

More accurate Jensen-type inequalities for signed measures characterized via Green function and applications

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Abstract: In this paper we derive several improved forms of the Jensen inequality, giving the necessary and sufficient conditions for them to hold in the case of the real Stieltjes measure not necessarily positive. The obtained relations are characterized via the Green function. As an application, our main results are employed for constructing some classes of exponentially convex functions and some Cauchy-type means.

Key words: Jensen inequality, Green function, refinement, convex function, exponentially convex function, Cauchy-type mean

1. Introduction

The Jensen inequality is one of the most important inequalities in mathematical analysis and its applications. Recently, Pečarić et al. [6] established conditions on a real Stieltjes measure $d\lambda$, not necessarily positive, under which the Jensen inequality and its reverse hold for a continuous convex function. These inequalities are characterized via the Green function $G : [\alpha, \beta] \times [\alpha, \beta] \rightarrow \mathbb{R}$ defined by

$$G(t, s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha} & \text{for } \alpha \leq s \leq t, \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha} & \text{for } t \leq s \leq \beta. \end{cases} \quad (1)$$

The corresponding result reads as follows: let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function, and let $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of a bounded variation such that $\lambda(a) \neq \lambda(b)$, and $\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)} \in [\alpha, \beta]$.

Then the following statements are equivalent:

(i) For every continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$ the following inequality holds:

$$\varphi \left(\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)} \right) \leq \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)}. \quad (2)$$

(ii) For all $s \in [\alpha, \beta]$ the following inequality holds:

$$G \left(\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}, s \right) \leq \frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)}. \quad (3)$$

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In addition, statements (i) and (ii) are also equivalent by changing the sign in both inequalities (2) and (3). Hence, the Jensen inequality (2) and its reverse are characterized via relation (3). It should be noticed here that in the case of a positive measure $d\lambda$, where λ is increasing and bounded, the inequality (2) reduces to the classical integral Jensen inequality.

There are several reverses of the Jensen inequality, one of the most significant of which is the Lah–Ribarić inequality. We single out the corresponding result for a real Stieltjes measure $d\lambda$ also derived in [6]: let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M]$, and let $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of a bounded variation such that $\lambda(a) \neq \lambda(b)$. Then the following statements are equivalent:

(i) For every continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$ the following inequality holds:

$$\frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} \leq \frac{M - \bar{g}}{M - m} \varphi(m) + \frac{\bar{g} - m}{M - m} \varphi(M), \tag{4}$$

where $\bar{g} = \frac{\int_a^b g(x) d\lambda(x)}{\int_a^b d\lambda(x)}$.

(ii) For all $s \in [\alpha, \beta]$ the following inequality holds:

$$\frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} \leq \frac{M - \bar{g}}{M - m} G(m, s) + \frac{\bar{g} - m}{M - m} G(M, s). \tag{5}$$

Motivated by [6], Jakšić et al. [4] obtained several reverses of (2) also characterized via the Green function. In particular, they showed that if (5) holds, then for every continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$ the following inequality holds:

$$\frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) \leq \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'_-(M) - \varphi'_+(m)), \tag{6}$$

provided that $g([a, b]) \subseteq [m, M] \subseteq (\alpha, \beta)$ and $\bar{g} \in [m, M]$.

The main objective of the present paper is to establish several refinements and reverses of inequalities (2) and (4). These more accurate relations will also be characterized via the Green function. In particular, we are going to show that the inequality (6) also holds if the same inequality holds with a Green function $G(x, \cdot)$ instead of a continuous convex function $\varphi(x)$.

The paper is divided into four sections, as follows: after this introduction, in Section 2, we give our main results. We give two improved forms of the Jensen inequality (2) and an improved form of the Lah–Ribarić inequality (4), all of which are characterized via the same inequality, but with a Green function instead of an arbitrary continuous convex function. As an application, we utilize established inequalities to construct some classes of exponentially convex functions and some Cauchy-type means. More precisely, in Section 3 we give mean value theorems arising from improved inequalities, which is the crucial step in obtaining the corresponding Cauchy-type means. Finally, combining our improved Jensen-type inequalities and a general exponential convexity method developed in [3], in Section 4 we obtain several classes of exponentially convex functions.

2. Main results

In this section we give improved versions of the Jensen inequality (2) and the Lah–Ribarič inequality (4); that is, we give the corresponding refinements and reverses. As we have already mentioned, these more accurate relations are also characterized by the Green formula. Contrary to (6), these new improvements are characterized via the same inequality, but with a Green function (1) instead of an arbitrary continuous convex function. Obviously, the Green function is continuous and convex in each variable. Roughly speaking, we are going to show that if the corresponding refinement or the reverse holds for the Green function with a fixed variable, then it holds for every continuous convex function.

The crucial step in establishing our results is the fact that every function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, can be represented as

$$\varphi(x) = \frac{\beta - x}{\beta - \alpha} \varphi(\alpha) + \frac{x - \alpha}{\beta - \alpha} \varphi(\beta) + \int_{\alpha}^{\beta} G(x, s) \varphi''(s) ds, \tag{7}$$

where the function G is defined by (1), which can be easily shown by integrating by parts (see also [8]).

Now our first result is an improvement of inequality (2). In particular, we show that the inequality (6) holds under a different condition also including the Green function. In order to shorten the notation, throughout this paper we use the notation $\bar{g} = \frac{\int_{\alpha}^{\beta} g(x) d\lambda(x)}{\int_{\alpha}^{\beta} d\lambda(x)}$.

