Meromorphic function and its difference operator share two sets with weight k

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Abstract: In this paper, we utilize Nevanlinna value distribution theory to study the uniqueness problem that a meromorphic function and its difference operator share two sets with weight k. Our results extend the previous results.

Key words: Meromorphic function, unicity, shared sets

1. Introduction and notation

In this paper, the term 'meromorphic function' always means meromorphic in the whole complex plane $\mathbb{C}$. Let $f(z)$ be a nonconstant meromorphic function, and we use the standard notation in Nevanlinna's theory of meromorphic functions such as $T(r,f), m(r,f), N(r,f)$, and $\overline{N}(r,f)$ (see, e.g., [2,6]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to +\infty$, possibly outside a set of r of finite measure. In addition, we use $N_k(r,1/f)$ to denote the counting function for the zeros of $f(z)$ with multiplicity m counted m times if $m \leq k$ and $k+1$ times if $m > k$.

We say that f and g share a CM (IM), if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities (ignoring multiplicities).

We also need the following definitions in this paper.

Definition 1 [3] Let k be a positive integer or infinity. For $a \in \hat{\mathbb{C}} (= \mathbb{C} \cup \{\infty\})$ we denote by $E_f(a,k)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

Definition 2 [3] Let k be a positive integer or infinity. If for $a \in \hat{\mathbb{C}}$, $E_f(a,k) = E_g(a,k)$, we say that f, g share the value a with weight k.

From these two definitions, we note that f, g share a CM or IM if and only if f, g share the value a with weight ∞ or 0, respectively.

Definition 3 [3] For $S \subset \hat{\mathbb{C}}$, we define $E_f(S,k)$ as $E_f(S,k) = \bigcup_{a \in S} E_f(a,k)$, where k is a positive integer or infinity.

In this paper, we assume that $S_1 = \{1, \omega, \cdots, \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega^n = 1$ and n is a positive integer.

Investigation of the uniqueness of meromorphic functions sharing sets is an important subfield of uniqueness theory. Yi [7], Li and Yang [5], and Yi and Yang [8] proved several results on the uniqueness problems of two meromorphic functions when they share two sets around 1995. In 2006, Lahiri and Banerjee [4] considered these problems with the idea of weighted sharing of sets.

In what follows, c always means a nonzero constant. For a meromorphic function $f(z)$, we denote its shift and difference operator by $f(z + c)$ and $\Delta_c f := f(z + c) - f(z)$, respectively. Recently, many papers mainly deal with some uniqueness questions for a meromorphic function that shares values or common sets with shift or its difference operator. We recall the following two results proved by Zhang [9] and Chen [1].

In 2010, Zhang [9] proved the following results on the relation between $f(z)$ and its shift $f(z + c)$ when they share two sets.

Theorem A *Let $c \in \mathbb{C}$. Suppose that $f(z)$ is a nonconstant meromorphic function with finite order such that $E_{f(z)}(S_j, \infty) = E_{f(z+c)}(S_j, \infty), (j = 1, 2)$. If $n \geq 4$, then $f \equiv t f(z + c)$, where $t^n = 1$.*

Recently, Chen [1] considered the relation between $f(z)$ and its operator $\Delta_c f$, and obtained the following result.

Theorem B *Let $c \in \mathbb{C}$. Suppose that $f(z)$ is a nonconstant meromorphic function with finite order such that $E_{f(z)}(S_1, 2) = E_{\Delta_c f}(S_1, 2)$ and $E_{f(z)}(S_2, \infty) = E_{\Delta_c f}(S_2, \infty)$. If $n \geq 7$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \neq -1$.*

From Theorem B, Chen [1] got two corollaries as follows.

