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Some notes on GQN rings

Long WANG*, **Junchao WEI**
School of Mathematics, Yangzhou University, Yangzhou, P.R. China

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Abstract: A ring R is called a generalized quasinormal ring (abbreviated as GQN ring) if $ea \in N(R)$ for each $e \in E(R)$ and $a \in N(R)$. The class of GQN rings is a proper generalization of quasinormal rings and NI rings. Many properties of quasinormal rings are extended to GQN rings. For a GQN ring R and $a \in R$, it is shown that: 1) if a is a regular element, then a is a strongly regular element; 2) if a is an exchange element, then a is clean; 3) if R is a semiperiodic ring with $J(R) \neq N(R)$, then R is commutative; 4) if R is an $MVNR$, then R is strongly regular.

Key words: GQN rings, (von Neumann) regular elements, NI rings, quasinormal rings, generalized GQN rings, semiperiodic rings, exchange rings

1. Introduction

All rings considered in this paper are associative with an identity. The symbols $J(R)$, $N(R)$, $U(R)$, and $E(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, and the set of all idempotent elements of R. For an element a of R, we write $l(a) = \{x \in R | xa = 0\}$ to denote the left annihilators of a. Write $Z_l(R) = \{a \in R | l(a) \text{ is an essential left ideal of } R\}$. It is easy to prove that $Z_l(R)$ is an ideal of R and call it the left singular ideal.

Recall that a ring R is called *quasinormal* [14] if for each $a \in N(R)$ and $e \in E(R)$, $ae = 0$ implies $eaRe = 0$. According to [14], the class of quasinormal rings is a proper generalization of *abelian* rings.

A ring R is called *(von Neumann) regular* [3] if for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring R is called *strongly regular* [8] if for every $a \in R$ there exists $b \in R$ such that $a = a^2b$. A ring R is called *left quasi-duo* [15] if every left ideal of R is an ideal, and R is said to be reduced if $N(R) = 0$. In the last decade, the strong regularity of regular rings that satisfy certain additional conditions has been studied by many authors. in [15, Theorem 2.7] it is shown that R is a strongly regular ring if and only if R is a left quasi-duo regular ring; In [8, Remark 2.13] it is shown that R is a strongly regular ring if and only if R is a reduced regular ring. In [14, Corollary 2.7] it is shown that R is a strongly regular ring if and only if R is a quasinormal regular ring. Recall that R is said to be generalized weakly symmetric (abbreviated as GWS) if $abc = 0$ implies $bac \in N(R)$. In [11, Corollary 3.2] it is shown that R is a strongly regular ring if and only if R is a GWS regular ring. This paper will continue the research in this area.

*Correspondence: wanglseu@hotmail.com

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A ring R is called *NI* if $N(R)$ forms an ideal of R and R is said to be *directly finite* if $ab = 1$ implies $ba = 1$. It is well known that NI rings are directly finite. In [14, Theorem 2.4], it is shown that quasinormal rings are directly finite.

A ring R is called a *generalized quasinormal ring* or GQN ring (abbreviated) if $ea \in N(R)$ for each $a \in N(R)$ and $e \in E(R)$. Clearly, *abelian* rings are GQN, but the converse is not true because $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ (where F is a field) is a GQN ring that is not abelian.

Since $N(R)$ is an ideal for an NI ring R, every NI ring is GQN, but the converse is not true by the following example.

Let $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \equiv d (mod 2)$ and $b \equiv c \equiv 0 (mod 2), a, b, c, d \in \mathbb{Z}\}$. Clearly, R is an abelian ring with $E(R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$, so R is GQN. Since $N(R)$ is not an ideal of R, R is not an NI ring.

Thus, the class of GQN rings is a proper generalization of both abelian rings and NI rings.

Following [5], an element x of R is called *clean* if $x = u + e$ for some $u \in U(R)$ and $e \in E(R)$. The ring R is said to be *clean* if all of its elements are clean. An element x of R is called *exchange* if there exists $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$. The ring R is said to be *exchange* if all of its elements are exchange. Clearly, clean elements are exchange, but the converse is not true unless one of the following conditions holds: 1) R is abelian [5]; 2) R is left quasi-duo [15]; 3) R is quasinormal [14]; 4) R is weakly normal [13]. Various weakened form conditions stimulate us to continue the study of this topic. In this paper, we first discuss the properties of GQN inherited from abelian rings and left quasi-duo rings. Next, with the help of GQN rings, we discuss the relations among left quasi-duo rings, exchange rings, abelian rings, and strongly regular rings.

