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Depth and Stanley depth of the path ideal associated to an n-cyclic graph

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Abstract: We compute the depth and Stanley depth for the quotient ring of the path ideal of length 3 associated to a n-cyclic graph, given some precise formulas for the depth when $n \not\equiv 1\,(\text{mod } 4)$, tight bounds when $n \equiv 1\,(\text{mod } 4)$, and for Stanley depth when $n \equiv 0, 3\,(\text{mod } 4)$, tight bounds when $n \equiv 1, 2\,(\text{mod } 4)$. We also give some formulas for the depth and Stanley depth of a quotient of the path ideals of length $n-1$ and n.

Key words: Stanley depth, Stanley inequality, path ideal, cyclic graph

1. Introduction

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K and M a finitely generated $\mathbb{Z}^n$-graded S-module. For a homogeneous element $u \in M$ and a subset $Z \subseteq \{x_1, \dots, x_n\}$, $uK[Z]$ denotes the K-subspace of M generated by all the homogeneous elements of the form uv, where v is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $uK[Z]$ is said to be a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where, as usual, $|Z|$ denotes the number of elements of Z. A Stanley decomposition of M is a decomposition of M as a finite direct sum of $\mathbb{Z}^n$-graded K-vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

where each $u_i K[Z_i]$ is a Stanley space of M. The number $\text{sdepth}_S(\mathcal{D}) = \min\{|Z_i| : i = 1, \dots, r\}$ is called the Stanley depth of decomposition $\mathcal{D}$ and the quantity

$$\text{sdepth}(M) := \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the Stanley depth of M. Stanley [13] conjectured that

$$\text{sdepth}(M) \geq \text{depth}(M)$$

for all $\mathbb{Z}^n$-graded S-modules M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals; see [6].

Herzog et al. [8] introduced a method to compute the Stanley depth of a factor of a monomial ideal, which was later developed into an effective algorithm by Rinaldo [12], implemented in CoCoA [5]. However, it

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is difficult to compute this invariant, even in some very particular cases. For instance, in [1] Biró et al. proved that $\operatorname{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$ where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of S and $\lceil \frac{n}{2} \rceil$ denotes the smallest integer $\geq \frac{n}{2}$. For an introduction to Stanley depth, we refer the reader to [7].

Let $I_{n,m}$ and $J_{n,m}$ be the path ideals of length m associated to the n-line, respectively n-cyclic, graph. Cimpoeaş [3] proved that $\operatorname{depth}(S/J_{n,2}) = \lceil \frac{n-1}{3} \rceil$ and when $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$, $\operatorname{sdepth}(S/J_{n,2}) = \lceil \frac{n-1}{3} \rceil$ and when $n \equiv 1 \pmod 3$, $\lceil \frac{n-1}{3} \rceil \leq \operatorname{sdepth}(S/J_{n,2}) \leq \lceil \frac{n}{3} \rceil$. In [4], he also showed that $\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = n + 1 - \lfloor \frac{n+1}{m+1} \rfloor - \lceil \frac{n+1}{m+1} \rceil$, where $\lfloor \frac{n+1}{m+1} \rfloor$ denotes the biggest integer $\leq \frac{n+1}{m+1}$. Using similar techniques, we prove that $\operatorname{sdepth}(S/J_{n,3}) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ and $n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil \leq \operatorname{sdepth}(S/J_{n,3}) \leq n + 1 - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$. Also, we prove that $\operatorname{depth}(S/J_{n,3}) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \not\equiv 1 \pmod 4$ and $n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil \leq \operatorname{depth}(S/J_{n,3}) \leq n + 1 - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \equiv 1 \pmod 4$. In Proposition 2.14, we prove that $\operatorname{sdepth}(J_{n,3}/I_{n,3}) = n + 1 - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for all $n \geq 4$. In the third section, we prove that $\operatorname{sdepth}(\frac{S}{J_{n,n-1}}) = \operatorname{depth}(\frac{S}{J_{n,n-1}}) = n - 2$ and $n - 3 \leq \operatorname{sdepth}(\frac{S}{J_{n,n-2}}), \operatorname{depth}(\frac{S}{J_{n,n-2}}) \leq n - 2$.

2. Depth and Stanley depth of the quotient of the path ideal with length 3

In this section, we will give some formulas for the depth and Stanley depth of quotient of the path ideals of length 3. We first recall some definitions about graphs and their path ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [16, 17].

