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Warped product spaces with Ricci conditions

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Abstract: In this paper, we study the Ricci soliton in the Riemannian products $M = R^n \times B$ and warped products $M = R \times_f B$ of the Euclidean space and Riemannian manifolds, and the gradient Ricci soliton in the warped products $M = S^1 \times_f B$ of 1-dimensional circle and Riemannian manifolds. Moreover, we introduce the concept of the generalized Ricci soliton and we suggest the method of construction of the Riemannian manifold (M, g) with a Ricci soliton g. Finally, we also study the Lorentzian warped products with the Ricci soliton.

Key words: Ricci curvature, warped product space, Ricci soliton

1. Introduction

The concept of a Ricci soliton was introduced by Hamilton [7], which is both a generalization of the Einstein metric and a special solution of the Ricci flow.

A Riemannian metric g on a complete Riemannian manifold M is called a Ricci soliton if there exists a smooth vector field X such that the Ricci tensor satisfies the following equation:

$$Ric + \frac{1}{2}\mathfrak{L}_X g = \rho g \tag{1.1}$$

for some constant ρ, where $\mathfrak{L}_X$ is the Lie derivative with respect to X [2, 3, 5, 6, 9]. It is said that (M, g) or M is a Ricci soliton if the metric g on M is a Ricci soliton. The Ricci soliton is called shrinking if $\rho > 0$, steady if $\rho = 0$, and expanding if $\rho < 0$. The metric of a Ricci soliton is useful in not only physics but also mathematics, and it is often referred to as quasi-Einstein [4]. If $X = \nabla h$ for some function h on M, then M is called a gradient Ricci soliton. In this case, equation (1.1) can be rewritten as:

$$Ric + Hess\ h = \rho g, \tag{1.2}$$

and h is called a potential function. It is well known that when $\rho \leq 0$ all compact solitons are necessarily Einstein [6], and a Ricci soliton on a compact manifold has a constant curvature in 2 dimensions [7] as well as in 3 dimensions [8]. Moreover, a Ricci soliton on a compact manifold is a gradient Ricci soliton [9, 14] and Vaghef and Razavi [15] studied the stability of compact Ricci solitons under Ricci flow.

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In [12], we studied the gradient Ricci soliton in the warped product space and obtained the criterion for the base space of the warped product space with a gradient Ricci soliton to be a gradient Ricci soliton or an Einstein space by taking the derivative twice of the warping function. However, there are no such criteria or methods of construction of a Ricci soliton for the Riemannian products or warped products as far as we know. Thus, it is natural to consider the case of noncompact Riemannian products or warped product spaces with a Ricci soliton.

From this point of view, we study the Riemannian product space $M = R^n \times B^m$ with a Ricci soliton and obtain the fact that M is a Ricci soliton if and only if B is a Ricci soliton. Furthermore, we get a necessary condition for the base space to be a Ricci soliton in the warped products $R \times_f B$ with a Ricci soliton. Moreover, we introduce a generalized Ricci soliton with the Ricci soliton warping function f, and using Theorem 3.7 we can suggest the method of construction of the warped products $R \times_f B$ to admit a Ricci soliton. Precisely, we proved that if B is a generalized Ricci soliton with the Ricci soliton warping function f, then $M = R \times_f B^m$ is a Ricci soliton.

On the other hand, Kenmotsu [10] gave a characterization of the warped product space $L \times_f CE^n$ by tensor equations. By use of the almost contact structure on $L \times_f CE^n$ introduced by Kenmotsu [10] and combining our theorems, we see that the warped product space $M = R \times_f B$ with Ricci soliton for the structure vector is, in fact, Einstein and also B is Einstein, where B is a Kaehler manifold and $f = ce^t$ for a constant c. In the consideration of the warped product space $M = S^1(k) \times_f B$ with a gradient Ricci soliton, we studied the relationship of the warping function f and the Einstein metric on B. Moreover, we clarify the function h appearing in equation (1.2) for the warped products $M = S^1 \times_f B$.

Finally, we study the Lorentzian warped product space $R \times_f B$ with a Ricci soliton and obtain the necessary condition of the base space to be a Ricci soliton. For the converse of this case, we can construct the Lorentzian warped product space $R \times_f B$ admitting a Ricci soliton when the base space is a generalized Lorentzian Ricci soliton with a Lorentzian Ricci soliton warping function f. Consequently, it is possible to construct a Ricci soliton on the Riemannian product space or the warped product space by use of our results, and not only the Riemannian case but also the Lorentzian case.

2. Ricci solitons in Riemannian product manifolds

Let (B, g) be an m-dimensional Riemannian manifold with a metric g and let $M = R^n \times B$ be the Riemannian product manifold with the metric $\tilde{g} = \begin{pmatrix} \delta_{uv} & 0 \\ 0 & g_{ab} \end{pmatrix}$, where the range of indices $u, v, w, \cdots$ is $\{1, 2, \cdots, n\}$ and the range of indices $a, b, c, \cdots$ is $\{n+1, \cdots, n+m\}$.

Then the Ricci curvature tensors $\tilde{S}$ and S of M and B, respectively, are given by $\tilde{S}_{ab} = S_{ab}$ and the others are zero.

