Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1144 – 1154

doi:10.3906/mat-1609-26

The role of the ideal elements in studying the structure of some ordered semigroups

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Received: 08.09.2016 • **Accepted/Published Online:** 27.10.2016 • **Final Version:** 28.09.2017

Abstract: The aim of writing this paper is given in the title. We want to show that not only the ideals but also the ideal elements play an essential role in studying the structure of some ordered semigroups. We first prove that a $\vee e$-semigroup S is a semilattice of left simple $\vee e$-semigroups if and only if it is decomposable into some pairwise disjoint left simple $\vee e$-subsemigroups of S indexed by a semilattice Y. Then we give an example of a semilattice of left simple $\vee e$-semigroups that leads to a characterization of the semilattices of left simple and the chains of left simple $\vee e$-semigroups in terms of left ideal elements.

Key words: $\vee e$-semigroup, left (right) ideal element, left (right) regular, semilattice (chain) of left simple $\vee e$-semigroups

1. Introduction and prerequisities

It is well known that the ideals of ordered semigroups play an essential role in studying their structure. Our aim is to show the role of the ideal elements in studying some ordered semigroups. An ordered groupoid (shortly *po*-groupoid) is an ordered set $(S, \leq)$ with a multiplication "." that is compatible with the ordering (that is, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$). The concept of the left (resp. right) ideal element of a *po*-groupoid S has been introduced in [1] as an element a of S such that $xa \leq a$ (resp. $ax \leq a$) for every $x \in S$. An element that is both a left and a right ideal element is called an ideal element. An ordered groupoid in which the multiplication is associative is called an ordered semigroup (*po*-semigroup) [1,3]. Every ordered set S such that, for any $a, b \in S$, the $\sup\{a, b\}$ exists in S is called an upper semilattice (or a $\vee$-semilattice). If S is a $\vee$-semigroup, a subset T of S is called a subsemigroup of S if, for every $a, b \in T$, we have $ab \in T$ and $a \vee b \in T$. If the $\vee$-subsemigroup T of S has a greatest element, say f, then we say that T is a $\vee e$-subsemigroup of S. If S and T are two $\vee e$-semigroups, by a homomorphism of S into T we clearly mean any mapping $\varphi : S \to T$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for every $x, y \in S$. A $\vee$-semigroup is a semigroup S and at the same time a $\vee$-semilattice such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for every $a, b, c \in S$ [1,3]. As one can easily see, every $\vee$-semigroup is a *po*-semigroup. By a $\vee e$-semigroup we mean a $\vee$-semigroup possessing a greatest element usually denoted by e (i.e. $e \geq a$ for all $a \in S$).

We consider a $\vee e$-semigroup S and introduce the concept of a congruence on S as an equivalence relation σ on S such that $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$, $(ca, cb) \in \sigma$ and $(a \vee c, b \vee c) \in \sigma$ for every $c \in S$. By a

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2010 *AMS Mathematics Subject Classification:* 06F05.

semilattice congruence on S, we mean a congruence σ on S such that $(a^2,a)\in\sigma$ and $(ab,ba)\in\sigma$ for all $a,b\in S$. A $\vee e$-semigroup S is said to be a semilattice of $\vee e$-semigroups of a given type, say $\mathcal{T}$, if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a $\vee e$-subsemigroup of S of type $\mathcal{T}$ for every $x\in S$. We first prove that a $\vee e$-semigroup S is a semilattice of $\vee e$-semigroups of type $\mathcal{T}$ if and only if it is decomposable into pairwise disjoint $\vee e$-subsemigroups S_α of S of type $\mathcal{T}$, indexed by a set Y that is semilattice under two operations "." and "$\vee$" satisfying $S_\alpha S_\beta\subseteq S_{\alpha\beta}$ and $S_\alpha\vee S_\beta\subseteq S_{\alpha\vee\beta}$ $\forall\ \alpha,\beta\in Y$, where $S_\alpha\vee S_\beta$ is the set $\{a\vee b\mid a\in S_\alpha,\, b\in S_\beta\}$. Then we prove that if a semigroup $(S,.)$ is a semilattice of left simple semigroups, then the set of nonempty subsets of S with the multiplication induced by the multiplication on S and the inclusion relation is a semilattice of left simple $\vee e$-semigroups. This being an example of a semilattice of left simple semigroups leads to a characterization of the semilattices of left simple and the chains of left simple $\vee e$-semigroups. A $\vee e$-semigroup S is said to be a chain of left simple semigroups if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ is a left simple $\vee e$-subsemigroup of S for every $x\in S$ and the set S/σ of all σ-classes of S endowed with the order relation $(x)_\sigma\le(y)_\sigma \Leftrightarrow (x)_\sigma=(xy)_\sigma$ is a chain. We prove that a $\vee e$-semigroup S is a semilattice of left simple $\vee e$-semigroups if and only if S is left regular and the left ideal elements of S are two-sited, equivalently, if the set of left ideal elements of S endowed with the operation "." of S is a semilattice (idempotent and commutative semigroup). We also prove that a $\vee e$-semigroup S is a semilattice of left simple $\vee e$-semigroups if and only if for any two left ideal elements a and b of S the element ab is the infimum of the elements a and b. Finally, we characterize the chain of left simple $\vee e$-semigroups in terms of left ideal elements. Characterizations of ordered semigroups that are semilattices of left simple semigroups using left ideals have been studied in [6,7]. A characterization of the chains of right simple semigroups in ordered semigroups in terms of right ideals has been given in [5]. In the present paper, the left (right) ideal elements instead of left (right) ideals play the essential role. The Theorem 7 in [8] can be also obtained either as an application of the Theorem 6 in [6] (see [6; Corollary 8]) or as an application of the Theorem 9 of the present paper.