Definition 2.1 *Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E. Then $G = (V, E)$ is called an n-line graph, denoted by L_n, if its edge set is given by $E = \{x_i x_{i+1} \mid 1 \leq i \leq n - 1\}$. Similarly, if $n \geq 3$, then $G = (V, E)$ is called an n-cyclic graph, denoted by C_n, if its edge set is given by $E = \{x_i x_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{x_n x_1\}$.*

Definition 2.2 *Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$. A path of length m in G is an alternating sequence of vertices and edges $w = \{x_i, e_i, x_{i+1}, \ldots, x_{i+m-2}, e_{i+m-2}, x_{i+m-1}\}$, where $e_j = x_j x_{j+1}$ is the edge joining x_j and x_{j+1}. A path of length m may also be denoted $\{x_i, \ldots, x_{i+m-1}\}$, the edges being evident from the context.*

Definition 2.3 *Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$. Then the path ideal of length m associated to G is the squarefree monomial ideal $I = (x_i \cdots x_{i+m-1} \mid \{x_i, \ldots, x_{i+m-1}\}$ is a path of length m in $G)$ of S.*

In this paper, we set $n \geq 3$ and consider the n-line graph L_n and n-cyclic graph C_n; their path ideals of length m are denoted by $I_{m,n}$ and $J_{m,n}$, respectively. Thus, we obtain that

$$I_{m,n} = (x_i \cdots x_{i+m-1} \mid 1 \leq i \leq n - m + 1),$$

and

$$J_{m,n} = I_{m,n} + (x_{n-m} \cdots x_n x_1, x_{n-m+1} \cdots x_n x_1 x_2, \ldots, x_n x_1 \cdots x_{m-1}).$$

Definition 2.4 *Let $(S,\mathfrak{m})$ be a local ring (or a Noetherian graded ring with $(S_0,\mathfrak{m}_0)$ local) and M a finite generated S-module with the property that $\mathfrak{m}M \subsetneq M$ (or a finite generated graded S-module with the property that $(\mathfrak{m}_0 \oplus \bigoplus_{i=1}^{\infty} S_i)M \subsetneq M$). Then the depth of M is defined as*

$$depth(M) = min\{i \mid Ext^i(S/\mathfrak{m}, M) \neq 0\}$$

(or $depth(M) = min\{i \mid Ext^i(S/(\mathfrak{m}_0 \oplus \bigoplus_{i=1}^{\infty} S_i), M) \neq 0\}$).

We recall the well-known depth lemma; see, for instance, [16, Lemma 1.3.9] or [15, Lemma 3.1.4].

Lemma 2.5 *(Depth lemma) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local; then:*

(i) $depth(M) \geq min\{depth(L), depth(N)\}$;

(ii) $depth(L) \geq min\{depth(M), depth(N)+1\}$;

(iii) $depth(N) \geq min\{depth(L)-1, depth(M)\}$.

Most of the statements of the above depth lemma are wrong if we replace depth by Stanley depth. Some counterexamples are given in [11, Example 2.5 and 2.6]. Rauf [11] proved the analog of Lemma 2.5 (i) for Stanley depth.

Lemma 2.6 *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded S-modules. Then*

$$sdepth(M) \geq min\{sdepth(L), sdepth(N)\}.$$

In [3], Cimpoeaş computed depth and Stanley depth for $S/J_{n,2}$.

Lemma 2.7 *(1) $depth(S/J_{n,2}) = \lceil \frac{n-1}{3} \rceil$;*

(2) $sdepth(S/J_{n,2}) = \lceil \frac{n-1}{3} \rceil$ for $n \equiv 0 \,(mod\ 3)$ or $n \equiv 2 \,(mod\ 3)$;

(3) $\lceil \frac{n-1}{3} \rceil \leq sdepth(S/J_{n,2}) \leq \lceil \frac{n}{3} \rceil$ for $n \equiv 1 \,(mod\ 3)$.

In [4], Cimpoeaş computed depth and Stanley depth for $S/I_{n,m}$, which generalizes [9, Lemma 2.8] and [14, Lemma 4].

Lemma 2.8 *$sdepth(S/I_{n,m}) = depth(S/I_{n,m}) = n+1-\lfloor \frac{n+1}{m+1} \rfloor - \lceil \frac{n+1}{m+1} \rceil$. In particular, $sdepth(S/I_{n,2}) = depth(S/I_{n,2}) = \lceil \frac{n}{3} \rceil$.*

Using these lemmas, we are able to prove the main result of this section.

Theorem 2.9 *(1) $depth(S/J_{n,3}) \geq n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$;*

(2) $sdepth(S/J_{n,3}) \geq n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$.

Proof These two results can be shown by similar arguments, so we only prove that $\operatorname{sdepth}(S/J_{n,3}) \geq n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$. Let S_t be the polynomial ring in t variables over a field. The case $n = 3$ is trivial. The cases $n = 4$ and $n = 5$ follow from Examples 2.10 and 2.11, respectively.