Suppose that B is a Ricci soliton. If we take $\widetilde{\rho} = \rho$ and a covector field $\tilde{U} = (\tilde{\xi}_u, \tilde{\xi}_a)$ on M by $\tilde{\xi}_u = \rho t_u, \tilde{\xi}_a = \xi_a$ on B, then we obtain

$$\begin{aligned} \tilde{S}_{ab} &= S_{ab} = \rho g_{ab} - \tfrac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a) = \tilde{\rho}\tilde{g}_{ab} - \tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a), \\ \tilde{S}_{au} &= 0 = -\tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_u + \tilde{\nabla}_u \tilde{\xi}_a), \\ \tilde{S}_{uv} &= 0 = \tilde{\rho}\delta_{uv} - \tfrac{1}{2}(\tilde{\nabla}_u \tilde{\xi}_v + \tilde{\nabla}_v \tilde{\xi}_u). \end{aligned} \tag{2.1}$$

Hence, we can see that if B is a Ricci soliton, then $M = R^n \times B$ is a Ricci soliton.

Conversely, if $M = R^n \times B$ is a Ricci soliton, then there exists a covector field $\tilde{V}$ on M such that

$$\widetilde{S}_{ij} = \widetilde{\rho}\, \widetilde{g}_{ij} - \frac{1}{2}(\widetilde{\nabla}_i \tilde{\xi}_j + \widetilde{\nabla}_j \tilde{\xi}_i)$$

for some constant function $\tilde{\rho}$ on M, where $\tilde{\xi}_i$ is a dual component of smooth vector field $\tilde{V}$ on M and the range of indices $i, j, k, \cdots$ is $\{1, 2, 3, \cdots, n, n+1, \cdots, n+m\}$. Thus, we have

$$\begin{aligned} S_{ab} = \tilde{S}_{ab} = \tilde{\rho} g_{ab} - \tfrac{1}{2}(\partial_a \tilde{\xi}_b + \partial_b \tilde{\xi}_a - 2 \left\{ \begin{array}{c} c \\ ab \end{array} \right\} \tilde{\xi}_c), \\ \partial_a \tilde{\xi}_u + \partial_u \tilde{\xi}_a = 0, \\ \tilde{\rho}\delta_{uv} - \tfrac{1}{2}(\partial_u \tilde{\xi}_v + \partial_v \tilde{\xi}_u) = 0. \end{aligned} \tag{2.2}$$

From the third equation of (2.2) with the case $u = v$, we obtain

$$\tilde{\xi}_u \quad = \quad \tilde{\rho} x_u + h^u_{(u)}, \tag{2.3}$$

where $h^u_{(u)}$ is a function on M having no x_u-variable. The third equation of (2.2) with the case $u \neq v$ and (2.3) give rise to

$$\partial_u h^v_{(v)} = -\partial_v h^u_{(u)}, \tag{2.4}$$

which means that $\partial_u h^v_{(v)}$ is a function on M having no x_u-variable and x_v-variable. Thus, we can put $H_{(u,v)} = \partial_u h^v_{(v)}$. Integrating this equation and using (2.3), we get

$$h^u_{(u)} = -H_{(u,v)} x_v + k^u_{(v)}, \tag{2.5}$$

where $k^u_{(v)}$ is a function on M having no x_v-variable. Then from equations (2.3) and (2.5), we obtain

$$\tilde{\xi}_u = \tilde{\rho} x_u - \sum_{u \neq v} H_{(u,v)} x_v + K^u_{(1,2,\cdots,n)}, (u \neq v). \tag{2.6}$$

On the other hand, the second equation of (2.2) and (2.6) give

$$\tilde{\xi}_a = -\sum_{u=1}^{n} \partial_a K^u_{(1,2,\cdots,n)} x_u + L^a_{(1,2,\cdots,n)} \tag{2.7}$$

and the first equation of (2.2) gives

$$S_{ab} - \tilde{\rho} g_{ab} = -\frac{1}{2}(\partial_a \tilde{\xi}_b + \partial_b \tilde{\xi}_a - 2 \left\{ \begin{array}{c} c \\ ab \end{array} \right\} \tilde{\xi}_c).$$

Hence, if we consider equations (2.6) and (2.7), we see that

$$\tilde{V} = (\tilde{\rho} x_1 - \sum_{1 \neq v} H_{(1,v)} x_v + K^1_{(1,2,\cdots,n)}, \cdots, \tilde{\rho} x_n - \sum_{n \neq v} H_{(n,v)} x_v + K^n_{(1,2,\cdots,n)},$$

$$-\sum_{u=1}^{n}\partial_{n+1}K^{u}_{(1,2,\cdots,n)}x_u+L^{n+1}_{(1,2,\cdots,n)},\cdots,-\sum_{u=1}^{n}\partial_{n+m}K^{u}_{(1,2,\cdots,n)}x_u+L^{n+m}_{(1,2,\cdots,n)}).$$