Theorem 2.1 *Let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M] \subseteq (\alpha, \beta)$, and let $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of a bounded variation such that $\lambda(a) \neq \lambda(b)$. If $\bar{g} \in [\alpha, \beta]$, then the following two statements are equivalent:*

(i) *The inequality*

$$\frac{\int_{\alpha}^{\beta} \varphi(g(x)) d\lambda(x)}{\int_{\alpha}^{\beta} d\lambda(x)} - \varphi(\bar{g}) \leq \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'_-(M) - \varphi'_+(m)) \tag{8}$$

holds for every continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$.

(ii) *The inequality*

$$\frac{\int_{\alpha}^{\beta} G(g(x), s) d\lambda(x)}{\int_{\alpha}^{\beta} d\lambda(x)} - G(\bar{g}, s) \leq \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (G'_x(M, s) - G'_x(m, s)) \tag{9}$$

holds for all $s \in [\alpha, \beta]$, where the function G is defined by (1).

In addition, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both relations (8) and (9).

Proof The first implication (i) \Rightarrow (ii) is trivial since the function $G(\cdot, s)$ is continuous and convex on $[\alpha, \beta]$, for every fixed value $s \in [\alpha, \beta]$.

Now we show that (ii) \Rightarrow (i). We first prove that the statement (ii) implies the relation (8) for the case of a convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$ such that $\varphi \in C^2([\alpha, \beta])$. Namely, since $\varphi \in C^2([\alpha, \beta])$, utilizing the representation formula (7), it follows that

$$\varphi'(x) = \frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha} + \int_{\alpha}^{\beta} G'_x(x, s) \varphi''(s) ds,$$

so the difference between the left-hand side and the right-hand side of the inequality (8) can be rewritten in the following form:

$$\begin{aligned} & \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'(M) - \varphi'(m)) \\ &= \int_\alpha^\beta \left[\frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G(\bar{g}, s) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (G'_x(M, s) - G'_x(m, s)) \right] \varphi''(s) ds. \end{aligned} \tag{10}$$

Now, since φ is in addition convex, it follows that $\varphi''(s) \geq 0$ for all $s \in [\alpha, \beta]$. Therefore, with the assumption (ii), it follows that the right-hand side of relation (10) is not greater than zero. This means that (8) holds for a convex function $\varphi \in C^2([\alpha, \beta])$.

Furthermore, it should be noticed that it is not necessary to demand the existence of the second derivative of the function φ (see [7, p. 172] and references therein). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The remaining part of the theorem referring to relations with a reversed sign of inequality is proved in the same way. □

Remark 2.2 *Observe that in the statement of Theorem 2.1 the interval $[m, M]$ belongs to the interior of the interval $[\alpha, \beta]$. This condition assures finiteness of the one-sided derivatives in (8). Without this assumption, these derivatives might be infinite.*

Remark 2.3 *If $\bar{g} \in [m, M]$, the inequality (8) represents the reverse of (2), while the inequality with the reversed sign represents the refinement of the Jensen inequality (2).*

Theorem 2.1 refers to a convex function φ . The same conclusion can be drawn for the case of a concave function.

Remark 2.4 *Suppose that the assumptions as in Theorem 2.1 are fulfilled. Then the following statements are equivalent:*

(i') *The reverse inequality in (8) holds for every continuous concave function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$.*

(ii') *The inequality (9) holds for all $s \in [\alpha, \beta]$.*

In addition, the statements (i') and (ii') are also equivalent by changing the sign of inequality in the corresponding relations.

Now we give another type of improvement of the Jensen inequality (2), which is characterized again via the Green function, but this time without one-sided derivatives.

Theorem 2.5 *Let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M]$, and let $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of a bounded variation such that $\lambda(a) \neq \lambda(b)$. If $\bar{g} \in [\alpha, \beta]$, then the following two statements are equivalent:*

(i) The inequality

$$\frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) \leq \max \left\{ \frac{M - \bar{g}}{M - m}, \frac{\bar{g} - m}{M - m} \right\} \cdot \left[\varphi(m) + \varphi(M) - 2\varphi\left(\frac{m + M}{2}\right) \right] \quad (11)$$

holds for every continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$.

(ii) The inequality

$$\frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G(\bar{g}, s) \leq \max \left\{ \frac{M - \bar{g}}{M - m}, \frac{\bar{g} - m}{M - m} \right\} \cdot \left[G(m, s) + G(M, s) - 2G\left(\frac{m + M}{2}, s\right) \right] \quad (12)$$

holds for all $s \in [\alpha, \beta]$, where the function G is defined by (1).

Furthermore, the statements (i) and (ii) are also equivalent with the reversed sign of inequality in (11) and (12).