We may assume that $n \geq 6$. Let $k = \lfloor \frac{n}{4} \rfloor$ and $\varphi(n) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$. One can easily see that

$$\varphi(n) = \begin{cases} n - 2k, & \text{if } n \equiv 0 \pmod 4; \\ n - 2k + 1, & \text{otherwise.} \end{cases}$$

We denote $u_i = x_i x_{i+1} x_{i+2}$ for $1 \leq i \leq n-2$, $u_{n-1} = x_{n-1} x_n x_1$, and $u_n = x_n x_1 x_2$. Set $L_0 = J_{n,3}$, $L_1 = (L_0 : x_n)$ and $U_1 = (L_0, x_n)$. Notice that $L_0 = (u_1, \ldots, u_n)$, $L_1 = (u_2, \ldots, u_{n-4}, \frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n})$, and $U_1 = (u_1, \ldots, u_{n-3}, x_n)$. Since $S/U_1 \simeq S_{n-1}/I_{n-1,3}$, we obtain $\operatorname{sdepth}(S/U_1) = \varphi(n)$ by Lemma 2.8.

We set $L_{j+1} = (L_j : x_{4j})$ and $U_{j+1} = (L_j, x_{4j})$ where $1 \leq j \leq k-3$. One can easily check that:

$$L_{j+1} = (\frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_6}{x_8}, \ldots, \frac{u_{4(j-1)}}{x_{4(j-1)}}, \frac{u_{4j-2}}{x_{4j}}, \ldots, \frac{u_{4j}}{x_{4j}}, u_{4j+2}, \ldots, u_{n-4}, \frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}),$$

and

$$U_{j+1} = (\frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_6}{x_8}, \ldots, \frac{u_{4(j-1)-2}}{x_{4(j-1)}}, \ldots, \frac{u_{4(j-1)}}{x_{4(j-1)}}, x_{4j}, u_{4j+1}, \ldots, u_{n-4}, \frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}),$$

where $x_0 = 1$ and $u_j = 0$ for $j \leq 0$.

We consider the following three cases:

(1). If $n = 4k$ or $n = 4k-1$, we denote $L_{j+1} = (L_j : x_{4j})$ and $U_{j+1} = (U_j, x_{4j})$ for $j = k-2, k-1$. We conclude $L_k \simeq J_{n-k,2}S$, $U_k = (x_{4(k-1)}, V_k)$ where $V_k = (\frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_6}{x_8}, \ldots, \frac{u_{4(k-2)}}{x_{4(k-2)}}, \frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n})$. Note that

$$V_k \simeq \begin{cases} I_{n-k-2,2}, & \text{if } n = 4k; \\ I_{n-k-1,2}, & \text{if } n = 4k-1. \end{cases}$$

Thus, by Lemmas 2.7, 2.8, and [8, Lemma 3.6], it follows that

$$\operatorname{sdepth}(S/L_k) = k + \operatorname{sdepth}(S_{n-k}/J_{n-k,2}) = k + k = \varphi(n),$$

and

$$\operatorname{sdepth}(\frac{S}{U_k}) = \begin{cases} (k+1) + \operatorname{sdepth}(\frac{S_{n-k-2}}{I_{n-k-2,2}}) = (k+1) + k = 1 + \varphi(n), & \text{if } n = 4k; \\ k + \operatorname{sdepth}(\frac{S_{n-k-1}}{I_{n-k-1,2}}) = k + k = \varphi(n), & \text{if } n = 4k-1. \end{cases}$$

(2). If $n = 4k-2$, we denote $L_{k-1} = (L_{k-2} : x_{4(k-2)})$, $U_{k-1} = (L_{k-2}, x_{4(k-2)})$, $L_k = (L_{k-1} : x_{4(k-1)-1})$, and $U_k = (U_{k-1}, x_{4(k-1)-1})$. We have $L_k \simeq J_{n-k,2}S$, $U_k = (x_{4(k-1)-1}, V_k)$ where $V_k = (\frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_6}{x_8}, \ldots, \frac{u_{4(k-2)}}{x_{4(k-2)}}, \frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}) \simeq I_{n-k,2}S$. Thus, by Lemma 2.8 and [8, Lemma 3.6], we obtain

$$\operatorname{sdepth}(S/U_k) = (k-1) + \operatorname{sdepth}(S_{n-k}/I_{n-k,2}) = (k-1) + k = \varphi(n).$$

Applying Lemma 2.7 and [8, Lemma 3.6], we get

$$\operatorname{sdepth}(S/L_k) = k + \operatorname{sdepth}(S_{n-k}/J_{n-k,2}) \geq k + (k-1) = \varphi(n),$$

and

$$\text{sdepth}\,(S/L_k) = k + \text{sdepth}\,(S_{n-k}/J_{n-k,2}) \leq k + k = 1 + \varphi(n).$$