If we take $V=(\xi_{n+1},\ \cdots,\ \xi_{n+m})$ such that $\xi_a=L^a_{(1,2,\cdots,n)}$, then we get

$$S_{ab}-\rho g_{ab}=-\frac{1}{2}(\nabla_a L^b_{(1,2,\cdots,n)}+\nabla_b L^a_{(1,2,\cdots,n)})$$

for $\rho=\tilde{\rho}$, because V is a smooth vector field on B and $S_{ab}-\rho g_{ab}$ is independent of R^n. Hence, we can state that if $M=R^n\times B$ is a Ricci soliton, then B is a Ricci soliton. Thus, we have:

Theorem 2.1 *$M=R^n\times B$ is a Ricci soliton if and only if B is a Ricci soliton.*

3. Ricci solitons in warped product manifolds

Consider the warped product space $M=R\times_f B$ with $\tilde{g}=\begin{pmatrix}1 & 0\\ 0 & f^2g\end{pmatrix}$, where $f:R\to R^+$ is a warping function and g is a Riemannian metric on B. Then the Ricci curvature tensors $\tilde{S}$ and S of M and B respectively are given by [1, 11, 12]

$$\begin{aligned}\tilde{S}_{ab} &= S_{ab}-ff_{11}g_{ab}-(m-1)f_1^2 g_{ab},\\ \tilde{S}_{a1} &= 0,\\ \tilde{S}_{11} &= -\tfrac{mf_{11}}{f},\end{aligned}\tag{3.1}$$

where $f_1=\frac{df}{dt}, f_{11}=\frac{d^2f}{dt^2}, m=dimB$, and the range of indices $a,b,c,\cdots$ is $\{2,3,\cdots,m+1\}$.

Let $\tilde{\xi}_i$ be the dual components of $\tilde{\xi}^i$ for any vector field $\tilde{W}=(\tilde{\xi}^1,\cdots,\tilde{\xi}^{m+1})$ on M; then the covariant derivatives are given by

$$\begin{aligned}\widetilde{\nabla}_a\widetilde{\xi_b} &= \partial_a\tilde{\xi}_b-\left\{\begin{matrix}e\\ ab\end{matrix}\right\}\widetilde{\xi_e}+ff^1g_{ab}\widetilde{\xi_1},\\ \widetilde{\nabla}_a\widetilde{\xi_1} &= \partial_a\tilde{\xi}_1-\tfrac{f_1}{f}\tilde{\xi}_a,\\ \widetilde{\nabla}_1\widetilde{\xi_a} &= \partial_1\tilde{\xi}_a-\tfrac{f_1}{f}\tilde{\xi}_a,\\ \widetilde{\nabla}_1\widetilde{\xi_1} &= \partial_1\tilde{\xi}_1,\end{aligned}\tag{3.2}$$

where $\tilde{\nabla}$ and ∇ are operators of the covariant derivatives on M and B, respectively.

Suppose that $M=R\times_f B$ is a Ricci soliton. Then there exists a vector field $\tilde{V}$ on M such that $\tilde{S}_{ij}=\tilde{\rho}\tilde{g}_{ij}-\frac{1}{2}(\tilde{\nabla}_i\tilde{\xi}_j+\tilde{\nabla}_j\tilde{\xi}_i)$ for some constant function $\tilde{\rho}$ on M, where $\tilde{\xi}_i$ are dual components of smooth vector field $\tilde{V}$ on M and the range of indices $i,j,k,\cdots$ is $\{1,2,3,\cdots,m+1\}$. Thus, we have

$$\begin{aligned}\tilde{S}_{ab} &= \tilde{\rho}f^2g_{ab}-\tfrac{1}{2}(\tilde{\nabla}_a\tilde{\xi}_b+\tilde{\nabla}_b\tilde{\xi}_a),\\ \tilde{S}_{a1} &= -\tfrac{1}{2}(\tilde{\nabla}_a\tilde{\xi}_1+\tilde{\nabla}_1\tilde{\xi}_a) = -\tfrac{1}{2}(\partial_a\tilde{\xi}_1+\partial_1\tilde{\xi}_a-\tfrac{2f_1}{f}\tilde{\xi}_a),\\ \tilde{S}_{11} &= \tilde{\rho}-\tfrac{1}{2}(\tilde{\nabla}_1\tilde{\xi}_1+\tilde{\nabla}_1\tilde{\xi}_1) = \tilde{\rho}-\partial_1\tilde{\xi}_1,\end{aligned}\tag{3.3}$$

where ρ is a constant on M.