Proof The proof is analogous to the proof of Theorem 2.1. We only prove the implication (ii) \Rightarrow (i). Namely, if $\varphi \in C^2([\alpha, \beta])$ is convex, then, utilizing (7), the difference between the left-hand side and the right-hand side of the inequality (11) can be rewritten in the following form:

$$\begin{aligned} & \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) - \max \left\{ \frac{M - \bar{g}}{M - m}, \frac{\bar{g} - m}{M - m} \right\} \cdot \left[\varphi(m) + \varphi(M) - 2\varphi\left(\frac{m + M}{2}\right) \right] \\ &= \int_\alpha^\beta \left[\frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G(\bar{g}, s) - \max \left\{ \frac{M - \bar{g}}{M - m}, \frac{\bar{g} - m}{M - m} \right\} \cdot \left[G(m, s) + G(M, s) - 2G\left(\frac{m + M}{2}, s\right) \right] \right] \varphi''(s) ds. \end{aligned} \quad (13)$$

Now, since φ is convex, the inequality (12) implies that the right-hand side of (13) is not greater than zero. In addition, the differentiability condition can be omitted by the same argumentation as in the proof of Theorem 2.1. This means that (11) holds. \square

Remark 2.6 Provided that $\bar{g} \in [m, M]$, the inequality (11) represents the reverse of the Jensen inequality (2), while the inequality with the reversed sign represents its refinement.

Remark 2.7 It was shown in [4] that the inequality (11) is also valid provided that (5) holds.

Our next theorem yields the refinement and the reverse of the Lah–Ribarić inequality (4).

Theorem 2.8 Let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M] \subseteq (\alpha, \beta)$, and let $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of a bounded variation such that $\lambda(a) \neq \lambda(b)$. Then the following two statements are equivalent:

(i) The inequality

$$\frac{M - \bar{g}}{M - m} \varphi(m) + \frac{\bar{g} - m}{M - m} \varphi(M) - \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} \leq \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'_-(M) - \varphi'_+(m)) \quad (14)$$

holds for every continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$.

(ii) *The inequality*

$$\frac{M - \bar{g}}{M - m}G(m, s) + \frac{\bar{g} - m}{M - m}G(M, s) - \frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} \leq \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (G'_x(M, s) - G'_x(m, s)) \tag{15}$$

holds for all $s \in [\alpha, \beta]$, where the function G is defined by (1).

The statements (i) and (ii) are also equivalent if we change the sign of inequality in both relations (14) and (15).

Proof The proof follows the lines of the proof of Theorem 2.1. We only prove that (ii) implies (i). Let $\varphi \in C^2([\alpha, \beta])$ be a convex function. Then, utilizing the representation formula (7), the difference between the left-hand side and the right-hand side of inequality (14) can be transformed in the following way:

$$\begin{aligned} & \frac{M - \bar{g}}{M - m}\varphi(m) + \frac{\bar{g} - m}{M - m}\varphi(M) - \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'(M) - \varphi'(m)) \\ &= \int_\alpha^\beta \left[\frac{M - \bar{g}}{M - m}G(m, s) + \frac{\bar{g} - m}{M - m}G(M, s) - \frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} \right. \\ & \quad \left. - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (G'_x(M, s) - G'_x(m, s)) \right] \varphi''(s) ds. \end{aligned} \tag{16}$$

Now, due to convexity of φ and taking into account that (ii) holds, it follows that the right-hand side of (16) is not greater than zero. This means that (14) holds for a convex function $\varphi \in C^2([\alpha, \beta])$. Now, in the same way as in Theorem 2.1, the differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials. \square

Remark 2.9 *It should be noticed here that in the previous theorem \bar{g} does not have to belong to the interval $[\alpha, \beta]$. In the case when $\bar{g} \in [m, M]$, the inequality (14) represents the reverse of the Lah–Ribarić inequality (4), while the inequality with the reversed sign represents its refinement.*

Remark 2.10 *Theorems 2.5 and 2.8 refer to a convex function φ . The case of the concave function is treated in the same way as in Remark 2.4.*

3. Mean-value theorems

The improved Jensen-type inequalities derived in Section 2 can be utilized in obtaining some means of Cauchy type. The crucial step in this direction is to establish mean-value theorems arising from Theorems 2.1, 2.5, and 2.8. The starting point in this direction is to construct the corresponding functionals as the differences between the right-hand sides and the left-hand sides of inequalities (8), (11), and (14).

As in the previous section, let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M] \subseteq (\alpha, \beta)$, and let $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of a bounded variation such that $\lambda(a) \neq \lambda(b)$. Motivated by inequalities (8), (11), and (14), for g, λ and for a continuous convex function $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$, we

define three functionals:

$$A_1(g, \lambda, \varphi) = \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'_-(M) - \varphi'_+(m)),$$

$$A_2(g, \lambda, \varphi) = \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) - \max \left\{ \frac{M - \bar{g}}{M - m}, \frac{\bar{g} - m}{M - m} \right\} \cdot \left[\varphi(m) + \varphi(M) - 2\varphi\left(\frac{m + M}{2}\right) \right],$$

$$A_3(g, \lambda, \varphi) = \frac{M - \bar{g}}{M - m} \varphi(m) + \frac{\bar{g} - m}{M - m} \varphi(M) - \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'_-(M) - \varphi'_+(m)),$$

where $\bar{g} \in [\alpha, \beta]$ for $A_1(g, \lambda, \varphi)$ and $A_2(g, \lambda, \varphi)$.