For the sake of completeness, let us give the following definitions regarding the nonordered semigroups needed in Theorem 7: If $(S,.)$ is a semigroup, a nonempty subset A of S is called a left (resp. right) ideal of S if $SA\subseteq A$ (resp. $AS\subseteq A$). We denote by $\mathcal{P}(S)$ the set of (all) nonempty subsets of S and by $L(A)$ the left ideal of S generated by A. An equivalence relation σ on S is called semilattice congruence if, for any $a,b,c\in S$, $(a,b)\in\sigma$ implies $(ac,bc)\in\sigma$ and $(ca,cb)\in\sigma$; $(a^2,a)\in\sigma$ and $(ab,ba)\in\sigma$. A semigroup $(S,.)$ is called a semilattice of semigroups of type $\mathcal{T}$ if S is the union of a semilattice Ω of semigroups S_α $(\alpha\in\Omega)$, where each S_α is of type $\mathcal{T}$ [2]. This is equivalent to saying that there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S of type $\mathcal{T}$ for every $x\in S$. A semigroup S is called left (resp. right) simple if S is the only left (resp. right) ideal of S, that is, if T is a left (resp. right) ideal of S, then $T=S$. A semigroup that is idempotent (that is, $a^2=a\ \forall\ a\in S$) and commutative is called semilattice. For further information we refer to [1,4,6] (for *po*-semigroups) and to [2] (for semigroups).

2. On semilattices of semigroups of type $\mathcal{T}$

A $\vee e$-semigroup is a semigroup S and at the same time a $\vee$-semilattice having a greatest element usually denoted by "e" such that

$$a(b\vee c)=ab\vee ac \text{ and } (a\vee b)=ac\vee bc$$

for every $a,b,c\in S$.

Definition 1. Let S be a $\vee e$-semigroup. An equivalence relation σ on S is called *congruence* if $(a,b)\in\sigma$ implies $(ac,bc)\in\sigma$, $(ca,cb)\in\sigma$ and $(a\vee c,b\vee c)\in\sigma$ for every $c\in S$.

Proposition 2. *Let S be a $\vee e$-semigroup and σ a congruence on S. Then the set S/σ of all σ-classes of S endowed with the operations*

$$(x)_\sigma.(y)_\sigma=(xy)_\sigma \text{ and } (x)_\sigma\vee(y)_\sigma=(x\vee y)_\sigma$$

is a $\vee e$-semigroup.

Proof. Since σ is a congruence on S, the operations "." and "$\vee$" on S/σ are well defined. The operation "." is associative and so $(S/\sigma,.)$ is a semigroup. The operation "$\vee$" is idempotent, that is $(x)_\sigma\vee(x)_\sigma=(x)_\sigma$ for all $x\in S$, commutative, and associative; thus the set S/σ endowed with the relation

$$(x)_\sigma\le(y)_\sigma \;\Leftrightarrow\; (x\vee y)_\sigma=(y)_\sigma$$

is an ordered set such that $\sup\{(x)_\sigma,(y)_\sigma\}=(x\vee y)_\sigma$ for all $x,y\in S$; in other words, S/σ is an upper semilattice. If e is the greatest element of S, then the class $(e)_\sigma$ is the greatest element of S/σ. Moreover, for all $x,y,z\in S$, we have

$$\begin{aligned}(x)_\sigma.\Big((y)_\sigma\vee(z)_\sigma\Big) &= (x)_\sigma.(y\vee z)_\sigma=\Big(x(y\vee z)\Big)_\sigma=(xy\vee xz)_\sigma\\ &= (xy)_\sigma\vee(xz)_\sigma=(x)_\sigma.(y)_\sigma\vee(x)_\sigma.(z)_\sigma.\end{aligned}$$

Similarly we get $\Big((x)_\sigma\vee(y)_\sigma\Big).(z)_\sigma=(x)_\sigma.(z)_\sigma\vee(y)_\sigma.(z)_\sigma$ and so S/σ is a $\vee e$-semigroup. □

Definition 3. Let S be a $\vee e$-semigroup. A congruence σ on S is called *semilattice congruence* if $(a^2,a)\in\sigma$ and $(ab,ba)\in\sigma$ for every $a,b\in S$.

We can easily prove that if σ is a semilattice congruence on S, then the σ-class $(x)_\sigma$ of S containing x $(x\in S)$ is a subsemigroup of S.

Definition 4. A $\vee e$-semigroup S is said to be a *semilattice of $\vee e$-semigroups of a given type, say $\mathcal{T}$*, if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a $\vee e$-subsemigroup of S of type $\mathcal{T}$ for every $x\in S$.

The next Theorem characterizes the semilattices of $\vee e$-semigroups of type $\mathcal{T}$ and so can be applied to semilattices of left simple $\vee e$-semigroups.

Theorem 5. *Let S be a $\vee e$-semigroup. The following are equivalent:*

1. *There exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ is a $\vee e$-subsemigroup of S of type $\mathcal{T}$ for every $x\in S$.*

2. *There exists a $\vee e$-semigroup Y that is a semilattice (i.e. idempotent and commutative semigroup) under two operations "." and "$\vee$" and a homomorphism $\varphi: S\to Y$ such that the set $\varphi^{-1}(\{\alpha\})$ is a $\vee e$-subsemigroup of S of type $\mathcal{T}$ for every $\alpha\in Y$.*

3. *There exists a semilattice Y under two operations "." and "$\vee$" and a family $\{S_\alpha\}_{\alpha\in Y}$ of $\vee e$-subsemigroups of S of type $\mathcal{T}$ such that*

$S_\alpha \cap S_\beta = \emptyset \quad \forall\, \alpha, \beta \in Y\,,\ \alpha \neq \beta\,;$

$S = \bigcup\limits_{\alpha \in Y} S_\alpha \ ;$

$S_\alpha S_\beta \subseteq S_{\alpha\beta} \quad \forall\, \alpha, \beta \in Y\,;$

$S_\alpha \vee S_\beta \subseteq S_{\alpha\vee\beta} \quad \forall\, \alpha, \beta \in Y$, where $S_\alpha \vee S_\beta := \{a \vee b \mid a \in S_\alpha,\, b \in S_\beta\}$.