(3). If $n = 4k-3$, we denote $L_{k-1} = (L_{k-2} : x_{4(k-2)-1})$, $U_{k-1} = (L_{k-2}, x_{4(k-2)-1})$, $L_k = (L_{k-1} : x_{4(k-1)-2})$, and $U_k = (U_{k-1}, x_{4(k-1)-2})$. We have $L_k \simeq J_{n-k,2}S$, $U_k = (x_{4(k-1)-2}, V_k)$ where $V_k = (\frac{u_2}{x_4}, \frac{u_3}{x_4}, \frac{u_4}{x_4}, \frac{u_6}{x_8}, \ldots, \frac{u_{4(k-3)}}{x_{4(k-3)}}, \frac{u_{4(k-2)-2}}{x_{4(k-2)-1}}, \frac{u_{4(k-2)-1}}{x_{4(k-2)-1}}, \frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}) \simeq I_{n-k,2}S$. Therefore, by Lemmas 2.7, 2.8, and [8, Lemma 3.6], we have

$$\text{sdepth}\,(S/L_k) = k + \text{sdepth}\,(S_{n-k}/J_{n-k,2}) = k + (k-1) = 1 + \varphi(n),$$

and

$$\text{sdepth}\,(S/U_k) = (k-1) + \text{sdepth}\,(S_{n-k}/I_{n-k,2}) = (k-1) + (k-1) = \varphi(n).$$

This shows that $\varphi(n) \leq \text{sdepth}\,(S/L_k) \leq 1 + \varphi(n)$ and $\text{sdepth}\,(S/U_k) \geq \varphi(n) \quad (*)$.

Consider the following short exact sequences:

$$0 \longrightarrow \tfrac{S}{L_1} \longrightarrow \tfrac{S}{L_0} \longrightarrow \tfrac{S}{U_1} \longrightarrow 0$$

$$0 \longrightarrow \tfrac{S}{L_2} \longrightarrow \tfrac{S}{L_1} \longrightarrow \tfrac{S}{U_2} \longrightarrow 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$0 \to \tfrac{S}{L_{k-1}} \to \tfrac{S}{L_{k-2}} \to \tfrac{S}{U_{k-1}} \to 0$$

$$0 \longrightarrow \tfrac{S}{L_k} \longrightarrow \tfrac{S}{L_{k-1}} \longrightarrow \tfrac{S}{U_k} \longrightarrow 0.$$

By Lemma 2.6 and $(*)$, we have

$$\begin{aligned}
\text{sdepth}\,(\frac{S}{J_{n,3}}) &= \text{sdepth}\,(\frac{S}{L_0}) \geq \min\{\text{sdepth}\,(\frac{S}{L_1}), \text{sdepth}\,(\frac{S}{U_1})\} \\
&\geq \min\{\text{sdepth}\,(\frac{S}{L_2}), \text{sdepth}\,(\frac{S}{U_2}), \text{sdepth}\,(\frac{S}{U_1})\} \\
&\geq \cdots \\
&\geq \min\{\text{sdepth}\,(\frac{S}{L_k}), \text{sdepth}\,(\frac{S}{U_k}), \text{sdepth}\,(\frac{S}{U_{k-1}}), \ldots, \text{sdepth}\,(\frac{S}{U_1})\} \\
&\geq \min\{\varphi(n), \text{sdepth}\,(\frac{S}{U_{k-1}}), \ldots, \text{sdepth}\,(\frac{S}{U_2}), \text{sdepth}\,(\frac{S}{U_1})\}.
\end{aligned}$$

To show $\text{sdepth}\,(\frac{S}{J_{n,3}}) \geq \varphi(n)$, it is enough to prove the claim below.

Claim: $\text{sdepth}\,(S/U_{j+1}) \geq \varphi(n)$ for all $1 \leq j \leq k-2$.

For any $1 \leq j \leq k-3$, we set $V_{j+1} = (\frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}, \frac{u_2}{x_4}, \ldots, \frac{u_{4(j-1)-2}}{x_{4(j-1)}}, \ldots, \frac{u_{4(j-1)}}{x_{4(j-1)}})$ and $W_{j+1} = (u_{4j+1}, \ldots, u_{n-4})$ where $x_0 = 1$ and $u_j = 0$ for $j \leq 0$. We have $\frac{S}{U_{j+1}} \simeq \frac{S/V_{j+1} \oplus S/W_{j+1}}{x_{4j}(S/V_{j+1} \oplus S/W_{j+1})}$. Since x_{4j} is regular on $S/V_{j+1} \oplus S/W_{j+1}$, by [10, Theorem 1.1] and [2, Theorem 1.3], we have

$$\text{sdepth}\,(\frac{S}{U_{j+1}}) = \text{sdepth}\,(\frac{S}{V_{j+1}} \oplus \frac{S}{W_{j+1}}) - 1 \geq \text{sdepth}\,(\frac{S}{V_{j+1}}) + \text{sdepth}\,(\frac{S}{W_{j+1}}) - n - 1.$$