Corollary A *Let $c \in \mathbb{C}$. Suppose that $f(z)$ is a nonconstant meromorphic function with finite order such that $E_{f(z)}(S_1, \infty) = E_{\Delta_c f}(S_1, \infty)$ and $E_{f(z)}(S_2, \infty) = E_{\Delta_c f}(S_2, \infty)$. If $n \geq 7$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \neq -1$.*

Corollary B *Let $c \in \mathbb{C}$. Suppose that $f(z)$ is a nonconstant entire function with finite order such that $E_{f(z)}(S_1, \infty) = E_{\Delta_c f}(S_1, \infty)$. If $n \geq 5$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \neq -1$.*

It is natural to ask the following questions about Theorem B:

(1) Is the condition "$f(z)$ has finite order" necessary?

(2) Can the assumption "$n \geq 7$" can be replaced by a weaker one?

(3) Theorem B just considers the function shares two sets with weight 2, and what will happen if the function shares the set with weight 1, and, in general, what will happen if the function shares the set with weight k, where k is a positive number or infinity.

In this paper we shall investigate the above problems and give an affirmative answer to the question.

Theorem 1 *Let $c \in \mathbb{C}$ and k be a positive number or infinity. Suppose that $f(z)$ is a nonconstant meromorphic function such that $E_{f(z)}(S_1, k) = E_{\Delta_c f}(S_1, k)$ and $E_{f(z)}(S_2, \infty) = E_{\Delta_c f}(S_2, \infty)$. If $n \geq 7$ when $k = 1$ or $n \geq 5$ when $k \geq 2$ then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \neq -1$.*

From Theorem 1, the following corollary follows directly.

Corollary 1 *Let $c \in \mathbb{C}$ and k be a positive number or infinity. Suppose that $f(z)$ is a nonconstant meromorphic function such that $E_{f(z)}(S_1, \infty) = E_{\Delta_c f}(S_1, \infty)$ and $E_{f(z)}(S_2, \infty) = E_{\Delta_c f}(S_2, \infty)$. If $n \geq 5$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \neq -1$.*

For the entire function, we also get the following result.

Theorem 2 *Let $c \in \mathbb{C}$ and k be a positive number or infinity. Suppose that $f(z)$ is a nonconstant entire function such that $E_{f(z)}(S_1, k) = E_{\Delta_c f}(S_1, k)$. If $n \geq 6$ when $k = 1$ or $n \geq 5$ when $k \geq 2$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \neq -1$.*

2. Preliminary results

For the proof of Theorem 1 and Theorem 2, we need the following results.

Lemma 1 [6] *Let f be a meromorphic functions in $\mathbb{C}$, and let n be a positive number. Then*

$$T(r, f^n) = nT(r, f) + S(r, f).$$

Lemma 2 [6] *Let f be a meromorphic functions in $\mathbb{C}$, and let k be a positive number. Then*

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f).$$

Lemma 3 *Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$, and let k be a positive number or infinity. If $E_F(1, k) = E_G(1, k)$ and $E_F(\infty, \infty) = E_G(\infty, \infty)$, then one of the following cases occurs:*

(i). $T(r, F) + T(r, G) \leq 2\{\overline{N}(r, F) + N_2(r, 1/F) + \overline{N}(r, G) + N_2(r, 1/G)\}$

$$+ \left[\frac{1}{k}\right] \{\overline{N}_{(k+1)}(r, \frac{1}{F-1}) + \overline{N}_{(k+1)}(r, \frac{1}{G-1})\} + S(r, F) + S(r, G).$$

(ii). $F \equiv G$ *or* $FG \equiv 1$.

Proof Let

$$\psi(z) = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}. \tag{2.1}$$

Since $E_F(1, k) = E_G(1, k)$, by a simple computation, we see that if z_0 is a simple zero of $F(z) - 1$ and $G(z) - 1$, then $\psi(z_0) = 0$.

Next we shall consider two cases.

Case 1. $\psi(z) \not\equiv 0$. Then

$$N_{1)}(r, \frac{1}{F-1}) = N_{1)}(r, \frac{1}{G-1}) \leq N(r, \frac{1}{\psi}) \leq T(r, \psi) + O(1)$$

$$\leq N(r, \psi) + S(r, F) + S(r, G), \tag{2.2}$$

where $N_{1)}(r, \frac{1}{F-1})$ is the counting function that only counts simple zeros of $F(z) - 1$ in $\{z : |z| \leq r\}$.