Using equations (3.1) and (3.3), we obtain

$$\begin{aligned} S_{ab} - ff_{11}g_{ab} - (m-1)f_1^2 g_{ab} &= \tilde{\rho} f^2 g_{ab} - \tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a), \\ \partial_a \tilde{\xi}_1 + \partial_1 \tilde{\xi}_a = \tfrac{2f_1}{f}\tilde{\xi}_a&, \\ \partial_1 \tilde{\xi}_1 = \tilde{\rho} + \tfrac{mf_{11}}{f}&. \end{aligned} \tag{3.4}$$

Assuming that $\tilde{\xi}_1 = \tilde{\xi}_1(t)$, then from the second equation of (3.4), we have $\partial_1 \tilde{\xi}_a = \frac{2f_1}{f}\tilde{\xi}_a$. This means that $\partial_1(ln\tilde{\xi}_a) = 2\partial_1 lnf$. Thus, we can put $\tilde{\xi}_a = f^2 C_a$, where C_a is a function on B. Moreover, we have

$$\tilde{\nabla}_b \tilde{\xi}_a = f^2(\nabla_b C_a) + ff^1 g_{ab}\tilde{\xi}_1(t). \tag{3.5}$$

Hence, the first equation of (3.4) and (3.5) give rise to

$$S_{ab} = \{ff_{11} + (m-1)f_1^2 + \tilde{\rho} f^2 - ff^1\tilde{\xi}_1(t)\}g_{ab} - \frac{f^2}{2}\{\nabla_a(C_b) + \nabla_b(C_a)\}. \tag{3.6}$$

If we take $\xi_b \equiv f^2 C_b$ on B, then we see that $\frac{f^2}{2}\{\nabla_a C_b + \nabla_b C_a\} = \frac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a)$. Since the coefficient of g_{ab} in (3.6) is a constant on B, we have:

Theorem 3.1 *If $M = R \times_f B$ is a Ricci soliton with $\tilde{\xi}_1 = \tilde{\xi}_1(t)$, then B is a Ricci soliton.*

In particular, if we consider the case that B is Kaehler manifold and $f = ce^t$ for a constant c, then $M = R \times_f B$ admits an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ such that [10]

$$\begin{aligned} (\nabla_X \phi)Y &= -\eta(Y)\phi X - \tilde{g}(X, \phi Y)\xi, \\ \nabla_X \eta &= X - \eta(X)\xi. \end{aligned} \tag{3.7}$$

Letting M be a Ricci soliton for a structure vector ξ, then $\tilde{S} + \frac{1}{2}\mathfrak{L}_\xi \tilde{g} = \rho\tilde{g}$ for a constant ρ. Since $\mathfrak{L}_\xi \tilde{g} = 2(\tilde{g} - \eta \otimes \eta)$ in M, $\tilde{S} = (\rho - 1)g + \eta \otimes \eta$; that is, M is η-Einstein with constant coefficients. The following is well known [10].

Lemma 3.2 *Let M be an almost contact Riemannian manifold satisfying (3.7). If M is η-Einstein with constant coefficients, then M is Einstein.*

From Lemma 3.2 and the above mentioned particular case, we can state:

Theorem 3.3 *Let B be a Kaehler manifold and $f = ce^t$ for a constant c. If $M = R \times_f B$ is a Ricci soliton for a structure vector ξ of an almost contact structure $(\phi, \xi, \eta, \tilde{g})$ that is induced on M, then M is Einstein.*

Related to the Riemannian manifold with a gradient Ricci soliton, we proved [12]:

Theorem 3.4 *Let $M = R \times_f B$ be a gradient Ricci soliton and $f''(t) \neq 0$. Then B is Einstein.*

Since Einstein is a generalization of the gradient Ricci soliton, if we combine Theorems 3.3 and 3.4, then we have:

Theorem 3.5 *Under the same assumptions as in Theorem 3.3, B is Einstein.*

On the other hand, for the converse of Theorem 3.1, we introduce the following definition as a generalization of the Ricci soliton.

Definition 3.6 *A Riemannian manifold B with a Riemannian metric g is called a generalized Ricci soliton if there exist smooth covector fields $V = (\eta_1)$ and $U = (\eta_a)$ on R and B, respectively, and a positive smooth function f on R such that*

$$S_{ab} = \{ff_{11} + (m-1)f_1^2 + \rho f^2 - ff_1\eta_1(t)\}g_{ab} - \frac{f^2}{2}(\nabla_a\eta_b + \nabla_b\eta_a) \tag{3.8}$$

for some constant ρ. In this case, we call f the Ricci soliton warping function. We easily see that the generalized Ricci soliton with $f^2 = 1$ becomes a Ricci soliton.

Now let g be a generalized Ricci soliton on B. From equation (3.8), we have:

$$\tilde{S} = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{1a} \\ \tilde{S}_{a1} & \tilde{S}_{ab} \end{pmatrix} = \begin{pmatrix} -\frac{mf_{11}}{f} & 0 \\ 0 & (\rho f^2 - ff_1\eta_1(t))g_{ab} - \frac{f^2}{2}(\nabla_a\eta_b + \nabla_b\eta_a) \end{pmatrix}.$$

If we take $\tilde{\rho} = \rho$, $\tilde{\eta}_1 = \int(\rho + \frac{mf_{11}}{f})dt$ with a real integral constant and $\tilde{\eta}_a = f^2\eta_a$, then we obtain

$$\tilde{\nabla}_1\tilde{\eta}_1 = \tilde{\rho} + \frac{mf_{11}}{f}, \tilde{\nabla}_a\tilde{\eta}_1 = -ff_1\eta_a, \tilde{\nabla}_1\tilde{\eta}_a = ff_1\eta_a, \tilde{\nabla}_a\tilde{\eta}_b = f^2(\nabla_a\eta_b) + ff^1\tilde{\eta}_1 g_{ab}.$$