On the other hand, $V_{j+1} \simeq I_{3j+1,2}S$, $W_{j+1} \simeq I_{n-4(j+1)+2,3}S$. Thus, by Lemma 2.8, we have

$$\operatorname{sdepth}(S/V_{j+1}) = [n-(3j+1)] + \lceil \frac{3j+1}{3} \rceil = n-2j$$

and

$$\begin{aligned}
\operatorname{sdepth}(S/W_{j+1}) &= [4(j+1)-2] + [n-4(j+1)+3] - \lfloor \frac{n-4(j+1)+3}{4} \rfloor \\
&- \lceil \frac{n-4(j+1)+3}{4} \rceil \\
&= n+1 - \lfloor \frac{n-4j-1}{4} \rfloor - \lceil \frac{n-4j-1}{4} \rceil \\
&= n+1+2j - \lfloor \frac{n-1}{4} \rfloor - \lceil \frac{n-1}{4} \rceil.
\end{aligned}$$

By some simple computations, we conclude that

$$\begin{aligned}
\operatorname{sdepth}(S/U_{j+1}) &\geq (n-2j) + (n+1+2j) - \lfloor \frac{n-1}{4} \rfloor - \lceil \frac{n-1}{4} \rceil - n - 1 \\
&= n - \lfloor \frac{n-1}{4} \rfloor - \lceil \frac{n-1}{4} \rceil \geq \varphi(n).
\end{aligned}$$

If $n \neq 4k-3$, we have $V_{k-1} = (\frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}, \frac{u_2}{x_4}, \ldots, \frac{u_{4(k-3)-2}}{x_{4(k-3)}}, \ldots, \frac{u_{4(k-3)}}{x_{4(k-3)}})$ and $W_{k-1} = (u_{4(k-2)+1}, \ldots, u_{n-4})$. It follows from similar arguments as above.

If $n = 4k-3$, we have $V_{k-1} = (\frac{u_{n-2}}{x_n}, \frac{u_{n-1}}{x_n}, \frac{u_n}{x_n}, \frac{u_2}{x_4}, \ldots, \frac{u_{4(k-3)-2}}{x_{4(k-3)}}, \ldots, \frac{u_{4(k-3)}}{x_{4(k-3)}})$ and $W_{k-1} = (u_{4(k-2)}, \ldots, u_{n-4})$. Note that $V_{k-1} \simeq I_{3(k-2)+1,2}S$ and $W_{k-1} \simeq I_{n-4(k-1)+3,3}S$. Thus, by Lemma 2.8, we obtain

$$\operatorname{sdepth}(S/V_{k-1}) = (n-(3(k-2)+1)) + \lceil \frac{3(k-2)+1}{3} \rceil = n-2k+4 = 2k+1$$

and

$$\begin{aligned}
\operatorname{sdepth}(S/W_{k-1}) &= (4(k-1)-3) + (n-4(k-1)+4) - \lfloor \frac{n-4(k-1)+4}{4} \rfloor \\
&- \lceil \frac{n-4(k-1)+4}{4} \rceil \\
&= n+1 - \lfloor \frac{n-4k+8}{4} \rfloor - \lceil \frac{n-4k+8}{4} \rceil \\
&= n+1 - \lfloor \frac{5}{4} \rfloor - \lceil \frac{5}{4} \rceil = n-2.
\end{aligned}$$

One can easily see that $\frac{S}{U_{k-1}} \simeq \frac{S/V_{k-1} \oplus S/W_{k-1}}{x_{4(k-2)-1}(S/V_{k-1} \oplus S/W_{k-1})}$. Since $x_{x_{4(k-2)-1}}$ is regular on $S/V_{k-1} \oplus S/W_{k-1}$, by [10, Theorem 1.1] and [2, Theorem 1.3], we have

$$\begin{aligned}
\operatorname{sdepth}(\frac{S}{U_{k-1}}) &= \operatorname{sdepth}(\frac{S}{V_{k-1}} \oplus \frac{S}{W_{k-1}}) - 1 \\
&\geq \operatorname{sdepth}(\frac{S}{V_{k-1}}) + \operatorname{sdepth}(\frac{S}{W_{k-1}}) - n - 1 \\
&= 2k + 1 + (n-2) - n - 1 \\
&= 2k - 2 = \varphi(n).
\end{aligned}$$