From (2.1), $E_F(1,k) = E_G(1,k)$ and $E_F(\infty,\infty) = E_G(\infty,\infty)$, we can easily verify that possible poles of ψ occur at: (1) multiple zeros of F and G; (2) the zeros of $F-1$ and $G-1$ with multiplicities $\geq k+1$; (3) the zeros of F' that are not the zeros of $F(F-1)$; (4) the zeros of G' that are not the zeros of $G(G-1)$.

Since all poles of ψ are simple, then we get

$$
\begin{aligned}
N(r,\psi) \leq & \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) \\
& + \frac{1}{2}\{\overline{N}_{(k+1}(r,\frac{1}{F-1}) + \overline{N}_{(k+1}(r,\frac{1}{G-1})\} + S(r,F) + S(r,G),
\end{aligned}
\tag{2.3}
$$

where $N_0(r,\frac{1}{F'})$ only counts the zeros of F' that are not those of $F(F-1)$.

On the other hand, by the second fundamental theorem, we have

$$
T(r,F) \leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{F-1}) - N_0(r,\frac{1}{F'}) + S(r,F),
\tag{2.4}
$$

and

$$
T(r,G) \leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{G-1}) - N_0(r,\frac{1}{G'}) + S(r,G).
\tag{2.5}
$$

It is easy to show that

$$
\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) = N_{1)}(r,\frac{1}{F-1}) + \frac{1}{2}\left\{N_2(r,\frac{1}{F-1}) + N_2(r,\frac{1}{G-1})\right\}.
\tag{2.6}
$$

Combining (2.4)–(2.6), we obtain

$$
\begin{aligned}
2\{T(r,F) + T(r,G)\} \leq & 2\{\overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{G})\} \\
& + 2N_{1)}(r,\frac{1}{F-1}) + N_2(r,\frac{1}{F-1}) + N_2(r,\frac{1}{G-1}) \\
& - 2N_0(r,\frac{1}{F'}) - 2N_0(r,\frac{1}{G'}) + S(r,F) + S(r,G).
\end{aligned}
\tag{2.7}
$$

Then from (2.2), (2.3), and (2.7), we get

$$
\begin{aligned}
2\{T(r,F) + T(r,G)\} \leq & 2\{\overline{N}(r,F) + N_2(r,\frac{1}{F}) + \overline{N}(r,G) + N_2(r,\frac{1}{G})\} \\
& + \overline{N}_{(k+1}(r,\frac{1}{F-1}) + \overline{N}_{(k+1}(r,\frac{1}{G-1}) \\
& + N_2(r,\frac{1}{F-1}) + N_2(r,\frac{1}{G-1}) + S(r,F) + S(r,G).
\end{aligned}
\tag{2.8}
$$

If $k \geq 2$, then

$$
N_2(r,\frac{1}{F-1}) + \overline{N}_{(k+1}(r,\frac{1}{F-1}) \leq N(r,\frac{1}{F-1}) \leq T(r,F) + O(1).
\tag{2.9}
$$

$$N_2(r, \frac{1}{G-1}) + \overline{N}_{(k+1}(r, \frac{1}{G-1}) \leq N(r, \frac{1}{G-1}) \leq T(r,G) + O(1). \tag{2.10}$$

Combining (2.8)–(2.10), we get

$$T(r,F) + T(r,G) \leq 2\{\overline{N}(r,F) + N_2(r, \frac{1}{F}) + \overline{N}(r,G) + N_2(r, \frac{1}{G})\} + S(r,F) + S(r,G).$$

If $k = 1$, note that

$$N_2(r, \frac{1}{F-1}) \leq T(r,F) + O(1). \tag{2.11}$$

$$N_2(r, \frac{1}{G-1}) \leq T(r,G) + O(1). \tag{2.12}$$

Combining (2.8), (2.11), and (2.12), we get

$$T(r,F) + T(r,G) \leq 2\{\overline{N}(r,F) + N_2(r,1/F) + \overline{N}(r,G) + N_2(r,1/G)\}$$

$$+ \overline{N}_{(k+1}(r, \frac{1}{F-1}) + \overline{N}_{(k+1}(r, \frac{1}{G-1}) + S(r,F) + S(r,G).$$