Therefore, we see that

$$\tilde{\rho}\tilde{g} - \frac{1}{2}\mathfrak{L}_X\tilde{g} = \begin{pmatrix} -\frac{mf_{11}}{f} & 0 \\ 0 & \rho f^2 g_{ab} - \frac{1}{2}(f^2(\nabla_a\eta_b + \nabla_b\eta_a) + 2ff^1 g_{ab}\tilde{\eta}_1) \end{pmatrix}.$$

Thus, we have $\tilde{S} = \tilde{\rho}\tilde{g} - \frac{1}{2}\mathfrak{L}_X\tilde{g}$, and so $R \times_f B$ is a Ricci soliton.

Theorem 3.7 *If B is a generalized Ricci soliton with a Ricci soliton warping function f, then $M = R \times_f B$ is a Ricci soliton.*

By use of Theorem 3.7, we can construct Riemannian manifolds (M, g) with Ricci soliton g.

4. Gradient Ricci solitons in the warped product spaces $M = S^1(k) \times_f B$

Consider the warped product space $M = S^1(k) \times_f B$ of the $S^1(k)$ and the Riemannian manifold B with the metric $\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & f^2\bar{g} \end{pmatrix}$, where g and $\bar{g}$ are the metrics on $S^1(k)$ and an m-dimensional Riemannian space B, respectively. For a local coodinate system $u^1 = t$ of $S^1(k)$ and the metric tensor g has the component $g_{11} = 1 + \frac{t^2}{k^2 - t^2}$. Similarly, for a local coordinate system (u^x) of B, the metric tensor $\bar{g}$ has the components $\bar{g}_{xy}$. Then, with respect to the local coordinate system (t, u^x) of M, the metric $\tilde{g}$ has the components $\tilde{g}_{ij}$. Throughout this paper, the range of indices are as follows:

$$x, y, z, \cdots = 2, 3, \cdots, m+1 \text{ and } i, j, k, \cdots = 1, 2, \cdots, m+1.$$

Let $\tilde{\nabla}, \nabla$, and $\bar{\nabla}$ be the covariant derivatives with respect to $\tilde{g}, g$, and $\bar{g}$, respectively. Then $\widetilde{\left\{ \begin{array}{c} 1 \\ 11 \end{array} \right\}} = \frac{t}{k^2-t^2}$, $\widetilde{\left\{ \begin{array}{c} x \\ y1 \end{array} \right\}} = \frac{f_1}{f}\delta_y^x$, $\widetilde{\left\{ \begin{array}{c} 1 \\ yz \end{array} \right\}} = -ff^1\bar{g_{yz}}$, $\widetilde{\left\{ \begin{array}{c} x \\ yz \end{array} \right\}} = \bar{\left\{ \begin{array}{c} x \\ yz \end{array} \right\}}$, and the others are zero, where $f_1 = \frac{df}{dt}, f^1 = f_1 g^{11}$. Hence, we have $\tilde{R}_{111}{}^1 = R_{111}{}^1 = 0$, $\tilde{R}_{1x1}{}^y = \frac{1}{f}(\nabla_1 f_1)\delta_x^y$, $\tilde{R}_{xyz}{}^w = \bar{R}_{xyz}{}^w - ||f_1||^2(\bar{g}_{tz}\delta_x^w - \bar{g}_{xz}\delta_y^w)$, and the others are zero. Thus, for the components of Ricci curvature tensors $\tilde{S}$, S, and $\bar{S}$ of M, $S^1(k)$, and B, respectively, we have:

$$\begin{array}{rcl} \tilde{S}_{11} & = & S_{11} - \frac{m}{f}(\nabla_1 f_1) = -\frac{m}{f}(\nabla_1 f_1), \\ \tilde{S}_{1x} & = & 0, \\ \tilde{S}_{xy} & = & \bar{S}_{xy} - f(\triangle f)\bar{g}_{xy} - (m-1)||f_1||^2\bar{g}_{xy}, \end{array} \tag{4.1}$$

where $\triangle f = \nabla_1 f^1$. Now suppose that M is a gradient Ricci solution. Then we have

$$\begin{array}{rcl} \tilde{S}_{11} & = & \tilde{\rho}\tilde{g}_{11} - \tilde{\nabla}_1\tilde{\nabla}_1 h = \tilde{\rho}(1 + \frac{t^2}{k^2-t^2}) - (\partial_1 h_1 - \frac{th_1}{k^2-t^2}), \\ \tilde{S}_{1x} & = & \tilde{\rho}\tilde{g}_{1x} - \tilde{\nabla}_1\tilde{\nabla}_x h = -\partial_1 h_x + \frac{f_1}{f}h_x, \\ \tilde{S}_{xy} & = & \tilde{\rho}\tilde{g}_{xy} - \tilde{\nabla}_x\tilde{\nabla}_y h = \tilde{\rho}f^2\bar{g}_{xy} - (\bar{\nabla}_x\bar{\nabla}_y h + ff^1h_1\bar{g}_{xy}), \end{array} \tag{4.2}$$