Taking into account Theorems 2.1, 2.5, and 2.8, it follows that:

- $A_1(g, \lambda, \varphi) \leq 0$ if for all $s \in [\alpha, \beta]$ the inequality (9) holds, and $A_1(g, \lambda, \varphi) \geq 0$ if for all $s \in [\alpha, \beta]$ the reverse inequality in (9) holds;
- $A_2(g, \lambda, \varphi) \leq 0$ if for all $s \in [\alpha, \beta]$ the inequality (12) holds, and $A_3(g, \lambda, \varphi) \geq 0$ if for all $s \in [\alpha, \beta]$ the reverse inequality in (12) holds;
- $A_3(g, \lambda, \varphi) \leq 0$ if for all $s \in [\alpha, \beta]$ the inequality (15) holds, and $A_2(g, \lambda, \varphi) \geq 0$ if for all $s \in [\alpha, \beta]$ the reverse inequality in (15) holds.

In the sequel we give two mean value theorems of Lagrange and Cauchy type for each of the functionals $A_i(g, \lambda, \varphi)$, $i = 1, 2, 3$.

Theorem 3.1 *Let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M] \subseteq (\alpha, \beta)$. Furthermore, let $\varphi : [\alpha, \beta] \rightarrow \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of bounded variation such that $\lambda(a) \neq \lambda(b)$, and let $\phi_0(t) = t^2$.*

If for all $s \in [\alpha, \beta]$ the inequality (9) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (9) holds, then there exists $\xi \in [\alpha, \beta]$ such that

$$A_1(g, \lambda, \varphi) = \frac{1}{2} \varphi''(\xi) A_1(g, \lambda, \phi_0). \tag{17}$$

If for all $s \in [\alpha, \beta]$ the inequality (12) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (12) holds, then there exists $\xi \in [\alpha, \beta]$ such that

$$A_2(g, \lambda, \varphi) = \frac{1}{2} \varphi''(\xi) A_2(g, \lambda, \phi_0). \tag{18}$$

If for all $s \in [\alpha, \beta]$ the inequality (15) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (15) holds, then there exists $\xi \in [\alpha, \beta]$ such that

$$A_3(g, \lambda, \varphi) = \frac{1}{2} \varphi''(\xi) A_3(g, \lambda, \phi_0). \tag{19}$$

Proof We show the relation (17). Following the assumptions of the theorem, we have that the function φ'' is continuous and

$$\frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G(\bar{g}, s) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (G'_x(M, s) - G'_x(m, s))$$

does not change the sign on $[\alpha, \beta]$. Moreover, utilizing the relation (10) and the integral mean-value theorem, it follows that there exists $\xi \in [\alpha, \beta]$ such that

$$\begin{aligned} & \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'(M) - \varphi'(m)) \\ &= \varphi''(\xi) \int_\alpha^\beta \left[\frac{\int_a^b G(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G(\bar{g}, s) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (G'_x(M, s) - G'_x(m, s)) \right] ds. \end{aligned} \tag{20}$$

Now a straightforward calculation yields

$$\int_\alpha^\beta G(t, s) ds = \int_\alpha^t \frac{(t - \beta)(s - \alpha)}{\beta - \alpha} ds + \int_t^\beta \frac{(s - \beta)(t - \alpha)}{\beta - \alpha} ds = \frac{1}{2}(t - \alpha)(t - \beta)$$

and

$$\int_\alpha^\beta G'_x(t, s) ds = \int_\alpha^t \frac{s - \alpha}{\beta - \alpha} ds + \int_t^\beta \frac{s - \beta}{\beta - \alpha} ds = t - \frac{1}{2}(\alpha + \beta).$$

Finally, calculating the integral on the right side of (20), we have

$$\begin{aligned} & \frac{\int_a^b \varphi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi(\bar{g}) - \frac{(M - \bar{g})(\bar{g} - m)}{M - m} (\varphi'(M) - \varphi'(m)) \\ &= \varphi''(\xi) \left[\frac{\int_a^b \int_\alpha^\beta G(g(x), s) ds d\lambda(x)}{\int_a^b d\lambda(x)} - \int_\alpha^\beta G(\bar{g}, s) ds \frac{(M - \bar{g})(\bar{g} - m)}{M - m} \int_\alpha^\beta (G'_x(M, s) - G'_x(m, s)) ds \right] \\ &= \frac{1}{2} \varphi''(\xi) \left[\frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} - \bar{g}^2 - 2(M - \bar{g})(\bar{g} - m) \right] \\ &= \frac{1}{2} \varphi''(\xi) A_1(g, \lambda, \phi_0), \end{aligned}$$

which proves the relation (17).

To prove relations (18) and (19), we proceed in the same way except that we utilize relations (13) and (16) instead of (10). □

Theorem 3.2 Let $g : [a, b] \rightarrow [\alpha, \beta]$ be a continuous function such that $g([a, b]) \subseteq [m, M] \subseteq (\alpha, \beta)$. Furthermore, let $\varphi, \psi : [\alpha, \beta] \rightarrow \mathbb{R}$, $\varphi, \psi \in C^2([\alpha, \beta])$, and $\lambda : [a, b] \rightarrow \mathbb{R}$ be a continuous function or a function of bounded variation such that $\lambda(a) \neq \lambda(b)$.