Case 2. $\psi(z) \equiv 0$. We deduce from (2.1) that

$$F \equiv \frac{AG+B}{CG+D}, \tag{2.13}$$

where A, B, C, D are finite complex numbers satisfying $AD - BC \neq 0$, and from (2.13) we have $T(r,F) = T(r,G)$

Next we consider three cases.

Case 1.1. $AC \neq 0$. From (2.13), we have $\overline{N}(r, 1/(F - A/C)) = \overline{N}(r,G)$. Then by the second fundamental theorem and $E_F(\infty, \infty) = E_G(\infty, \infty)$, we obtain

$$T(r,F) \leq \overline{N}(r,F) + \overline{N}(r, \frac{1}{F - \frac{A}{C}}) + \overline{N}(r, \frac{1}{F}) + S(r,F)$$

$$\leq \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}(r, \frac{1}{F}) + S(r,F),$$

which reveals (i) of Lemma 3.

Case 1.2. $A \neq 0, C = 0$. Then $F \equiv (AG + B)/D$. We consider two subcases.

Case 1.2.1. $B \neq 0$. Then by the second fundamental theorem,

$$T(r,F) \leq \overline{N}(r,F) + \overline{N}(r, \frac{1}{F - \frac{B}{D}}) + \overline{N}(r, \frac{1}{F}) + S(r,F)$$

$$\leq \overline{N}(r,F) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F}) + S(r,F),$$

which reveals (i) of Lemma 3.

Case 1.2.2. $B = 0$. Then $F \equiv \frac{A}{D}G$. Since $E_F(1,k) = E_G(1,k)$, if $F(z) \neq 1$, then by the second fundamental theorem

$$T(r,F) \leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{F}) + S(r,F)$$

$$\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + S(r,F),$$

which reveals (i) of Lemma 3.

If there exists a point z_0 such that $F(z_0) = G(z_0) = 1$. Thus $\frac{A}{D} = 1$, which yields $F \equiv G$.

Case 1.3. $A = 0, C \neq 0$, Then $F \equiv \frac{B}{CG+D}$. We can similarly prove that $D = 0, \frac{B}{C} = 1$, and thus $FG \equiv 1$.

By the above discussion, Case (i) or (ii) must occur. This completes the proof of Lemma 3.

$\square$

3. Proof of Theorem 1 and Theorem 2

Since the proofs of Theorem 1 and Theorem 2 are similar, we only prove Theorem 1.

Denote $g = \Delta_c f$. By the condition that $E_{f(z)}(S_1,k) = E_{\Delta_c f}(S_1,k)$, we see that f^n and g^n share 1 with weight k, that is $E_{f^n}(1,k) = E_{g^n}(1,k)$. Since $E_{f(z)}(\infty,\infty) = E_{\Delta_c f}(\infty,\infty)$, we also have $E_{f^n}(\infty,\infty) = E_{g^n}(\infty,\infty)$.

We suppose that

$$T(r,f^n) + T(r,g^n) \leq 2\{\overline{N}(r,f) + 2N(r,1/f) + \overline{N}(r,g) + 2N(r,1/g)\}$$

$$+ \left[\frac{1}{k}\right]\left\{\overline{N}_{(k+1}(r,\frac{1}{f^n-1}) + \overline{N}_{(k+1}(r,\frac{1}{g^n-1})\right\} + S(r). \tag{3.1}$$