Proof. $1 \Longrightarrow 2$. Let σ be a semilattice congruence on S such that $(x)_\sigma$ is a $\vee e$-subsemigroup of S of type $\mathcal{T}$ for every $x \in S$. As we have already seen, the set $Y := S/\sigma$ endowed with the operation "." (resp. "$\vee$") defined in Proposition 2 is a semigroup and $(Y, \vee)$ is a semilattice (i.e. idempotent and commutative). The semigroup $(Y, .)$ is also a semilattice. Indeed: Let $x, y \in S$. Since $(x^2, x) \in \sigma$, we have $(x)_\sigma.(x)_\sigma = (x^2)_\sigma = (x)_\sigma$ and so $(S/\sigma, .)$ is idempotent. Since $(xy, yx) \in \sigma$, we have $(x)_\sigma.(y)_\sigma = (xy)_\sigma = (yx)_\sigma = (y)_\sigma.(x)_\sigma$ and so $(S/\sigma, .)$ is commutative. For the mapping $\varphi : S \to Y \mid x \to (x)_\sigma$ (of the $\vee e$-semigroup S into the $\vee e$-semigroup Y), and any $x, y \in S$, we have $\varphi(xy) = (xy)_\sigma = (x)_\sigma (y)_\sigma = \varphi(x).\varphi(y)$ and

$\varphi(x \vee y) = (x \vee y)_\sigma = (x)_\sigma \vee (y)_\sigma = \varphi(x) \vee \varphi(y)$.

Let now $\alpha \in Y$. Then $\alpha = (x)_\sigma$ for some $x \in S$, and $\varphi^{-1}(\{\alpha\}) = (x)_\sigma$. Indeed: If $t \in \varphi^{-1}(\{\alpha\})$, then $t \in S$, $\varphi(t) = \alpha$. Since $t \in S$, we have $\varphi(t) = (t)_\sigma$. Since $(t)_\sigma = \alpha = (x)_\sigma$, we have $t \in (x)_\sigma$. If now $t \in (x)_\sigma$, then $t \in S$, $\varphi(t) = (t)_\sigma = (x)_\sigma = \alpha$ and so $t \in \varphi^{-1}(\{\alpha\})$. Since $(x)_\sigma$ is a $\vee e$-subsemigroup of S of type $\mathcal{T}$, $\varphi^{-1}(\{\alpha\})$ is a $\vee e$-subsemigroup of S of type $\mathcal{T}$ as well.

$2 \Longrightarrow 3$. We put $\varphi^{-1}(\{\alpha\})$ for each $\alpha \in Y$. Then $\{S_\alpha\}_{\alpha \in Y}$ is a family of $\vee e$-subsemigroups of S of type $\mathcal{T}$. Moreover, the following assertions are satisfied:

(a) Let $\alpha, \beta \in Y$, $\alpha \neq \beta$. Then $\varphi^{-1}(\{\alpha\}) \cap \varphi^{-1}(\{\beta\}) = \emptyset$. Indeed, if $t \in \varphi^{-1}(\{\alpha\}) \cap \in \varphi^{-1}(\{\beta\})$, then $\varphi(t) = \alpha$, $\varphi(t) = \beta$ and so $\alpha = \beta$, which is impossible.

(b) $S = \bigcup\limits_{\alpha \in Y} \varphi^{-1}(\{\alpha\})$. Indeed: If $\alpha \in Y$, then $\{\alpha\} \subseteq Y$, so $\varphi^{-1}(\{\alpha)\} \subseteq S$, and $\bigcup\limits_{\alpha \in Y} \varphi^{-1}(\{\alpha\}) \subseteq S$. If $x \in S$, then for the element $\alpha := \varphi(x) \in Y$, we have $x \in \varphi^{-1}(\{\alpha\}) \subseteq \bigcup\limits_{\alpha \in Y} \varphi^{-1}(\{\alpha\})$.

(c) Let $\alpha, \beta \in Y$. Then $\varphi^{-1}(\{\alpha\})\varphi^{-1}(\{\beta\}) \subseteq \varphi^{-1}(\{\alpha\beta\})$. Indeed: If $t = xy$, for some $x \in \varphi^{-1}(\{\alpha\})$, $y \in \varphi^{-1}(\{\beta\})$, then $\varphi(t) = \varphi(xy) = \varphi(x)\varphi(y) = \alpha\beta$; thus $t \in \varphi^{-1}(\{\alpha\beta\})$.

(d) Let $\alpha, \beta \in Y$, $a \in \varphi^{-1}(\{\alpha\})$ and $b \in \varphi^{-1}(\{\beta\})$. Then $a \vee b \in \varphi^{-1}(\{\alpha \vee \beta\})$. Indeed, since $\varphi(a) = \alpha$, $\varphi(b) = \beta$, we have $\alpha \vee \beta = \varphi(a) \vee \varphi(b) = \varphi(a \vee b)$ and so $a \vee b \in \varphi^{-1}(\{\alpha \vee \beta\})$.

$3 \Longrightarrow 1$. We define a relation σ on S as follows:

$$\sigma := \{(x, y) \in S \times S \mid \exists\, \alpha \in Y : x, y \in S_\alpha\}.$$

(i) σ is a semilattice congruence on S. In fact: σ is clearly reflexive and symmetric. Let $(x, y) \in \sigma$, $(y, z) \in \sigma$. Suppose $\alpha \in Y$ such that $x, y \in S_\alpha$ and $\beta \in Y$ such that $y, z \in S_\beta$. Since $y \in S_\alpha \cap S_\beta$, we have $\alpha = \beta$, $x, z \in S_\alpha$ and $(x, z) \in \sigma$ and so σ is transitive. Let $(x, y) \in \sigma$ and $z \in S$. If $\alpha \in Y$ such that $x, y \in S_\alpha$ and $\beta \in Y$ such that $z \in S_\beta$, then $xz, yz \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $x \vee z, y \vee z \in S_\alpha \vee S_\beta \subseteq S_{\alpha\vee\beta}$, where $\alpha\beta \in Y$ and $\alpha \vee \beta \in Y$ and so $(xz, yz) \in \sigma$ and $(x \vee z, y \vee z) \in \sigma$. Similarly $(zx, zy) \in \sigma$. If $x \in S$ and $\alpha \in Y$ such that $x \in S_\alpha$, then $x^2 \in S_\alpha S_\alpha \subseteq S_{\alpha^2} = S_\alpha$. Since $x^2, x \in S_\alpha$, $\alpha \in Y$, we have $(x^2, x) \in \sigma$. If $x, y \in S$ and $\alpha, \beta \in Y$ such that $x \in S_\alpha$, $y \in S_\beta$, then $xy \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $yx \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$. Since $xy, yx \in S_{\alpha\beta}$ and $\alpha\beta \in Y$, we have $(xy, yx) \in \sigma$.