This completes the proof. □

Example 2.10 *Let $J_{4,3} = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_1, x_4x_1x_2) \subset S = K[x_1, \ldots, x_4]$. Note that $4 - \lfloor \frac{4}{4} \rfloor - \lceil \frac{4}{4} \rceil = 2$. Set $L_1 = (J_{4,3} : x_4)$ and $U_1 = (J_{4,3}, x_4)$. Since $L_1 = (x_1x_2, x_2x_3, x_3x_1) = J_{3,2}S$ and $U_1 = (x_1x_2x_3, x_4)$, thus $S/U_1 = K[x_1, x_2, x_3]/(x_1x_2x_3)$. By Lemmas 2.7, 2.8, and [8, Lemma 3.6], we have sdepth$(S/L_1) =$ depth$(S/L_1) = 1 + \lceil \frac{3-1}{3} \rceil = 2$ and sdepth$(S/U_1) = 2$. Applying Lemma 2.6 to the short exact sequence*

$$0 \longrightarrow S/L_1 \longrightarrow S/J_{4,3} \longrightarrow S/U_1 \longrightarrow 0,$$

we obtain depth$(\frac{S}{J_{4,3}}) = 2$ and sdepth$(\frac{S}{J_{4,3}}) \geq 2$. By [2, Proposition 2.7], it follows that sdepth$(\frac{S}{J_{4,3}}) \leq$ sdepth$(S/L_1) = 2$. Thus, sdepth$(S/J_{4,3}) = 2$.

Example 2.11 *Let $J_{5,3} = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2) \subset S = K[x_1, \ldots, x_5]$. Note that $5 - \lfloor \frac{5}{4} \rfloor - \lceil \frac{5}{4} \rceil = 2$. Set $L_1 = (J_{5,3} : x_5)$ and $U_1 = (J_{5,3}, x_5)$. Since $L_1 = (x_3x_4, x_4x_1, x_1x_2) \simeq I_{4,2}S$ and $U_1 = (x_1x_2x_3, x_2x_3x_4, x_5)$, thus $S/U_1 = S_4/I_{4,3}$, and by Lemma 2.8 and [8, Lemma 3.6], we have sdepth$(S/L_1) =$ depth$(S/L_1) = 1 + \lceil \frac{4}{3} \rceil = 3$ and sdepth$(S/U_1) = 5 - \lfloor \frac{5}{4} \rfloor - \lceil \frac{5}{4} \rceil = 2$. Using Lemmas 2.5 and 2.6 on the short exact sequence*

$$0 \longrightarrow S/L_1 \longrightarrow S/J_{5,3} \longrightarrow S/U_1 \longrightarrow 0,$$

we obtain depth$(S/J_{5,3}) \geq 2$ and sdepth$(S/J_{5,3}) \geq 2$.

As a consequence of Theorem 2.9, one has the following results.

Corollary 2.12 *(1) sdepth$(S/J_{n,3}) \leq n + 1 - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \equiv 1 \,(mod\ 4)$ or $n \equiv 2 \,(mod\ 4)$;*

(2) sdepth$(S/J_{n,3}) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \equiv 0 \,(mod\ 4)$ or $n \equiv 3 \,(mod\ 4)$.

Proof Set $\varphi(n) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$. From the proof of Theorem 2.9, we see that sdepth$(S/L_k) \leq 1 + \varphi(n)$ for $n \equiv 1 \,(\text{mod } 4)$ or $n \equiv 2 \,(\text{mod } 4)$, and otherwise sdepth$(S/L_k) = \varphi(n)$. These are direct consequences of [2, Proposition 2.7]. □

Corollary 2.13 *(1) depth$(S/J_{n,3}) \leq n + 1 - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \equiv 1 \,(mod\ 4)$,*

(2) depth$(S/J_{n,3}) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ for $n \not\equiv 1 \,(mod\ 4)$.

Proof Set $\varphi(n) = n - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$. Replacing the Stanley depth by depth in the proof of Theorem 2.9, we see that $\text{depth}\,(S/L_k) = 1 + \varphi(n)$ for $n \equiv 1\,(\text{mod } 4)$, and otherwise $\text{depth}\,(S/L_k) = \varphi(n)$. These are direct consequences of [11, Corollary 1.3]. □

Proposition 2.14 *sdepth* $(J_{n,3}/I_{n,3}) = n + 1 - \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ *for all* $n \geq 4$.

Proof One can easily check that $J_{4,3}/I_{4,3} \simeq x_1x_3x_4K[x_1,x_3,x_4] \oplus x_1x_2x_4K[x_1,x_2,x_4]$. Thus, $\text{sdepth}\,(J_{4,3}/I_{4,3}) = 3$, as required. Similarly, for $n = 5$, we have $J_{5,3}/I_{5,3} \simeq x_1x_4x_5K[x_1,x_4,x_5] \oplus x_1x_2x_5K[x_1,x_2,x_5] \oplus x_1x_2x_4x_5K[x_1,x_2,x_4,x_5]$; for $n = 6$, we have $J_{6,3}/I_{6,3} \simeq x_1x_5x_6K[x_1,x_3,x_5,x_6] \oplus x_1x_2x_6K[x_1,x_2,x_4,x_6] \oplus x_1x_2x_5x_6K[x_1,x_2,x_5,x_6]$; and for $n = 7$, we get $J_{7,3}/I_{7,3} \simeq x_1x_6x_7K[x_1,x_3,x_4,x_6,x_7] \oplus x_1x_2x_7K[x_1,x_2,x_4,x_5,x_7] \oplus x_1x_2x_6x_7K[x_1,x_2,x_4,x_6,x_7]$.