Set

$$\phi = \frac{(f^n)'}{f^n(f^n-1)} - \frac{(g^n)'}{g^n(g^n-1)}. \tag{3.2}$$

If $\phi \not\equiv 0$. Let z_0 be a pole of f; then z_0 is a pole of g, and so z_0 is a zero of ϕ with multiplicity at least $n-1$, and so we get

$$\overline{N}(r,f) = \overline{N}(r,g) \leq \frac{1}{n-1}N(r,\frac{1}{\phi}) \leq \frac{1}{n-1}N(r,\phi) + S(r). \tag{3.3}$$

where $S(r) = S(r,f) + S(r,g)$

From (3.2), $E_{f^n}(1,k) = E_{g^n}(1,k)$ and $E_f(\infty,\infty) = E_g(\infty,\infty)$, we can easily verify that possible poles of ϕ occur at: (1) zeros of f and g; (2) the zeros of $f^n - 1$ and $g^n - 1$ with multiplicities $\geq k+1$. Thus we have

$$N(r,\phi) \leq \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + \frac{1}{2}\overline{N}_{(k+1}(r,\frac{1}{f^n-1}) + \frac{1}{2}\overline{N}_{(k+1}(r,\frac{1}{g^n-1}) + S(r).$$

Noticing that

$$\overline{N}_{(k+1}(r, \frac{1}{f^n - 1}) \leq \frac{1}{k}N(r, \frac{1}{f'}), \quad \overline{N}_{(k+1}(r, \frac{1}{g^n - 1}) \leq \frac{1}{k}N(r, \frac{1}{g'}).$$

With Lemma 2, it is easy to obtain

$$N(r, \phi) \leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \frac{1}{2k}N(r, \frac{1}{f'}) + \frac{1}{2k}N(r, \frac{1}{g'}) + S(r)$$

$$\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \frac{1}{2k}(N(r, \frac{1}{f}) + \overline{N}(r, f)) + \frac{1}{2k}(N(r, \frac{1}{g}) + \overline{N}(r, g)) + S(r).$$

(3.4)

From (3.3) and (3.4), we get

$$\overline{N}(r, f) = \overline{N}(r, g) \leq \frac{k}{(n-1)k - 1}\left\{\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \frac{1}{2k}N(r, \frac{1}{f}) + \frac{1}{2k}N(r, \frac{1}{g})\right\} + S(r).$$ (3.5)

On the other hand, from (3.1) and Lemma 1, we have

$$nT(r, f) + nT(r, g) \leq 2\{\overline{N}(r, f) + 2N(r, 1/f) + \overline{N}(r, g) + 2N(r, 1/g)\}$$

$$+ \left[\frac{1}{k}\right]\left\{\frac{1}{k}(N(r, \frac{1}{f}) + \overline{N}(r, f)) + \frac{1}{k}(N(r, \frac{1}{g}) + \overline{N}(r, g))\right\} + S(r).$$

(3.6)

When $k = 1, n \geq 7$, from (3.6), we get

$$nT(r, f) + nT(r, g) \leq 5N(r, \frac{1}{f}) + 3\overline{N}(r, f) + 5N(r, \frac{1}{g}) + 3\overline{N}(r, g) + S(r).$$ (3.7)

By (3.5) and (3.7), we obtain

$$(n - 5)\{T(r, f) + T(r, g)\} \leq \frac{9}{n - 2}\left\{N(r, \frac{1}{f}) + N(r, \frac{1}{g})\right\} + S(r)$$

$$\leq \frac{9}{n - 2}\{T(r, f) + T(r, g)\} + S(r),$$

a contradiction.

When $k \geq 2, n \geq 5$, from (3.6), we get

$$nT(r, f) + nT(r, g) \leq 4\left\{N(r, \frac{1}{f}) + N(r, \frac{1}{f})\right\} + 2\left\{\overline{N}(r, f) + \overline{N}(r, g)\right\} + S(r).$$ (3.8)

By (3.5) and (3.8), we obtain

$$(n - 4)\{T(r, f) + T(r, g)\} \leq \frac{2(k + 1)}{(n-1)k - 1}\left\{N(r, \frac{1}{f}) + N(r, \frac{1}{g})\right\} + S(r)$$

$$\leq \frac{2(k + 1)}{(n-1)k - 1}\{T(r, f) + T(r, g)\} + S(r).$$

Noticing that $k \geq 2, n \geq 5$, a contradiction.