(ii) Let $x \in S$. Then $(x)_\sigma$ is a $\vee e$-subsemigroup of S of type $\mathcal{T}$. In fact: Since $x \in S$, $x \in S_\alpha$ for some $\alpha \in Y$. We have $(x)_\sigma = S_\alpha$. Indeed: If $y \in (x)_\sigma$, then $(y,x) \in \sigma$ and so there exists $\beta \in Y$ such that $y,x \in S_\beta$. Since $x \in S_\alpha \cap S_\beta$, we have $\alpha = \beta$, and $y \in S_\alpha$. If $y \in S_\alpha$, then $y,x \in S_\alpha$ $(\alpha \in Y)$, then $(x,y) \in \sigma$ and $y \in (y)_\sigma = (x)_\sigma$. $\square$

3. On semilattices and chains of left simple $\vee e$-semigroups

We deal here with a decomposition of left simple $\vee e$-semigroups into left simple components using the left ideal elements.

Definition 6. [4] An element a of a $\vee e$-semigroup S is called a *left* (resp. *right*) *ideal element* if $ea \le a$ (resp. $ae \le a$). A $\vee e$-semigroup S is called *left simple* if for every left ideal element a of S we have $a = e$.

Theorem 7. *If a semigroup $(S,.)$ is a semilattice of left simple semigroups, then the $\vee e$-semigroup $(\mathcal{P}(S),.,\subseteq)$ is a semilattice of left simple $\vee e$-semigroups.*

Proof. Let σ be a semilattice congruence on $(S,.)$ such that $(x)_\sigma$ is a left simple subsemigroup of $(S,.)$ for every $x \in S$. The set $\mathcal{P}(S)$ of all subsets of S with the multiplication on $\mathcal{P}(S)$ induced by the multiplication of S and the inclusion relation forms a $\vee e$-semigroup (where S is the greatest element of $\mathcal{P}(S)$). That is, $(\mathcal{P}(S),.,\subseteq)$ is a $\vee e$-semigroup. Let $\mathcal{L}$ be the equivalence relation on $\mathcal{P}(S)$ defined by

$$\mathcal{L} := \{(A,B) \in \mathcal{P}(S) \times \mathcal{P}(S) \mid L(A) = L(B)\}.$$

First of all, we prove that $SX \cup X = SX$, equivalently, $X \subseteq SX$ for every $X \subseteq S$. In fact, let $X \subseteq S$ and $x \in X$. The set $SX \cap (x)_\sigma$ is a left ideal of $((x)_\sigma,.)$. This is because it is a nonempty subset of $(x)_\sigma$ (as $x^2 \in SX \cap (x)_\sigma$) and

$$(x)_\sigma(SX \cap (x)_\sigma) \subseteq (x)_\sigma SX \cap (x)_\sigma^2 \subseteq S^2X \cap (x)_\sigma \subseteq SX \cap (x)_\sigma.$$

Since $((x)_\sigma,.)$ is left simple, we have $SX \cap (x)_\sigma = (x)_\sigma$; thus $x \in SX$. Hence we have

$$\mathcal{L} = \{(A,B) \in \mathcal{P}(S) \times \mathcal{P}(S) \mid SA = SB\}.$$

The following assertions are satisfied:

(1) $\mathcal{L}$ is a semilattice congruence on $(\mathcal{P}(S),.,\subseteq)$. In fact:

Let $(A,B) \in \mathcal{L}$ and $C \in \mathcal{P}(\mathcal{S})$. Since $SA = SB$, we have $SAC = SBC$ and so $(AC,BC) \in \mathcal{L}$. Since $S(A \cup C) = SA \cup SC = SB \cup SC = S(B \cup C)$, we have $(A \cup C, B \cup C) \in \mathcal{L}$. We also have $(CA,CB) \in \mathcal{L}$, that is $S(CA) = S(CB)$. Indeed: it is enough to prove that $SC = SCS$. Then we have

$$\begin{aligned} S(CA) &= (SC)A = (SCS)A = (SC)(SA) = (SC)(SB) \\ &= (SCS)B = (SC)B = S(CB). \end{aligned}$$

We have $SC = SCS$. In fact: Let $x \in S$, $y \in C$. The set $SCS \cap (xy)_\sigma$ is a left ideal of $(xy)_\sigma$. This is because $\emptyset \ne SCS \cap (xy)_\sigma \subseteq (xy)_\sigma$ (as $xy^2 \in SCS$, $xy^2 \in (xy^2)_\sigma = (x)_\sigma(y^2)_\sigma = (x)_\sigma(y)_\sigma = (xy)_\sigma$) and

$$\begin{aligned} (xy)_\sigma(SCS \cap (xy)_\sigma) &\subseteq (xy)_\sigma SCS \cap (xy)_\sigma^2 \subseteq S^2CS \cap (xy)_\sigma \\ &\subseteq SCS \cap (xy)_\sigma. \end{aligned}$$

Since $(xy)_\sigma$ is left simple, we have $SCS \cap (xy)_\sigma = (xy)_\sigma$ and so $xy \in SCS$, and $SC \subseteq SCS$. On the other hand, $CS \subseteq SC$. Indeed: Let $x \in C$, $y \in S$. The set $SC \cap (xy)_\sigma$ is a left ideal of $(xy)_\sigma$. This is because

$\emptyset \neq SC \cap (xy)_\sigma \subseteq (xy)_\sigma$ (since $yx \in SC,\ yx \in (xy)_\sigma$) and

$(xy)_\sigma(SC \cap (xy)_\sigma) \subseteq (xy)_\sigma SC \cap (xy)_\sigma^2 \subseteq S^2C \cap (xy)_\sigma \subseteq SC \cap (xy)_\sigma$.

