded minimal non-metahamiltonian $p$-group is metabelian (see Theorem 4). A group $G$ is said to be $d$-maximal if $d(H) < d(G)$ for any proper subgroup $H$ of $G$, where $d(G)$ denotes the cardinality of a minimal generating set of $G$.

Therefore, we are going to deal with the conditions that make the answer to this question be no.

Recall that a group $G$ is called *locally graded* if every finitely generated nontrivial subgroup of $G$ has a proper subgroup of finite index. Locally graded groups form a wide class of generalized soluble groups, containing in particular all locally (soluble-by-finite) groups.

Most of our notation is standard and can be found in [16].

## 2. The main conclusions

Now we are ready to give and prove Theorem 1, Theorem 2, and Theorem 3.

**Theorem 1** *Let $G$ be a locally graded minimal non-metahamiltonian group.*

*(i) If $G$ is perfect, then the Frattini factor group of $G$, i.e. $G/\Phi(G)$, is isomorphic to $A_5$.*

*(ii) If $G$ is not perfect, then $G$ is a soluble group with derived length of at most $4$.*

**Proof**

(i) Let $G$ be perfect, i.e. $G' = G$.

Then $\bar{G} = G/\Phi(G)$ is a simple nonabelian group by [6, Theorem 3.4] and [9, Lemma 3.2], which is of course minimal non-metahamiltonian. Since $\bar{G}$ is a finite nonabelian simple group in which every proper subgroup is soluble with derived length of at most 3 by [6, Theorem 3.4], then to complete the proof, it is enough to show that $PSL(3,3)$, $Sz(q)$, where $q = 2^p$ for any odd prime integer $p$, $PSL(2,p)$, where $p > 3$ is any prime integer such that $p^2 + 1 \equiv 0 \pmod 5$, $PSL(2,2^p)$, and $PSL(2,3^p)$, where $p$ is any odd integer, are not minimal non-metahamiltonian groups by [22, Corollary 1] and [8, Proposition 2.2].

Since $PSL(3,3)$ contains a subgroup of derived length 5 (see [8, pp. 4-5]), then $PSL(3,3)$ is not a minimal non-metahamiltonian group.

By [20, Theorem 9], $Sz(q)$ contains a non-metabelian Frobenius group $F$ of order $q^2(q-1)$ and $Sz(q)$ has only one abelian subgroup of order dividing $q^2(q-1)$ that is a cyclic group of order $q-1$. It follows that the commutator subgroup of $F$ has not prime-power order. Therefore, $Sz(q)$ is not a minimal non-metahamiltonian group by [6, Theorem 3.4].

$PSL(2,p)$, where $p > 3$ is any prime integer $p^2+1 \equiv 0 \pmod{5}$ and $p^2-1 \equiv 0 \pmod{16}$, has a proper subgroup isomorphic to $S_4$ by [21, Theorem 6.25 and Theorem 6.26]. Therefore, $PSL(2,p)$ is not a minimal non-metahamiltonian group by $S_4$ with a nonnormal subgroup of order 8 to isomorphic $D_8$.

$PSL(2,p)$, where $p > 3$ is any prime integer $p^2+1 \equiv 0 \pmod{5}$ and $p^2-1 \not\equiv 0 \pmod{16}$, $PSL(2,2^p)$, and $PSL(2,3^p)$, where $p$ is any odd integer, contain a maximal subgroup that is dihedral and has a proper nonabelian subgroup that is not normal (see [7]), and so these groups cannot be minimal non-metahamiltonian groups.

(ii) If $G$ is not perfect, i.e. $G' \neq G$, then $G'$ is metahamiltonian by the hypothesis and so $G$ is a soluble group with derived length of at most 4 by [6, Theorem 3.4].

□

**Corollary** *Let $G$ be a simple locally graded minimal non-metahamiltonian group. Then $G$ is isomorphic to $A_5$.*

**Theorem 2** *Let $G$ be a locally graded minimal non-metahamiltonian $p$-group, for a prime integer $p$. If every proper subgroup of $G$ is metacyclic, then $G$ is metabelian.*

**Proof** First, let $p = 2$. Assume that $G$ is not metacyclic. Since $G$ is a minimal non-metahamiltonian group by the hypothesis, the order of $G$ must be $2^5$ by [2, Theorem 66.1]. Also, every subgroup of order 8 is abelian by the proof of [2, Theorem 66.1]. However, since $G$ is not metahamiltonian, that is a contradiction. Therefore, $G$ is metacyclic and so it is metabelian.

Now let $p > 2$. In the case of $d(G) \geq 4$, since every proper subgroup is metacyclic by the hypothesis, then $G$ is metabelian by [1, Theorem 4]. Let $d(G) \leq 3$. Since every proper subgroup is metacyclic by the hypothesis, then every proper subgroup can be generated by two elements by [12, Theorem 2.3.1]. If $d(G) = 2$, then $G$ is metabelian by [3, Theorem 4]. If $d(G) = 3$, since every proper subgroup of $G$ can be generated by two elements, $G$ is $d$-maximal and therefore $G$ is metabelian by $p > 2$ and by [13, Theorem]. This completes the proof of this theorem. □

**Theorem 3** *Let $G$ be a locally graded minimal non-metahamiltonian $p$-group for a prime integer $p$. If $G$ is $d$-maximal, then $G$ is metabelian.*

**Proof** Suppose that $G$ is $d$-maximal.

For $p > 2$, since $G$ is a finite $p$-group, then $G$ is metabelian by [13, Theorem].

Now let be $p = 2$. Assume that $G$ is not metabelian. Then $G'$ cannot be generated by a commutative set. Hence, there exist elements $a, b, c$, and $d$ of $G$ such that $[[a,b],[c,d]] \neq 1$, which implies that $< a,b,c,d >$ is a non-metabelian finite $p$-group, and then $G =< a,b,c,d >$ by [14, Theorem 1]. It follows that $d(G) \leq 4$. If $d(G) \leq 3$, since $G$ is a $d$-maximal 2-group, then it follows by [10, Lemma A2] and [4, Theorem 3.2] that the nilpotent class of $G$ is at most 2, which implies a contradiction by $G$ not being metabelian. Hence, we obtain $d(G) = 4$. It follows that the nilpotent class of $G$ is at most 2 by [11, Proposition 5.1], which implies a contradiction again by $G$ not being metabelian. This completes the proof. □

We would like to make a note here that the generalized dihedral group for the direct product of $Z_4$ and $Z_4$ shows that the converse of the statement in Theorem 3 is not true in general.

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