If for all $s \in [\alpha, \beta]$ the inequality (9) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (9) holds, then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{A_1(g, \lambda, \varphi)}{A_1(g, \lambda, \psi)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \quad A_1(g, \lambda, \psi) \neq 0. \tag{21}$$

If for all $s \in [\alpha, \beta]$ the inequality (12) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (12) holds, then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{A_2(g, \lambda, \varphi)}{A_2(g, \lambda, \psi)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \quad A_2(g, \lambda, \psi) \neq 0. \tag{22}$$

If for all $s \in [\alpha, \beta]$ the inequality (15) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (15) holds, then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{A_3(g, \lambda, \varphi)}{A_3(g, \lambda, \psi)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \quad A_3(g, \lambda, \psi) \neq 0. \tag{23}$$

Proof We prove (21) only. Define a function χ as a linear combination of functions φ and ψ by $\chi(t) = A_1(g, \lambda, \psi) \cdot \varphi(t) - A_1(g, \lambda, \varphi) \cdot \psi(t)$. Applying the relation (17) to the function χ , after a straightforward calculation we obtain that there exists $\xi \in [\alpha, \beta]$ such that

$$\left(A_1(g, \lambda, \psi) \frac{\varphi''(\xi)}{2} - A_1(g, \lambda, \varphi) \frac{\psi''(\xi)}{2} \right) A_1(g, \lambda, \phi_0) = 0,$$

where ϕ_0 stands for the quadratic function $\phi_0(t) = t^2$. Since $A_1(g, \lambda, \phi_0) \neq 0$ (otherwise we would have a contradiction with $A_1(g, \lambda, \psi) \neq 0$), we get the assertion of the theorem. □

4. Applications to exponential convexity

In this section we are going to use Theorems 2.1, 2.5, and 2.8 to construct some new classes of exponentially convex functions as well as some interesting Cauchy-type means. These results rely on a general method of constructing exponentially convex functions and means of Cauchy type developed in [3].

Exponentially convex functions were invented by Bernstein in [2] as a subclass of convex functions in a given open interval. These functions have many nice properties; for example, they are analytical on their domain. Although we need only a few of these properties we point out here that a good reference on general results about exponential convexity is [1]. For some recent results about exponential convexity, the reader is referred to [3] and [5].

From now on, I stands for an open interval in \mathbb{R} . Recall that a function $f : I \rightarrow \mathbb{R}$ is n -exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^n p_i p_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all $p_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \dots, n$. In addition, $f : I \rightarrow \mathbb{R}$ is n -exponentially convex if it is n -exponentially convex in the Jensen sense and continuous on I .

Clearly, 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Additionally, n -exponentially convex functions in the Jensen sense are k -exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

A function $f : I \rightarrow \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n -exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. A function $f : I \rightarrow \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 4.1 *Some examples of exponentially convex functions are (for more details, see [3]):*

(i) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = ce^{kx}$, where $c \geq 0$ and $k \in \mathbb{R}$;

(ii) $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $f(x) = x^{-k}$, where $k > 0$;

(iii) $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $f(x) = e^{-k\sqrt{x}}$, where $k > 0$.

It is well known that a positive function $f : I \rightarrow \mathbb{R}$ is log-convex in the Jensen sense on I if and only if it is 2-exponentially convex in the Jensen sense on I , that is, if and only if the relation

$$\rho^2 f(x) + 2\rho\tau f\left(\frac{x+y}{2}\right) + \tau^2 f(y) \geq 0 \tag{24}$$

holds for every $\rho, \tau \in \mathbb{R}$ and $x, y \in I$. If such a function is additionally continuous, then it is log-convex on I .

We need the following characterization of a convex function (see, e.g., [7, p.2]): if $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \rightarrow \mathbb{R}$ is convex if and only if the following inequality holds:

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0. \tag{25}$$

Moreover, we utilize the following property of a convex function $f : I \rightarrow \mathbb{R}$ (see [7, p.2]): if $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{26}$$

When dealing with functions with different degrees of smoothness, divided differences are found to be very useful. The second-order divided difference of a function $f : I \rightarrow \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$\begin{aligned} [y_i] f &= f(y_i), & i = 0, 1, 2 \\ [y_i, y_{i+1}] f &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, & i = 0, 1 \\ [y_0, y_1, y_2] f &= \frac{[y_1, y_2] f - [y_0, y_1] f}{y_2 - y_0}. \end{aligned} \tag{27}$$

A function $f : I \rightarrow \mathbb{R}$ is convex if and only if for every choice of three mutually different points $y_0, y_1, y_2 \in I$, $[y_0, y_1, y_2] f \geq 0$ holds.

Now we use an idea from [3] to give an elegant method of producing exponentially convex functions by applying functionals A_i , $i = 1, 2, 3$, defined in the previous section, to a given family of functions with the same property.

In order to simplify our results, we define functionals Φ_i $i = 1, 2, 3$, by the following:

$$\Phi_1(g, \lambda, \varphi) = \begin{cases} -A_1(g, \lambda, \varphi), & \text{if for all } s \in [\alpha, \beta] \text{ inequality (9) holds;} \\ A_1(g, \lambda, \varphi), & \text{if for all } s \in [\alpha, \beta] \text{ the reverse inequality in (9) holds,} \end{cases}$$

$$\Phi_2(g, \lambda, \varphi) = \begin{cases} -A_2(g, \lambda, \varphi), & \text{if for all } s \in [\alpha, \beta] \text{ inequality (12) holds;} \\ A_2(g, \lambda, \varphi), & \text{if for all } s \in [\alpha, \beta] \text{ the reverse inequality in (12) holds,} \end{cases}$$

$$\Phi_3(g, \lambda, \varphi) = \begin{cases} -A_3(g, \lambda, \varphi), & \text{if for all } s \in [\alpha, \beta] \text{ inequality (15) holds;} \\ A_3(g, \lambda, \varphi), & \text{if for all } s \in [\alpha, \beta] \text{ the reverse inequality in (15) holds.} \end{cases}$$

Under the appropriate assumptions on functions g , λ , and φ , as in Theorems 2.1, 2.5, and 2.8, we now have that $\Phi_i(g, \lambda, \varphi) \geq 0$, $i = 1, 2, 3$, whenever they are defined. The following result yields a class of exponentially convex functions obtained via functionals $\Phi_i(g, \lambda, \varphi)$, $i = 1, 2, 3$.