Thus we get $SC \cap (xy)_\sigma = (xy)_\sigma$, and $xy \in SC$. Since $CS \subseteq SC$, we have $SCS \subseteq S^2C \subseteq SC$, and $SC = SCS$.

Let $A \in \mathcal{P}(S)$. Then $(A^2, A) \in \mathcal{L}$, that is $SA^2 = SA$. In fact: Let $x \in S$, $y \in A$. The set $SA^2 \cap (xy)_\sigma$ is a left ideal of $(xy)_\sigma$. Indeed,

$\emptyset \neq SA^2 \cap (xy)_\sigma \subseteq (xy)_\sigma$ (since $xy^2 \in SA^2$, $xy^2 \in (xy)_\sigma$) and

$(xy)_\sigma(SA^2 \cap (xy)_\sigma) \subseteq (xy)_\sigma SA^2 \cap (xy)_\sigma^2 \subseteq S^2A^2 \cap (xy)_\sigma \subseteq SA^2 \cap (xy)_\sigma$.

Then $SA^2 \cap (xy)_\sigma = (xy)_\sigma$, and $xy \in SA^2$. Thus we have $SA \subseteq SA^2 \subseteq SA$, and $SA^2 = SA$.

Let $A, B \in \mathcal{P}(S)$. Then $(AB, BA) \in \mathcal{L}$, that is $S(AB) = S(BA)$. In fact: Let $x \in S$, $a \in A$, $b \in B$. The set $SBA \cap (xab)_\sigma$ is a left ideal of $(xab)_\sigma$. Indeed,

$\emptyset \neq SBA \cap (xab)_\sigma \subseteq (xab)_\sigma$ (since $xba \in SBA$, $xba \in (xba)_\sigma)$) and

$(xab)_\sigma(SBA \cap (xab)_\sigma) \subseteq (xab)_\sigma SBA \cap (xab)_\sigma^2 \subseteq SBA \cap (xab)_\sigma$.

Then $SBA \cap (xab)_\sigma = (xab)_\sigma$, and $xab \in SBA$. Thus we have $SAB \subseteq SBA$. By symmetry, we get $SBA \subseteq SAB$; thus $S(AB) = S(BA)$.

(2) $(A)_\mathcal{L}$ is a left simple $\vee e$-subsemigroup of $(\mathcal{P}(S), ., \subseteq)$ for every $A \in \mathcal{P}(\mathcal{S})$.

In fact: Let $A \in \mathcal{P}(\mathcal{S})$ and $X, Y \in (A)_\mathcal{L}$. Since $X\mathcal{L}A$, $Y\mathcal{L}A$, we have $XY\mathcal{L}AY$, $AY\mathcal{L}A^2$ and then $XY\mathcal{L}A^2$. Since $XY\mathcal{L}A^2$ and $A^2\mathcal{L}\mathcal{A}$, we have $XY\mathcal{L}A$ and so $XY \in (A)_\mathcal{L}$, and $(A)_\mathcal{L}$ is a subsemigroup of $\mathcal{P}(S)$. Again since $X\mathcal{L}A$, $Y\mathcal{L}A$, we have $(X \cup Y)\mathcal{L}(A \cup Y)$ and $(A \cup Y)\mathcal{L}(A \cup A) = A$; then $(X \cup Y)\mathcal{L}A$, and $X \cup Y \in (A)_\mathcal{L}$ and so $(A)_\mathcal{L}$ is a $\vee$-subsemigroup of $\mathcal{P}(S)$. The set SA is the greatest element of $(A)_\mathcal{L}$. In fact: First of all, $SA \in (A)_\mathcal{L}$, that is $S(SA) = SA$. Indeed: Let $x \in S$. The set $S^2 \cap (x)_\sigma$ is a left ideal of $(x)_\sigma$. This is because $\emptyset \neq S^2 \cap (x)_\sigma \subseteq (x)_\sigma$ (since $x^2 \in S^2$, $x^2 \in (x)_\sigma$) and $(x)_\sigma(S^2 \cap (x)_\sigma) \subseteq (x)_\sigma S^2 \cap (x)_\sigma^2 \subseteq S^2 \cap (x)_\sigma$, then $S^2 \cap (x)_\sigma = (x)_\sigma$, and $x \in S^2$; thus $S^2 = S$ and so $S(SA) = SA$. Moreover, $X \subseteq SA$ for every $X \in (A)_\mathcal{L}$. Indeed, if $X \in (A)_\mathcal{L}$, then $SX = SA$, from which $X \subseteq X \cup SX = SX = SA$.

Finally, $(A)_\mathcal{L}$ is left simple. In fact: Let L be a left ideal element of $(A)_\mathcal{L}$. Then $L = SA$. Indeed: Since $L \in (A)_\mathcal{L}$, we have $L\mathcal{L}A$, then $SL = SA$. On the other hand,

$$\begin{aligned} L &\subseteq L \cup SL = SL = SL^2 \text{ (since } L\mathcal{L}L^2) \\ &= (SL)L = (SA)L \\ &\subseteq L \text{ (since } L \text{ is a left ideal of } (A)_\mathcal{L}); \end{aligned}$$

thus we have $L = SL = SA$. □

We are ready now to study the semilattices of left simple $\vee e$-semigroups. We give some properties that characterize the semilattices of left simple $\vee e$-semigroups. Then we apply our results to semigroups.

Definition 8. [4] A $\vee e$-semigroup S is called *left* (resp. *right*) *regular* if $a \leq ea^2$ (resp. $a \leq a^2e$) for every $a \in S$. It is called *left* (resp. *right*) *duo* if the left (resp. right) ideal elements of S are two-sided.

In the following, $l(a)$ denotes the left ideal element of S generated by a ($a \in S$); as one can easily prove, this is the element $ea \vee a$. The right ideal element of S generated by a, denoted by $r(a)$, is the element $ae \vee a$. We denote by F_l (resp. F_r) the set of (all) left (resp. right) ideal elements of S (cf. also [4]).