Therefore, by Lemma 3, we get either $f^n \equiv g^n$ or $f^n g^n \equiv 1$.

If $\phi \equiv 0$. It follows from (3.2) that

$$f^n = \frac{Ag^n + B}{Cg^n + D},$$ (3.9)

where A, B, C, D are finite complex numbers satisfying $AD - BC \neq 0$, and from (3.9) we have $T(r, f) = T(r, g)$

Next we consider three cases.

Case 1. $AC \neq 0$. From (3.9) and $E_{f^n}(\infty, \infty) = E_{g^n}(\infty, \infty)$, we have $\overline{N}(r, 1/(f^n - A/C)) = \overline{N}(r, g^n) = \overline{N}(r, g) = \overline{N}(r, f)$. Then by the second fundamental theorem, we obtain

$$nT(r, f) \leq \overline{N}(r, f^n) + \overline{N}(r, \frac{1}{f^n - \frac{A}{C}}) + \overline{N}(r, \frac{1}{f^n}) + S(r, f)$$

$$\leq \overline{N}(r, f) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + S(r, f),$$

which contradicts $n \geq 5$.

Case 2. $A \neq 0, C = 0$. Then $f^n \equiv (Ag^n + B)/D$. We consider two subcases.

Case 2.1. $B \neq 0$. Then by the second fundamental theorem,

$$nT(r, f) \leq \overline{N}(r, f^n) + \overline{N}(r, \frac{1}{f^n - \frac{B}{D}}) + \overline{N}(r, \frac{1}{f^n}) + S(r, f)$$

$$\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g}) + S(r, g),$$

which contradicts $n \geq 5$.

Case 2.2. $B = 0$. Then $f^n \equiv \frac{A}{D} g^n$. Since $E_{f^n}(1, k) = E_{g^n}(1, k)$, by the second fundamental theorem that there must exist a point z_0 such that $f^n(z_0) = g^n(z_0) = 1$. Thus $\frac{A}{D} = 1$, which yields $f^n \equiv g^n$.

Case 3. $A = 0, C \neq 0$. Then $f^n \equiv \frac{B}{Cg^n + D}$. We can similarly prove that $D = 0, \frac{B}{C} = 1$, and thus $f^n g^n \equiv 1$.

As discussed above, we get if $n \geq 7$ when $k = 1$; $n \geq 5$ when $k \geq 2$; then $f^n \equiv g^n$ or $f^n g^n \equiv 1$. That is $f^n \equiv (\Delta_c f)^n$ or $f^n (\Delta_c f)^n \equiv 1$

If $f^n(\Delta_c f)^n \equiv 1$. That is

$$f^n(z)[f(z + c) - f(z)]^n \equiv 1.$$ (3.10)

From (3.10) and $E_{f(z)}(\infty, \infty) = E_{\Delta_c f}(\infty, \infty)$, we obtain $f(z) \neq 0$ and $f(z) \neq \infty$, and so $f(z) = e^{h(z)}$, where $h(z)$ is a nonconstant entire function.

By (3.10), we get

$$f(z)[f(z + c) - f(z)] \equiv t,$$ (3.11)

where $t^n = 1$.

From (3.11) and $f(z) = e^{h(z)}$, we obtain

$$e^{h(z)}[e^{h(z+c)} - e^{h(z)}] \equiv t.$$

That is

$$e^{h(z)h(z+c)} - e^{2h(z)} \equiv t.$$

Since $e^{h(z)h(z+c)} \neq 0$, we easily get $e^{2h(z)} \neq -t$, and obviously $e^{2h(z)} \neq 0, \infty$. Then by Picard Theorem, we get $e^{2h(z)} \equiv C_1$; then $h \equiv C_2$, where C_1 and C_2 are two constants, a contradiction.

Therefore, we get $f^n \equiv (\Delta_c f)^n$. Then there exists a constant $t \in \mathbb{C}$ such that $\Delta_c f \equiv tf(z)$.

This completed the proof of Theorem 1.

Acknowledgments

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