for some constant $\tilde{\rho}$ and some function h on M. Considering $\nabla_1 f_1 = \partial_1 f_1 - \frac{t}{k^2-t^2}f_1$, and comparing (4.1) and (4.2), we obtain

$$\begin{array}{rcl} h_{11} - \frac{t}{k^2-t^2}h_1 & = & \tilde{\rho}(1 + \frac{t^2}{k^2-t^2}) + \frac{m}{f}(f_{11} - \frac{t}{k^2-t^2}f_1), \\ \partial_1 h_x & = & \frac{f_1}{f}h_x, \\ \bar{S}_{xy} & = & \{f(\triangle f) + (m-1)||f_1||^2 + \tilde{\rho}f^2 - ff^1h_1\}\bar{g}_{xy} - \bar{\nabla}_x\bar{\nabla}_y h, \end{array} \tag{4.3}$$

where we put $h_{11} = \partial_1 h_1$ and $f_{11} = \partial_1 f_1$.

Supposing that $h_x \neq 0$, then we have $\frac{\partial_1 h_x}{h_x} = \frac{f_1}{f}$. Hence, $\ln h_x = \ln f + C_x(u^2, \cdots, u^{m+1})$, where C_x is some function on B. Therefore, we have $h_x = fe^{C_x(u^2,\cdots,u^{m+1})}$ and that $h = fD(u^2, \cdot, u^{m+1}) + E(t)$, where $D(u^2, \cdot, u^{m+1})$ and $E(t)$ are some functions on B and $S^1(k)$, respectively. Here, $D(u^2, \cdot, u^{m+1})$ is not constant because $h_x \neq 0$. Thus, we have

$$\begin{array}{c} h_1 = f_1 D(u^2, \cdot, u^{m+1}) + E_1(t), \\ h_{11} = f_{11} D(u^2, \cdot, u^{m+1}) + E_{11}(t). \end{array} \tag{4.4}$$

Substituting (4.4) to the first equation of (4.3) leads to

$$\{(k^2-t^2)f_{11} - tf_1\}D(u^2, \cdot, u^{m+1}) = \tilde{\rho}k^2 + \frac{m}{f}\{(k^2-t^2)f_{11} - tf_1\} - (k^2-t^2)E_{11}(t) + tE_1(t). \tag{4.5}$$

The right-hand side of (4.5) is a function of t and the function $D(u^2, \cdot, u^{m+1})$ is independent of t and nonconstant. Thus, we have

$$(k^2-t^2)f_{11} - tf_1 = 0. \tag{4.6}$$

The general solution of (4.6) is $f = \alpha sin^{-1}\frac{t}{k} + \beta$. Hence, we can state that if $f \neq \alpha sin^{-1}\frac{t}{k} + \beta$, then $h_x = 0$, and h is of the form $h = h(t)$, $\bar{\nabla}_x\bar{\nabla}_y h = 0$, and hence we see that $\bar{S}_{xy} = A(t)\bar{g}_{xy}$ from the third equation of

(4.3), where $A(t) = f(\nabla f) + (m-1)||f_1||^2 + \tilde{\rho} f^2 - f f^1 h_1$ is constant on B. We conclude that B is an Einstein space. Thus, we have:

Theorem 4.1 *If the warped product space $M = S^1(k) \times_f B$ is a gradient Ricci soliton and $f \neq \alpha sin^{-1}\frac{t}{k} + \beta$, then B is an Einstein space.*

On the other hand, from (4.5) and (4.6), we have

$$(k^2 - t^2)E_{11}(t) - tE_1(t) = \tilde{\rho} k^2. \tag{4.7}$$

If we put $z = E_1(t)$, then we obtain

$$\sqrt{k^2 - t^2}\; z' - \frac{t}{\sqrt{k^2 - t^2}}\; z = \frac{\tilde{\rho} k^2}{\sqrt{k^2 - t^2}}. \tag{4.8}$$

Since the left-hand side of (4.8) is equal to $(\sqrt{k^2 - t^2}\; z)'$, equation (4.8) leads to

$$(\sqrt{k^2 - t^2}\; z)' = \frac{\tilde{\rho} k^2}{\sqrt{k^2 - t^2}}, \tag{4.9}$$

that is,

$$\sqrt{k^2 - t^2}\; z = \tilde{\rho} k^2 \int \frac{1}{\sqrt{k^2 - t^2}} dt. \tag{4.10}$$

Hence, we get

$$E_1 = z = \frac{\tilde{\rho} k^2}{\sqrt{k^2 - t^2}} sin^{-1}\frac{t}{k} = \tilde{\rho} k^2 (sin^{-1}\frac{t}{k})' sin^{-1}\frac{t}{k},$$

so that

$$E(t) = \frac{\tilde{\rho} k^2}{2}(sin^{-1}\frac{t}{k})^2. \tag{4.11}$$

Therefore, the function h becomes

$h = fD(u^1, \cdots, u^{m+1}) + E(t) = (\alpha sin^{-1}\frac{t}{k} + \beta)D(u^2, \cdots, u^{m+1}) + \frac{c}{2}(sin^{-1}\frac{t}{k})^2$, where $D(u^2, \cdots, u^{m+1})$ is nonconstant and independent of t, and α and β are constants.