Theorem 9. *Let S be a $\vee e$-semigroup. The following are equivalent:*

(i) *S is a semilattice of left simple $\vee e$-semigroups.*

(ii) *S is left regular and left duo.*

(iii) *For every $a, b \in F_l$, the $\inf\{a,b\}(:= a \wedge b)$ exists in S, and we have $a \wedge b = ab$.*

(iv) *$(F_l, .)$ is a semilattice.*

Proof. (i) $\Longrightarrow$ (ii). Let σ be a semilattice congruence on S such that $(y)_\sigma$ is a left simple $\vee e$-subsemigroup of S for every $y \in S$. Let $x \in S$ and f the greatest element of $(x)_\sigma$. Since $f, x^2 \in (x)_\sigma$, we have $fx^2 \in (x)_\sigma$. Since $f(fx^2) = f^2x^2 \leq fx^2$, fx^2 is a left ideal element of $(x)_\sigma$. Since $(x)_\sigma$ is left simple, we have $fx^2 = f$. Thus we have $x \leq f = fx^2 \leq ex^2$, and S is left regular. Let now a be a left ideal element of S and f the greatest element of $(a)_\sigma$. Since $fa \leq ea \leq a$, the element a is a left ideal element of $(a)_\sigma$. Since $(a)_\sigma$ is left simple, we have $a = f$. Since $ea \leq a \leq ea^2 \leq ea$, we have $ea = a$; then $(a)_\sigma = (ea)_\sigma = (ae)_\sigma$. Since $ae \in (a)_\sigma$, we get $ae \leq f = a$, and a is a right ideal element of S.

(ii) $\Longrightarrow$ (iii). Let $a, b \in F_l$. Since $a \in F_l \subseteq F_r$, we have $ab \leq ae \leq a$. Since $b \in F_l$, we have $ab \leq eb \leq b$. Let now $t \in S$ such that $t \leq a$ and $t \leq b$. Since S is left regular, we have $t \leq et^2 \leq eab \leq ab$. Thus the element ab is the infimum of the elements a and b in S.

(iii) $\Longrightarrow$ (iv). Since $e \in F_l$, F_l is a nonempty subset of S. Let now $a, b \in F_l$. Then $e(ab) = (ea)b \leq ab$ and so $ab \in F_l$, and F_l is a subsemigroup of S. Moreover, by (iii), $\inf\{a,b\} = ab$ and $\inf\{b,a\} = ba$. Since $\inf\{a,b\} = \inf\{b,a\}$, we have $ab = ba$ and so F_l is commutative. By (iii), we also get $a = \inf\{a,a\} = a^2$ and so F_l is idempotent. Thus $(F_l, .)$ is a semilattice.

(iv) $\Longrightarrow$ (i). We consider the equivalence relation on S defined as follows:

$$\mathcal{L} := \{(x,y) \in S \times S \mid l(x) = l(y)\}.$$

We remark that $l(x) = ex$ for every $x \in S$. Indeed, let $x \in S$. Since $l(x)$ is a left ideal element of S, by (iv), we have

$$\begin{aligned} l(x) &= (l(x))^2 = (ex \vee x)(ex \vee x) = exex \vee xex \vee ex^2 \vee x^2 \\ &\leq ex \leq ex \vee x = l(x), \end{aligned}$$

and so $l(x) = ex$. Thus we have

$$\mathcal{L} = \{(x,y) \in S \times S \mid ex = ey\}.$$

Then we have the following:

(1) $\mathcal{L}$ is a semilattice congruence on S.

In fact: Let $(a,b) \in \mathcal{L}$, $c \in S$. Since $ea = eb$, we have $(ea)c = (eb)c$; then $e(ac) = e(bc)$, and $(ac, bc) \in \mathcal{L}$. On the other hand, by (iv), we have

$$\begin{aligned} e(ca) &\leq ecl(a) = (ec)ea = (ec)eb = ((ec)e)b = (e(ec))b \\ &= e^2cb \leq e(cb), \end{aligned}$$

by symmetry, $e(cb) \leq e(ca)$; thus $e(ca) = e(cb)$ and $(ca, cb) \in \mathcal{L}$. Moreover, since $e(a \vee c) = ea \vee ec = eb \vee ec = e(b \vee c)$, we get $(a \vee c, b \vee c) \in \mathcal{L}$.
Let now $a, b \in S$. Since $ea \in F_l$, by (iv), we have $ea = (ea)^2$. Then we have

$$ea = (ea)(ea) = ((ea)e)a = (e(ea))a = e^2a^2 \leq ea^2 \leq ea,$$

and so $ea^2 = ea$ and $(a^2, a) \in F_l$. Also

$$\begin{aligned} e(ab) &= (eab)(eab) \leq eb(ea)e = ebe(ea) \text{ (since } ea, e \in F_l) \\ &\leq (eb)ea = e(eb)a \leq e(ba), \end{aligned}$$

by symmetry $e(ba) \leq e(ab)$; then $(ab, ba) \in \mathcal{L}$.

(2) $(x)_{\mathcal{L}}$ is a left simple $\vee e$-subsemigroup of S for every $x \in S$.
In fact: Let $x \in S$. The (nonempty set) $(x)_{\mathcal{L}}$ is a $\vee$-subsemigroup of S. Indeed: Let $a, b \in (x)_{\mathcal{L}}$. Since $a\mathcal{L}x$, $b\mathcal{L}x$, we have $ab\mathcal{L}xb$, $xb\mathcal{L}x^2$. Since $x^2\mathcal{L}x$, we get $ab\mathcal{L}x$, and $ab \in (x)_{\mathcal{L}}$. On the other hand, $a\mathcal{L}x$ and $b\mathcal{L}x$ imply $a \vee b\mathcal{L}x \vee b$ and $x \vee b\mathcal{L}x^2$; besides, we have $x^2\mathcal{L}x$ and so we get $a \vee b\mathcal{L}x$ and $a \vee b \in (x)_{\mathcal{L}}$. The element ex is the greatest element of $(x)_{\mathcal{L}}$. Indeed: Since $e \in F_l$, by (iv), we have $e^2 = e$. Then $e(ex) = e^2x = ex$ and so $(ex, x) \in \mathcal{L}$ and $ex \in (x)_{\mathcal{L}}$. If $a \in (x)_{\mathcal{L}}$, then $a\mathcal{L}x$ i.e. $ea = ex$ and so $a \leq l(a) = ea = ex$. Moreover $(x)_{\mathcal{L}}$ is left simple. Indeed: Let a be a left ideal element of $(x)_{\mathcal{L}}$. Since $a \in (x)_{\mathcal{L}}$, we have $a\mathcal{L}x$; then $ea = ex$. Since $a\mathcal{L}a^2$, we have