2. Some properties of GQN rings

Theorem 2.1 *(1) The following conditions are equivalent for a ring R:*

(a) R is a quasinormal ring;

(b) $ae = 0$ implies $eaN(R)e = 0$ for each $a \in N(R)$ and $e \in E(R)$;

(c) $ae = 0$ implies $eaN(R)e = 0$ for each $a \in R$ and $e \in E(R)$.

(2) If R is a quasinormal ring, then R is GQN.

(3) R is a GQN if and only if $ae \in N(R)$ for each $a \in N(R)$ and $e \in E(R)$.

(4) Let R be a GQN ring. If $a, b \in R$ and $e \in E(R)$, then $ea(1 - e)be \in N(R)$.

Proof (1) $(a) \Longrightarrow (b)$ is clear.

$(b) \Longrightarrow (c)$ Let $a \in R$ and $e \in E(R)$ with $ae = 0$. Then $ea \in N(R)$ and $(ea)e = 0$, by (b), and one obtains $e(ea)N(R)e = 0$; that is, $eaN(R)e = 0$.

$(c) \Longrightarrow (a)$ Let $e \in E(R)$ and $x, a \in R$. Write $h = (1 - e)ae$. Then $h^2 = 0$ and $h = (1 - e)he$. Since $(x(1 - e))e = 0$, by (c), $e(x(1 - e))N(R)e = 0$, and this gives $ex(1 - e)he = 0$, so $ex(1 - e)ae = exh = 0$ for each $x, a \in R$. Thus, $eR(1 - e)Re = 0$, by [14, Theorem 2.1], and R is quasinormal.

(2) Let $e \in E(R)$ and $a \in N(R)$. Since R is a quasinormal ring and $(a(1-e))e = 0$, $ea(1-e)N(R)e = 0$ by (1). Since $a \in N(R)$, $a^n = 0$ for some $n \geq 1$ and $a^i \in N(R)$ for all $1 \leq i \leq n$. Hence, $ea(1-e)a^ie = 0$, and

this gives $eaea^ie = ea^{i+1}e$ for all i; it follows that $(ea)^{i+1} = ea^iea$ and in particular one obtains $(ea)^{n+1} = 0$, so $ea \in N(R)$. Hence, R is GQN.

(3) If $ea \in N(R)$ for $a \in R$ and $e \in E(R)$, then we have $(ea)^n = 0$ for $n \in \mathbb{N}$. Hence, $(ae)^{n+1} = a(ea)^n e = 0$. This shows that $ae \in N(R)$.

(4) Clearly, $(1-e)be \in N(R)$. Let $g = e + ea(1-e)$. Then $eg = g$ and $ge = e$, so $g^2 = gg = g(eg) = (ge)g = eg = g$, and this implies $g \in E(R)$. Since R is a GQN ring, $g(1-e)be \in N(R)$. Hence, $ea(1-e)be \in N(R)$. □

By Theorem 2.1 and [14, Theorem 2.1], we have the following corollary.

Corollary 2.2 *R is a quasinormal ring if and only if $ae = 0$ implies $eaE(R)e = 0$ for $a \in R$ and $e \in E(R)$.*

Proof In the proof of $(c) \Longrightarrow (a)$ in Theorem 2.1(1), substituting g for h, where $g = e+h$ and $h = (1-e)ae$ (clearly, $g^2 = g$), one can finish the proof. □

Example 2.3 *Let F be a field and $R = \{\begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$. Then R is not quasinormal by [14]. Since $N(R) = \{\begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ is an ideal of R, R is NI, so R is GQN. Hence, the converse of Theorem 2.1(2) is not true in general.*

An element a of a ring R is called (strongly) regular if $a \in aRa$ ($a \in a^2R \cap Ra^2$), and a is said to be unit-regular if $a = aua$ for some $u \in U(R)$. According to [6], strongly regular $\Longrightarrow$ unit-regular $\Longrightarrow$regular. A ring R is called regular if every element of R is regular; R is called strongly regular if every element of R is strongly regular.