For the construction of the model space of the warped product space $M = S^1(k) \times_f B$ with gradient Ricci soliton, we consider $f(t) = \alpha sin^{-1}\frac{t}{k} + \beta$ and $h = h(t)$ as a potential function (1.2). Then the following relations have to hold:

$$\tilde{\nabla}_1 h_1 = \frac{\tilde{\rho} k^2}{k^2 - t^2}, \tag{4.12}$$

$$\bar{S}_{xy} - (m-1)\frac{\alpha^2}{k^2}\bar{g} = \rho^2(\alpha sin^{-1}\frac{t}{k} + \beta)^2\bar{g} - \tilde{\nabla}_x\tilde{\nabla}_y h, \tag{4.13}$$

where $h_1 = \partial_t h$ and $m = dimB$.

Since $\tilde{\nabla}_1 h_1 = \partial_1 h_1 - \frac{t}{k^2-t^2} h_1$, equation (4.12) is rewritten as

$$\partial_1 h_1 - \frac{t}{k^2-t^2} h_1 = \frac{\tilde{\rho} k^2}{k^2-t^2}. \tag{4.14}$$

If we put $z = h_1$, then we get

$$\sqrt{k^2-t^2} z' - \frac{t}{\sqrt{k^2-t^2}} z = \frac{\tilde{\rho} k^2}{\sqrt{k^2-t^2}}, \tag{4.15}$$

which is the same type as (4.8). Hence, we obtain $h(t) = \frac{\tilde{\rho} k^2}{2} (sin^{-1} \frac{t}{k})^2$, so if we take the fiber B as an Einstein manifold with the scalar curvature

$$m((m-1)\frac{\alpha^2}{k^2} + \tilde{\rho}^2 (\alpha sin^{-1} \frac{t}{k} + \beta)^2), \tag{4.16}$$

then relation (4.13) is satisfied. Hence, we have:

Theorem 4.2 *Let $f = \alpha sin^{-1} \frac{t}{k} + \beta$ and B be an Einstein manifold with scalar curvature (4.16). Then the warped product space $S^1(k) \times_f B$ admits a gradient Ricci soliton having the potential function $h = \frac{\tilde{\rho} k^2}{2} (sin^{-1} \frac{t}{k})^2$.*

Hence, if we use Theorem 4.2, then we can construct a Riemannian manifold with gradient Ricci soliton in the warped product space.

5. Ricci solitons in Lorentzian warped product spaces

The metric in the Lorentzian warped product space $M = R \times_f B$ is given by $\tilde{g} = \begin{pmatrix} -1 & 0 \\ 0 & f^2 g \end{pmatrix}$, where $f : R \to R^+$ is a warping function, and g is the Riemannian metric on B. It is well known that $\widetilde{\left\{ \begin{matrix} 1 \\ ab \end{matrix} \right\}} = f f_1 g_{ab}$, $\widetilde{\left\{ \begin{matrix} a \\ b1 \end{matrix} \right\}} = \frac{f_1}{f} \delta^a_b$, $\widetilde{\left\{ \begin{matrix} a \\ bc \end{matrix} \right\}} = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\}$, and the others are zero, where the range of indices $a, b, c, \cdots$ is $\{2, 3, \cdots, m+1\}$ and $m = dim B$.

The curvatures $\tilde{K}$ and K of M and B are given by $\tilde{K}_{dcb}{}^a = K_{dcb}{}^a + f_1^2(\delta^a_d g_{cb} - \delta^a_c g_{db})$, $\tilde{K}_{1ab}{}^1 = f f_{11} g_{ab}$, $\tilde{K}_{b11}{}^a = -\frac{f_{11}}{f} \delta^a_b$, and the others are zero. Moreover, the Ricci curvature tensors $\tilde{S}$ and S of M and B, respectively, are reduced to [11, 13]

$$\begin{aligned} \tilde{S}_{ab} &= S_{ab} + f f_{11} g_{ab} + (m-1) f_1^2 g_{ab}, \\ \tilde{S}_{a1} &= 0, \\ \tilde{S}_{11} &= -\tfrac{m f_{11}}{f}. \end{aligned} \tag{5.1}$$

Suppose that $M = R \times_f B$ is a Lorentzian Ricci soliton. Then there exists a vector field $\tilde{V}$ on M such that

$$\tilde{S}_{ij} = \tilde{\rho} \tilde{g}_{ij} - \frac{1}{2}(\tilde{\nabla}_i \tilde{\xi}_j + \tilde{\nabla}_j \tilde{\xi}_i)$$

for some constant $\tilde{\rho}$ on M, where $\tilde{\xi}_i$ are dual components of smooth vector field $\tilde{V}$ on M. Thus, we have

$$\begin{array}{rcl} \tilde{S}_{ab} &=& \tilde{\rho} f^2 g_{ab} - \frac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a), \\ \tilde{S}_{a1} &=& -\frac{1}{2}(\partial_a \tilde{\xi}_1 + \partial_1 \tilde{\xi}_a - \frac{2f_1}{f}\tilde{\xi}_a), \\ \tilde{S}_{11} &=& -\tilde{\rho} - \partial_1 \tilde{\xi}_1, \end{array} \tag{5.2}$$

where ρ is a constant on M.