$$a \leq l(a) = ea = ea^2 = (ea)a = (ex)a.$$

Since a is a left ideal element of $(x)_{\mathcal{L}}$ and ex the greatest element of $(x)_{\mathcal{L}}$, we have $(ex)a \leq a$. Then we have $a = ea = ex$. □

The characterization of a semigroup that is both left regular and left duo as a semilattice of simple semigroups given in [8] can be also obtained as an application of the Theorem 9 of the present paper.

Corollary 10. ([8; the Theorem]) *Let $(S,.)$ be a semigroup. The following are equivalent:*

(i) *S is a semilattice of left simple semigroups.*

(ii) *S is left regular and every left ideal of S is two-sided.*

(iii) *For all left ideals A, B of S, we have $A \cap B = AB$.*

(iv) *The set of left ideals of S is a semilattice.*

Proof. (i) $\Longrightarrow$ (ii). Let $(S,.)$ be a semilattice of left simple semigroups. By Theorem 7, the $\vee e$-semigroup $(\mathcal{P}(S), ., \subseteq)$ is a semilattice of left simple $\vee e$-semigroups. By Theorem 9(i)$\Rightarrow$(ii), the $\vee e$-semigroup $(\mathcal{P}(S), ., \subseteq)$ is left regular and every left ideal element of $(\mathcal{P}(S), ., \subseteq)$ is two-sided. Then the semigroup $(S,.)$ is left regular and every left ideal of $(S,.)$ is two-sided.

(ii) $\Longrightarrow$ (iii). Let $(S,.)$ be left regular and every left ideal of S is two-sided. Then $(\mathcal{P}(S), ., \subseteq)$ is left regular and every left ideal element of $\mathcal{P}(S)$ is two-sided. By Theorem 9(ii)$\Rightarrow$(iii), for all left ideal elements A, B of $\mathcal{P}(S)$ we have $\inf\{A, B\} = AB$. Then for all left ideals A, B of S we have $\inf\{A, B\} = AB$. Clearly, $\inf\{A, B\} = A \cap B$.

(iii) $\Longrightarrow$ (iv). Suppose for all left ideals A, B of S we have $A \cap B = AB$. Then for all left ideal elements A, B of $\mathcal{P}(S)$ we have $\inf\{A, B\} = A \cap B = AB$. By Theorem 9(iii)$\Rightarrow$(iv), the set of left ideal elements of $\mathcal{P}(S)$ is a semilattice. Then the set of left ideal elements of S is a semilattice.

(iv) $\Longrightarrow$ (i). Suppose the set of left ideals of S is a semilattice. Then the set of left ideal elements of $\mathcal{P}(S)$ is a semilattice. According to the proof of Theorem 9(iv)$\Rightarrow$(i), the relation $\mathcal{L}$ on $\mathcal{P}(S)$ defined by

$$\mathcal{L} := \{(A, B) \mid L(A) = L(B)\}$$

is equal to

$$\mathcal{L} = \{(A, B) \mid SA = SB\}$$

and it is a semilattice congruence on $\mathcal{P}(S)$ such that $(A)_{\mathcal{L}}$ is a left simple $\vee e$-subsemigroup of $\mathcal{P}(S)$ for every $A \in \mathcal{P}(S)$. We consider the relation $\mathcal{L}_S$ on S defined by

$$\mathcal{L}_S := \{(a, b) \mid (\{a\}, \{b\}) \in \mathcal{L}\}.$$

One can easily prove that $\mathcal{L}_S$ is a semilattice congruence on S and that the set $(a)_{\mathcal{L}_S}$ is a subsemigroup of S for every $a \in S$. Moreover, the class $(a)_{\mathcal{L}_S}$ is a left simple subsemigroup of S. Indeed: Let L be a left ideal of $(a)_{\mathcal{L}_S}$ and $x \in (a)_{\mathcal{L}_S}$. Take an element $z \in L$ $(L \neq \emptyset)$. We prove that $x = (tz)z$ for some $t \in S$ and also that $tz \in (a)_{\mathcal{L}_S}$. Then we have $x \in (a)_{\mathcal{L}_S} L \subseteq L$, and the proof is complete.

Since $x \in (a)_{\mathcal{L}_S}$, we have $(x, a) \in \mathcal{L}_S$; then $(\{x\}, \{a\}) \in \mathcal{L}$ and so $Sx = S\{x\} = S\{a\} = Sa$. Since $z \in (a)_{\mathcal{L}_S}$ and $(a)_{\mathcal{L}_S}$ is a subsemigroup of S, we have $z^2 \in (a)_{\mathcal{L}_S}$; then $(z^2, a) \in \mathcal{L}_S$, $(\{z^2\}, \{a\}) \in \mathcal{L}$ and so $S\{z^2\} = S\{a\}$ and $Sz^2 = Sa$. Since $\{x\} \in \mathcal{P}(S)$ and $\mathcal{L}$ is a semilattice congruence on $\mathcal{P}(S)$, we have $(\{x\}^2, \{x\}) \in \mathcal{L}$; then $L(\{x^2\}) = L(\{x\}$, that is $\{x^2\} \cup S\{x^2\} = \{x\} \cup S\{x\}$. Thus we get $x \in x \cup Sx = x^2 \cup Sx^2 \subseteq Sx$. Therefore, we have $x \in Sx = Sa = Sz^2$ and so $x = (tz)z$ for some $t \in S$. Moreover, since $\mathcal{L}_S$ is a semilattice congruence on S, we have

$$tz \in (t)_{\mathcal{L}_S}(z)_{\mathcal{L}_S} = (t)_{\mathcal{L}_S}(z^2)_{\mathcal{L}_S} = (tz^2)_{\mathcal{L}_S} = (x)_{\mathcal{L}_S} = (a)_{\mathcal{L}_S},$$

and so $tz \in (a)_{\mathcal{L}_S}$. □

Next, we characterize the chains of left simple $\vee e$-semigroups in terms of left ideal elements.