Now, assume $n \geq 8$, and let $u \in J_{n,3}$ be a monomial such that $u \notin I_{n,3}$. It follows that $u = x_1x_{n-1}x_nv_1$ or $u = x_1x_2x_nv_2$, with $v_1 \in K[x_1,\ldots,x_{n-3},x_{n-1},x_n]$ and $v_2 \in K[x_1,x_2,x_4,\ldots,x_n]$. We can write $v_1 = x_1^{\alpha}x_{n-1}^{\beta}x_n^{\gamma}w$ with $w \in K[x_2,\ldots,x_{n-3}]$. Since $u \notin I_{n,3}$, it follows that $w \notin (x_2x_3, x_3x_4x_5, \ldots, x_{n-5}x_{n-4}x_{n-3})$. Similarly, we can write $v_2 = x_1^{\alpha}x_2^{\beta}x_n^{\gamma}w$ with $w \in K[x_4,\ldots,x_{n-1}]$. Since $u \notin I_{n,3}$, it follows that $w \notin (x_4x_5x_6, \ldots, x_{n-4}x_{n-3}x_{n-2}, x_{n-2}x_{n-1})$. Therefore, we have the S-module isomorphism:

$$\begin{aligned} J_{n,3}/I_{n,3} &\simeq x_1x_2x_n\Big(\frac{K[x_4,\ldots,x_{n-2}]}{(x_4x_5x_6,\ldots,x_{n-4}x_{n-3}x_{n-2})}\Big)[x_1,x_2,x_n] \\ &\oplus x_1x_{n-1}x_n\Big(\frac{K[x_3,\ldots,x_{n-3}]}{(x_3x_4x_5,\ldots,x_{n-5}x_{n-4}x_{n-3})}\Big)[x_1,x_{n-1},x_n] \\ &\oplus x_1x_2x_{n-1}x_n\Big(\frac{K[x_4,\ldots,x_{n-3}]}{(x_4x_5x_6,\ldots,x_{n-5}x_{n-4}x_{n-3})}\Big)[x_1,x_2,x_{n-1},x_n]. \end{aligned}$$

Therefore, by Lemma 2.8 and [8, Lemma 3.6], we obtain

$$\begin{aligned} \text{sdepth}\,\Big(\frac{J_{n,3}}{I_{n,3}}\Big) &= \min\{3+(n-4)-\lfloor\frac{n-4}{4}\rfloor-\lceil\frac{n-4}{4}\rceil, 4+(n-5)-\lfloor\frac{n-5}{4}\rfloor-\lceil\frac{n-5}{4}\rceil\} \\ &= \min\{n+1-\lfloor\frac{n}{4}\rfloor-\lceil\frac{n}{4}\rceil, n+1-\lfloor\frac{n-1}{4}\rfloor-\lceil\frac{n-1}{4}\rceil\} \\ &= n+1-\lfloor\frac{n}{4}\rfloor-\lceil\frac{n}{4}\rceil. \end{aligned}$$

□

3. Depth and Stanley depth of the quotient of the path ideal of length $n-1$ or $n-2$

In this section, we will give some formulas for depth and Stanley depth of the quotient of the path ideal of length $n-1$ or $n-2$.

Proposition 3.1 *sdepth* $(S/J_{n,n-1}) =$ *depth* $(S/J_{n,n-1}) = n-2$.

Proof We apply induction on n. The case $n = 3$ follows from Lemma 2.7. Assume now that $n \geq 4$. Since $J_{n,n-1} = (\prod_{i=1}^{n-1} x_i, \prod_{i=2}^{n} x_i, (\prod_{i=3}^{n} x_i)x_1, \ldots, (\prod_{i=k}^{n} x_i)(\prod_{i=1}^{k-2} x_i), \ldots, x_n\prod_{i=1}^{n-2} x_i)$, we obtain

$(J_{n,n-1} : x_n) = (\prod_{i=1}^{n-2} x_i, \prod_{i=2}^{n-1} x_i, (\prod_{i=3}^{n-1} x_i)x_1, \ldots, (\prod_{i=k}^{n-1} x_i)(\prod_{i=1}^{k-2} x_i), \ldots, x_{n-1}\prod_{i=1}^{n-3} x_i) = J_{n-1,n-2}S$,

$(J_{n,n-1}, x_n) = (\prod_{i=1}^{n-1} x_i, x_n)$. Hence, we get $S/(J_{n,n-1} : x_n) = (S_{n-1}/J_{n-1,n-2})[x_n]$. Using the induction hypothesis and [8, Lemma 3.6], we conclude

$$\text{sdepth}\,(S/(J_{n,n-1} : x_n)) = 1 + \text{sdepth}\,(S_{n-1}/J_{n-1,n-2}) = n-2,$$

and

$$\text{depth}\,(S/(J_{n,n-1} : x_n)) = 1 + \text{depth}\,(S_{n-1}/J_{n-1,n-2}) = n-2.$$