Theorem 4.2 *Let $\Omega = \{\varphi_p : p \in I\}$ be a family of functions $\varphi_p : [\alpha, \beta] \rightarrow \mathbb{R}$, $\varphi_p \in C([\alpha, \beta])$, such that the function $p \mapsto [y_0, y_1, y_2]\varphi_p$ is n -exponentially convex (resp. exponentially convex) in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in [\alpha, \beta]$. Then the functions $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are n -exponentially convex (resp. exponentially convex) in the Jensen sense on I . In addition, if $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are continuous on I , then they are n -exponentially convex (resp. exponentially convex) on I .*

Proof For $q_j \in \mathbb{R}$, $j = 1, \dots, n$, we define the function

$$h(x) = \sum_{j,k=1}^n q_j q_k \varphi_{\frac{p_j+p_k}{2}}(x),$$

where $p_j, p_k \in I$, $1 \leq j, k \leq n$, and $\varphi_{\frac{p_j+p_k}{2}} \in \Omega$. Clearly, h is continuous on $[\alpha, \beta]$ since it is the linear combination of continuous functions. Since $p \mapsto [y_0, y_1, y_2]\varphi_p$ is n -exponentially convex in the Jensen sense by the assumption, for every three mutually different points $y_0, y_1, y_2 \in [\alpha, \beta]$ we have

$$[y_0, y_1, y_2] h = \sum_{j,k=1}^n q_j q_k [y_0, y_1, y_2] \varphi_{\frac{p_j+p_k}{2}} \geq 0,$$

which implies that h is also convex on $[\alpha, \beta]$. It follows that $\Phi_i(g, \lambda, h) \geq 0$ and therefore

$$\sum_{j,k=1}^n q_j q_k \Phi_i\left(g, \lambda, \varphi_{\frac{p_j+p_k}{2}}\right) \geq 0.$$

Hence, the functions $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are n -exponentially convex in the Jensen sense on I . In addition, assuming the continuity, the functions $p \mapsto \Phi_i(g, \lambda, \varphi_p)$ are n -exponentially convex. \square

The following consequence of Theorem 4.2 is very useful for constructing some Cauchy-type means expressed via functionals $\Phi_i(g, \lambda, \varphi)$, $i = 1, 2, 3$.

Corollary 4.3 Let $\Omega = \{\varphi_p : p \in I\}$ be a family of functions $\varphi_p : [\alpha, \beta] \rightarrow \mathbb{R}$, $\varphi_p \in C([\alpha, \beta])$, such that the function $p \mapsto [y_0, y_1, y_2]\varphi_p$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in [\alpha, \beta]$. Then the following statements hold:

(i) If the functions $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are continuous on I , then they are 2-exponentially convex on I . If $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are in addition positive, then they are also log-convex on I , and for $r, s, t \in I$ such that $r < s < t$ we have

$$(\Phi_i(g, \lambda, \varphi_s))^{t-r} \leq (\Phi_i(g, \lambda, \varphi_r))^{t-s} (\Phi_i(g, \lambda, \varphi_t))^{s-r}. \tag{28}$$

(ii) If the functions $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$ are positive and differentiable on I , then for every $p, q, u, v \in I$ such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(g, \Phi_i, \Omega) \leq \mu_{u,v}(g, \Phi_i, \Omega), \tag{29}$$

where

$$\mu_{p,q}(g, \Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(g, \lambda, \varphi_p)}{\Phi_i(g, \lambda, \varphi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp \left(\frac{\frac{d}{dp} \Phi_i(g, \lambda, \varphi_p)}{\Phi_i(g, \lambda, \varphi_p)} \right), & p = q \end{cases} \tag{30}$$

for $\varphi_p, \varphi_q \in \Omega$.

Proof (i) The first statement in (i) is an immediate consequence of Theorem 4.2, while the log-convexity is an immediate consequence of (24). Since $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are positive, then considering (25) with $f(x) = \log \Phi_i(g, \lambda, \varphi_x)$ and $r, s, t \in I$, $r < s < t$, it follows that

$$(t - s) \log \Phi_i(g, \lambda, \varphi_r) + (r - t) \log \Phi_i(g, \lambda, \varphi_s) + (s - r) \log \Phi_i(g, \lambda, \varphi_t) \geq 0,$$

which is equivalent to (28).