From equations (5.1) and (5.2), we obtain

$$\begin{array}{rcl} S_{ab} + f f_{11} g_{ab} + (m-1) f_1^2 g_{ab} &=& \tilde{\rho} f^2 g_{ab} - \frac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a), \\ \partial_a \tilde{\xi}_1 + \partial_1 \tilde{\xi}_a &=& \frac{2f_1}{f}\tilde{\xi}_a, \\ \partial_1 \tilde{\xi}_1 &=& -\tilde{\rho} + \frac{m f_{11}}{f}. \end{array} \tag{5.3}$$

The third equation of (5.3) gives

$$\tilde{\xi}_1 = -\tilde{\rho} t + \int \frac{m f_{11}}{f} dt. \tag{5.4}$$

Assuming that $\tilde{\xi}_1 = \tilde{\xi}_1(t)$ and considering the second equation of (5.3) and (5.4), we have $\partial_1 \tilde{\xi}_a = \frac{2f_1}{f}\tilde{\xi}_a$. Thus, we can have $\tilde{\xi}_a = f^2 C_a$, where C_a is a function on B. Moreover, we have

$$\tilde{\nabla}_b \tilde{\xi}_a = f^2(\nabla_b C_a) - f f^1 g_{ab} \tilde{\xi}_1(t). \tag{5.5}$$

Hence, the first equation of (5.3) and (5.5) give rise to

$$S_{ab} = \{-f f_{11} - (m-1) f_1^2 + \tilde{\rho} f^2 + f f^1 \tilde{\xi}_1(t)\} g_{ab} - \frac{f^2}{2}\{\nabla_a(C_b) + \nabla_b(C_a)\}. \tag{5.6}$$

If we take $\xi_b \equiv f^2 C_b$ on B, then we see that B is a Ricci soliton from equation (5.6). Thus, we have:

Theorem 5.1 *If $M = R \times_f B$ is a Ricci soliton with $\tilde{\xi}_1 = \tilde{\xi}_1(t)$, then B is a Ricci soliton.*

For the converse of Theorem 5.1, we introduce the following definition as a generalization of the Ricci soliton.

Definition 5.2 *A Riemannian manifold B with a Riemannian metric g is called a generalized Lorentzian Ricci soliton if there exist smooth covector fields $V = (\eta_1)$ and $U = (\eta_a)$ on R and B, respectively, and a positive smooth function f on R such that*

$$S_{ab} = \{-f f_{11} - (m-1) f_1^2 + \rho f^2 + f f_1 \eta_1(t)\} g_{ab} - \frac{f^2}{2}(\nabla_a \eta_b + \nabla_b \eta_a) \tag{5.7}$$

for some constant ρ. In this case, we call f the Ricci soliton warping function. We easily see that the generalized Lorentzian Ricci soliton with $f^2 = 1$ becomes a Ricci soliton.

Now let B be a generalized Lorentzian Ricci soliton. Then we have, from equation (5.7),

$$\tilde{S} = \begin{pmatrix} -\frac{mf_{11}}{f} & 0 \\ 0 & (\rho f^2 + ff_1\eta_1(t))g_{ab} - \frac{f^2}{2}(\nabla_a\eta_b + \nabla_b\eta_a) \end{pmatrix}.$$

If we take $\tilde{\rho} = \rho$, $\tilde{\eta}_1 = \int(-\rho + \frac{mf_{11}}{f})dt$ with a real integral constant and $\tilde{\eta}_a = f^2\eta_a$, then we obtain

$$\tilde{\nabla}_1\tilde{\eta}_1 = -\tilde{\rho} + \frac{mf_{11}}{f}, \tilde{\nabla}_a\tilde{\eta}_1 = -ff_1\eta_a, \tilde{\nabla}_1\tilde{\eta}_a = ff_1\eta_a, \tilde{\nabla}_a\tilde{\eta}_b = f^2(\nabla_a\eta_b) - ff^1\tilde{\eta}_1 g_{ab}.$$

Therefore, we see that

$$\tilde{\rho}\tilde{g} - \frac{1}{2}\mathfrak{L}_X\tilde{g} = \begin{pmatrix} -\frac{mf_{11}}{f} & 0 \\ 0 & \rho f^2 g_{ab} - \frac{1}{2}(f^2(\nabla_a\eta_b + \nabla_b\eta_a) - ff^1 g_{ab}\tilde{\eta}_1) \end{pmatrix}.$$

Thus, we have $\tilde{S} = \tilde{\rho}\tilde{g} - \frac{1}{2}\mathfrak{L}_X\tilde{g}$, and so $R \times_f B$ is a Ricci soliton.

Theorem 5.3 *If B is a generalized Lorentzian Ricci soliton with a Lorentzian Ricci soliton warping function f, then $M = R \times_f B$ is a Lorentzian Ricci soliton.*

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