On the other hand, we obtain $\text{sdepth}\,(S/(J_{n,n-1}, x_n)) = n-2$ by [10, Theorem 1.1]. By applying Lemmas 2.5 and 2.6 to the exact sequence

$$0 \longrightarrow S/(J_{n,n-1} : x_n) \xrightarrow{\cdot x_n} S/J_{n,n-1} \longrightarrow S/(J_{n,n-1}, x_n) \longrightarrow 0,$$

we obtain $\text{depth}\,(S/J_{n,n-1}) \geq n-2$ and $\text{sdepth}\,(S/J_{n,n-1}) \geq n-2$. Therefore, it follows that $\text{sdepth}\,(S/J_{n,n-1}) = n-2$ by [2, Proposition 2.7]. □

Proposition 3.2 *(1)* $n-3 \leq \textit{sdepth}\,(S/J_{n,n-2}) \leq n-2$,

(2) $n-3 \leq \textit{depth}\,(S/J_{n,n-2}) \leq n-2$.

Proof The case $n=3$ is trivial. The case $n=4$ follows from Lemma 2.7. We may assume that $n \geq 5$. Set $L_0 = J_{n,n-2}$, $L_j = (L_{j-1} : x_{n-j+1})$ and $U_j = (L_{j-1}, x_{n-j+1})$ for all $1 \leq j \leq n-4$. We conclude that $L_0 = (\prod_{i=1}^{n-2} x_i, \prod_{i=2}^{n-1} x_i, \prod_{i=3}^{n} x_i, (\prod_{i=4}^{n} x_i)x_1, \ldots, (\prod_{i=k}^{n} x_i)(\prod_{i=1}^{k-3} x_i), \ldots, x_n \prod_{i=1}^{n-3} x_i)$,
$L_1 = (\prod_{i=3}^{n-1} x_i, (\prod_{i=4}^{n-1} x_i)x_1, \ldots, (\prod_{i=k}^{n-1} x_i)(\prod_{i=1}^{k-3} x_i), \ldots, x_{n-1}\prod_{i=1}^{n-4} x_i, \prod_{i=1}^{n-3} x_i)$, and $U_1 = (\prod_{i=1}^{n-2} x_i, \prod_{i=2}^{n-1} x_i, x_n)$. Since $S/U_1 = S_{n-1}/I_{n-1,n-2}$, we obtain $\text{sdepth}\,(S/U_1) = \text{depth}\,(S/U_1) = n-3$ by Lemma 2.8. For any $1 \leq j \leq n-4$, by some simple computations, one can see that

$$L_j = (\prod_{i=3}^{n-j} x_i, (\prod_{i=4}^{n-j} x_i)x_1, \ldots, (\prod_{i=k}^{n-j} x_i)(\prod_{i=1}^{k-3} x_i), \ldots, x_{n-j} \prod_{i=1}^{n-j-3} x_i, \prod_{i=1}^{n-j-2} x_i),$$

and $U_j = (U_{j-1}, x_{n-j+1}) = (\prod_{i=1}^{n-j-1} x_i, x_{n-j+1})$. In particular, $L_{n-4} = (x_3x_4, x_4x_1, x_1x_2)$ and $S/L_{n-4} \simeq (S_4/I_{4,2})[x_5, \ldots, x_n]$. Therefore, by Lemma 2.8 and [8, Lemma 3.6], we get $\text{sdepth}\,(S/L_{n-4}) = \text{depth}\,(S/L_{n-4}) = (n-4) + 5 - \lfloor \frac{5}{3} \rfloor - \lceil \frac{5}{3} \rceil = n-2$. On the other hand, we obtain $\text{sdepth}\,(S/U_j) = \text{depth}\,(S/U_j) = n-2$ by [10, Theorem 1.1]. By applying Lemmas 2.5 and 2.6 on the exact sequences

$$0 \longrightarrow S/L_j \xrightarrow{\cdot x_{n-j+1}} S/L_{j-1} \longrightarrow S/U_j \longrightarrow 0 \quad \text{for } 1 \leq j \leq n-4,$$

we conclude $\text{depth}\,(S/J_{n,n-2}) \geq n-3$ and $\text{sdepth}\,(S/J_{n,n-2}) \geq n-3$.

On the other hand, by [10, Theorem 1.1] and [2, Proposition 2.7], we have $\text{depth}\,(S/J_{n,n-2}) \leq \text{depth}\,(S/L_{n-4})$ and $\text{sdepth}\,(S/J_{n,n-2}) \leq \text{sdepth}\,(S/L_{n-4})$. This completes the proof. □

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