(ii) Since by (i) the functions $p \mapsto \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are log-convex on I , that is, the functions $p \mapsto \log \Phi_i(g, \lambda, \varphi_p)$, $i = 1, 2, 3$, are convex on I , utilizing (26) for $p \leq u$, $q \leq v$, $p \neq q$, $u \neq v$, we obtain

$$\frac{\log \Phi_i(g, \lambda, \varphi_p) - \log \Phi_i(g, \lambda, \varphi_q)}{p - q} \leq \frac{\log \Phi_i(g, \lambda, \varphi_u) - \log \Phi_i(g, \lambda, \varphi_v)}{u - v}, \tag{31}$$

from which we obtain $\mu_{p,q}(g, \Phi_i, \Omega) \leq \mu_{u,v}(g, \Phi_i, \Omega)$. The cases $p = q$ and $u = v$ follow from (31) as the limit cases. □

Remark 4.4 The value $[y_0, y_1, y_2]f$ is independent of the order of the points y_0, y_1 , and y_2 . This definition may be extended to include the case in which some or all of the points coincide (see [7, p. 16]). Taking the limit $y_1 \rightarrow y_0$ in (27), we get

$$\lim_{y_1 \rightarrow y_0} [y_0, y_1, y_2]f = [y_0, y_0, y_2]f = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0,$$

provided that f' exists. Furthermore, taking the limits $y_i \rightarrow y_0, i = 1, 2$, in (27), we get

$$\lim_{y_2 \rightarrow y_0} \lim_{y_1 \rightarrow y_0} [y_0, y_1, y_2]f = [y_0, y_0, y_0]f = \frac{f''(y_0)}{2},$$

provided that f'' exists. Taking into account the above discussion and assuming the differentiability of a family φ_p , it is obvious that the results from Theorem 4.2 and Corollary 4.3 still hold when some or all of points $y_0, y_1, y_2 \in [\alpha, \beta]$ coincide.

To conclude the paper, we vary the choice of a family $\Omega = \{\varphi_p : p \in I\}$, presenting several families of functions that fulfill the conditions of Theorem 4.2 and Corollary 4.3. In such a way we are going to construct several examples of exponentially convex functions, as well as some related Cauchy-type means.

Example 4.5 Let $\Omega_1 = \{\varphi_p : \mathbb{R}^+ \rightarrow \mathbb{R} : p \in \mathbb{R}\}$ be a family of functions defined by

$$\varphi_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p \neq 0, 1; \\ -\log x, & p = 0; \\ x \log x, & p = 1. \end{cases}$$

Since $\varphi_p''(x) = x^{p-2}, p \neq 0, 1$, it follows that the function φ_p is convex on \mathbb{R}^+ , so that $\Phi_i(g, \lambda, \varphi_p) \geq 0, i = 1, 2, 3$. Due to Remark 4.1 it follows that $p \mapsto \varphi_p''(x)$ is exponentially convex, and utilizing [3, Corollary 3.6.], we then have that $p \mapsto [y_0, y_1, y_2]\varphi_p$ is exponentially convex. Therefore, a family of functions Ω_1 fulfills conditions as in Theorem 4.2, providing a class of exponentially convex functions. Namely, the mappings $p \mapsto \Phi_i(g, \lambda, \varphi_p), i = 1, 2, 3$, are exponentially convex in the Jensen sense. In addition, these mappings are obviously continuous, so they are exponentially convex.

Next, our intention is to construct some Cauchy-type means in connection with family Ω_1 . For that, we consider restrictions of functions φ_p on bounded interval $[\alpha, \beta] \subseteq \mathbb{R}^+$. Now, employing Corollary 4.3 for this family of functions, the expressions $\mu_{p,q}(g, \Phi_i, \Omega_1), i = 1, 2, 3$, become

$$\mu_{p,q}(g, \Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g, \lambda, \varphi_p)}{\Phi_i(g, \lambda, \varphi_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp \left(\frac{1-2p}{p(p-1)} - \frac{\Phi_i(g, \lambda, \varphi_0 \varphi_p)}{\Phi_i(g, \lambda, \varphi_p)} \right), & p = q \neq 1, 0; \\ \exp \left(1 - \frac{\Phi_i(g, \lambda, \varphi_0^2)}{2\Phi_i(g, \lambda, \varphi_0)} \right), & p = q = 0; \\ \exp \left(-1 - \frac{\Phi_i(g, \lambda, \varphi_0 \varphi_1)}{2\Phi_i(g, \lambda, \varphi_1)} \right), & p = q = 1, \end{cases}$$

satisfying the monotonicity property as in the corollary.

Now, by virtue of Theorem 3.2, we show that $\mu_{p,q}(g, \Phi_i, \Omega_1), i = 1, 2, 3$, represent means of a function g . More precisely, considering relations (21), (22), and (23) with $\varphi = \varphi_p \in \Omega_1$ and $\psi = \varphi_q \in \Omega_1$, it follows that there exist $\xi_i \in [\alpha, \beta], i = 1, 2, 3$, such that $\xi_i^{p-q} = \frac{\Phi_i(g, \lambda, \varphi_p)}{\Phi_i(g, \lambda, \varphi_q)}$. Since the function $\xi \mapsto \xi^{p-q}$ is invertible for $p \neq q$, we then have

$$\alpha \leq \left(\frac{\Phi_i(g, \lambda, \varphi_p)}{\Phi_i(g, \lambda, \varphi_q)} \right)^{\frac{1}{p-q}} \leq \beta, \text{ for } i = 1, 2, 3,$$

that is,

$$\alpha = \min_{t \in [a,b]} \{g(t)\} \leq \left(\frac{\Phi_i(g, \lambda, \varphi_p)}{\Phi_i(g, \lambda, \varphi_q)} \right)^{\frac{1}{p-q}} \leq \max_{t \in [a,b]} \{g(t)\} = \beta,$$

provided that $[\alpha, \beta]$ is the image of function g . This shows that $\mu_{p,q}(g, \Phi_i, \Omega_1)$ are means of function g .