Definition 11. A $\vee e$-semigroup S is called a *chain of left simple semigroups* if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ is a left simple $\vee e$-subsemigroup of S for every $x \in S$ and the set $S/\sigma := \{(x)_\sigma \mid x \in S\}$ endowed with the relation

$$(x)_\sigma \leq (y)_\sigma \iff (x)_\sigma = (xy)_\sigma$$

is a chain. (Recall that, since σ is a semilattice congruence on S, the relation "$\leq$" is an order relation on S/σ).

Theorem 12. *Let S be a $\vee e$-semigroup. The following are equivalent:*

(i) *S is a chain of left simple $\vee e$-semigroups.*

(ii) *The set F_l of the left ideal elements of S endowed with the relation*

$$\preceq := \{(a, b) \mid a, b \in F_l,\ a = ab = ba\}$$

is a chain.

Proof. (i) $\Longrightarrow$ (ii). Let σ be a semilattice congruence on S such that $(x)_\sigma$ is a left simple $\vee e$-subsemigroup of S for every $x \in S$, and $(S/\sigma, \leq)$ is a chain. Since S is a semilattice of left simple $\vee e$-semigroups, by Theorem 9(i)$\Rightarrow$(iv), $(F_l, .)$ is a semilattice. Then we have the following:

(1) $(F_l, \preceq)$ is an ordered set. Indeed: If $a \in F_l$, then $a = a^2$ and so $(a,a) \in \preceq$. If $(a,b) \in \preceq$ and $(b,a) \in \preceq$, then $a = ab = ba$ and $b = ba = ab$ and so $a = b$. If $(a,b) \in \preceq$ and $(b,c) \in \preceq$, then $a = ab = ba$ and $b = bc = cb$; then

$ac = (ab)c = a(bc) = ab = a$ and $ca = c(ba) = (cb)a = ba = a$,

and so $a = ac = ca$, and $(a,c) \in \preceq$.

(2) $(F_l, \preceq)$ is a chain. Indeed: Let $a, b \in F_l$. As $(a)_\sigma, (b)_\sigma \in S/\sigma$ and $(S/\sigma, \leq)$ is a chain, we have $(a)_\sigma = (ab)_\sigma$ or $(b)_\sigma = (ba)_\sigma$. Let $(a)_\sigma = (ab)_\sigma$ and f the greatest element of $(ab)_\sigma$. Since $a, ab \in (ab)_\sigma$, $fa \leq ea \leq a$ and $f(ab) = (fa)b \leq ab$, the elements a and ab are left ideal elements of $(ab)_\sigma$. Since $(ab)_\sigma$ is left simple, we have $ab = f = a$. Since F_l is a semilattice, we get $ab = ba$; thus we have $a = ab = ba$ and $(a,b) \in \preceq$. If $(b)_\sigma = (ba)_\sigma$ then, by symmetry, we obtain $(b,a) \in \preceq$.

(ii) $\Longrightarrow$ (i). $(F_l, .)$ is a semilattice. In fact: Since $e \in F_l$ $(: ee \leq e)$, the set F_l is a nonempty subset of S. If $a, b \in F_l$, then $e(ab) = (ea)b \leq ab$ and so $ab \in F_l$. If $a \in F_l$, then $(a,a) \in \preceq$ and so $a^2 = a$ and if $a, b \in F_l$, then $(a,b) \in \preceq$ or $(b,a) \in \preceq$, that is $a = ab = ba$ or $b = ba = ab$ and so $ab = ba$. Thus $(F_l, .)$ is a commutative and idempotent semigroup, which means that it is a semilattice.

According to the proof of (iv) $\Rightarrow$ (i) of the Theorem 9, the relation

$$\mathcal{L} = \{(a,b) \in S \times S \mid ea = eb\}$$

is a semilattice congruence on S and $(x)_\mathcal{L}$ is a left simple $\vee e$-subsemigroup of S for every $x \in S$. It remains to prove that the set $S/\mathcal{L}$ endowed with the order relation "$\leq$" is a chain. Let now $x, y \in S$. Since $ex, ey \in F_l$ and $(F_l, \preceq)$ is a chain, we have $(ex, ey) \in \preceq$ or $(ey, ex) \in \preceq$. If $(ex, ey) \in \preceq$, then $ex = exey\,(= eyex)$. Since $ex, e \in F_l$ and $(F_l, .)$ is a semilattice, we have

$$exey = (ex)ey = e(ex)y = e^2xy = exy,$$

and so $ex = exy$; then $(x, xy) \in \mathcal{L}$ and $(x)_\mathcal{L} = (xy)_\mathcal{L}$. If $(ey, ex) \in \preceq$, by symmetry, we get $(y)_\mathcal{L} = (yx)_\mathcal{L} = (xy)_\mathcal{L}$. Hence we have $(x)_\mathcal{L} = (xy)_\mathcal{L}$ or $(y)_\mathcal{L} = (xy)_\mathcal{L}$, and $(S/\mathcal{L}, \leq)$ is a chain. $\square$

The right analogues of the above results also hold.

Problem. Using a computer find the (nonisomorphic) left regular and left duo $\vee e$-semigroups of order 5.

Acknowledgment

I would like to express my best thanks to the Editor-in-Chief of the journal Professor Dr Adnan Tercan for editing and communicating the paper, and to the two anonymous referees for reading the paper carefully (something lately not very usual) and their prompt reply.

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