Another class of Cauchy-type means arises from the previous relation by imposing an additional parameter $r \neq 0$. Namely, considering the previous relation with $g^r, \frac{p}{r}, \frac{q}{r}$, instead of g, p, q , respectively, we have

$$\min_{t \in [a,b]} \{(g(t))^r\} \leq \left(\frac{\Phi_i(g^r, \lambda, \varphi_p)}{\Phi_i(g^r, \lambda, \varphi_q)} \right)^{\frac{r}{p-q}} \leq \max_{t \in [a,b]} \{(g(t))^r\}, \quad \text{for } i = 1, 2, 3.$$

The previous relation establishes a generalized class of means defined by

$$\mu_{p,q;r}(g, \Phi_i, \Omega_1) = \begin{cases} \left(\mu_{\frac{p}{r}, \frac{q}{r}}(g^r, \Phi_i, \Omega_1) \right)^{\frac{1}{r}}, & r \neq 0; \\ \mu_{p,q}(\log g, \Phi_i, \Omega_1), & r = 0. \end{cases}$$

These new means are also monotone, i.e. $\mu_{p,q;r}(g, \Phi_i, \Omega_1) \leq \mu_{u,v;r}(g, \Phi_i, \Omega_1)$, $i = 1, 2, 3$, whenever $p \leq u, q \leq v$. This follows by virtue of the monotonicity of a class $\mu_{p,q}(g, \Phi_i, \Omega_1)$.

Example 4.6 Let $\Omega_2 = \{\psi_p : \mathbb{R} \rightarrow \mathbb{R} : p \in \mathbb{R}\}$ be a family of functions defined by

$$\psi_p(x) = \begin{cases} \frac{1}{p^2} e^{px}, & p \neq 0; \\ \frac{1}{2} x^2, & p = 0. \end{cases}$$

Since $\psi_p''(x) = e^{px}$, it follows that ψ_p is convex on \mathbb{R} and the function $p \mapsto \psi_p''(x)$ is exponentially convex. Furthermore, utilizing the same argumentation as in Example 4.5, we obtain that the mappings $p \mapsto \Phi_i(g, \lambda, \psi_p)$, $i = 1, 2, 3$, are exponentially convex.

Now, to construct Cauchy-type means in connection with family Ω_2 , we consider restrictions of functions ψ_p on a bounded interval $[\alpha, \beta] \subseteq \mathbb{R}$. In this setting, the expressions $\mu_{p,q}(g, \Phi_i, \Omega_2)$, $i = 1, 2, 3$, (see Corollary 4.3) become

$$\mu_{p,q}(g, \Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(g, \lambda, \psi_p)}{\Phi_i(g, \lambda, \psi_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp \left(\frac{\Phi_i(g, \lambda, id \cdot \psi_p)}{\Phi_i(g, \lambda, \psi_p)} - \frac{2}{p} \right), & p = q \neq 0; \\ \exp \left(\frac{1}{3} \frac{\Phi_i(g, \lambda, id \cdot \psi_0)}{\Phi_i(g, \lambda, \psi_0)} \right), & p = q = 0, \end{cases}$$

where id stands for an identity function. In this case the relations (21), (22), and (23) with $\varphi = \psi_p \in \Omega_2$ and $\psi = \psi_q \in \Omega_2$ yield the estimate

$$\alpha \leq \log \mu_{p,q}(g, \Phi_i, \Omega_2) \leq \beta, \quad \text{for } i = 1, 2, 3,$$

that is,

$$\alpha = \min_{t \in [a,b]} \{g(t)\} \leq \log \mu_{p,q}(g, \Phi_i, \Omega_2) \leq \max_{t \in [a,b]} \{g(t)\} = \beta, \quad \text{for } i = 1, 2, 3,$$

provided that $[\alpha, \beta]$ is the image of the function g . This shows that $\log \mu_{p,q}(g, \Phi_i, \Omega_2)$, $i = 1, 2, 3$, are means of the function g and they are also monotone.

References

- [1] Akhiezer NI. The Classical Moment Problem and Some Related Questions in Analysis. Edinburgh, UK: Oliver and Boyd, 1965.
- [2] Bernstein SN. Sur les fonctions absolument monotones. Acta Math 1929; 52: 1-66 (in French).
- [3] Jakšetić J, Pečarić J. Exponential convexity method. J Convex Anal 2013; 20: 181-197.
- [4] Jakšić R, Pečarić J, Rodić Lipanović M. Reverses of the Jensen-type inequalities for signed measures. Abstr Appl Anal 2014; 2014: 626359.
- [5] Khan AR, Pečarić J, Rodić Lipanović M. n -exponential convexity for Jensen-type inequalities. J Math Inequal 2013; 7: 313-335.
- [6] Pečarić J, Perić I, Rodić Lipanović M. Uniform treatment of Jensen type inequalities. Math Reports 2014; 16: 183-205.
- [7] Pečarić JE, Proschan F, Tong YL. Convex Functions, Partial Orderings, and Statistical Applications. London, UK: Academic Press, 1992.
- [8] Widder DV. Completely convex functions and Lidstone series. T Am Math Soc 1942; 51: 387-398.