Theorem 2.4 *Let R be a GQN ring and $a \in R$. If a is a regular element, then a is a strongly regular element.*

Proof Let $a = aba$ for some $b \in R$. Write $e = ba$. Then

$$e^2 = e; a = ae. \tag{2.1}$$

Write $h = a - ea$. Then

$$he = h; eh = 0; h^2 = 0. \tag{2.2}$$

Set $g = e + h$. Then

$$eg = e; ge = g; g^2 = g. \tag{2.3}$$

Let $t = eb(1-e)$. Then

$$et = t; te = 0; t^2 = 0. \tag{2.4}$$

Since R is a GQN ring, $gt \in N(R)$. Hence, there exists a positive integer m such that $(gt)^m = 0$.

Since $gt = et + ht = t + (1-e)ab(1-e)$ and $(gt)^m = t((1-e)ab(1-e))^{m-1} + ((1-e)ab(1-e))^m$, $t((1-e)ab(1-e))^{m-1} = 0$.

If $m = 1$, then $t = 0$; that is, $eb(1-e) = 0$, so $eb = ebe$ and $a = aba = (ae)ba = a(eb)a = a(ebe)a = aeb^2a^2 \in Ra^2$.

If $m = 2$, then $t(1-e)ab(1-e) = 0$, so $tab(1-e) = 0$ and $tab = tabe$. Hence, $ta = taba = tabea$, and this gives $eb(1-e)a = tabea$ and $eba = ebea + tabea$. Thus, $a = aeba = aebea + atabea \in Ra^2$.

If $m > 2$, then there exist $c, d \in R$ such that $((1-e)ab(1-e))^{m-1} = ab + deab + cabe$ because $ab \in E(R)$ and $a = aba$. Hence, $tab = -tdeab - tcabe$ and $ta = taba = -tdeaba - tcabea = -tdea - tcabea$, and this implies $eb(1-e)a = xea$ where $x = -td - tcab$. Thus, $eba = ebea + xea$ and $a = aeba = aebea + axea \in Ra^2$. Hence, in any case, we have $a \in Ra^2$. Similarly, by Theorem 2.1(3), one can show that $a \in a^2R$. □

Corollary 2.5 *(1) R is a strongly regular ring if and only if R is a GQN regular ring.*

(2) R is a strongly regular ring if and only if R is a quasinormal regular ring [14, Corollary 2.7].

(3) R is a strongly regular ring if and only if R is an NI regular ring.

(4) If R is an exchange GQN ring, then R has stable range 1.

Proof Since strongly regular rings are *Abel*, NI, and regular, (1) is an immediate result of Theorem 2.4.

(2) and (3) are direct corollaries of (1).

(4) It is an immediate corollary of Theorem 2.4.

Recall that a ring R is directly finite if $ab = 1$ implies $ba = 1$ for all $a, b \in R$, and R is said to be $n$regular [12] if every element of $N(R)$ is regular. A ring R is called *reduced* if for $N(R) = 0$.

Corollary 2.6 *(1) GQN rings are directly finite.*

(2) Quasinormal rings are directly finite [14, Theorem 2.4].

(3) NI rings are directly finite.

Proof (1) Let $a, b \in R$ with $ab = 1$. Then $a = aba$. Since R is GQN, by Theorem 2.4, there exists $c \in R$ such that $a = ca^2$; this gives $1 = ab = ca^2b = ca$ and $b = 1b = cab = c$. Hence, $ba = ca = 1$, and this shows that R is directly finite.

(2) and (3) are direct corollaries of (1). □

Also by [14, Theorem 2.8], [12, Theorem 2.9], and Theorem 2.4, we have the following corollary.

Corollary 2.7 *The following conditions are equivalent for a ring R:*

(1) R is a reduced ring;

(2) R is a GQN n-regular ring;

(3) R is an NI n-regular ring;

(4) R is a quasinormal n-regular ring.

Recall that a ring R is nil-semicommutative [7] if $ab \in N(R)$ implies $arb \in N(R)$ for all $a, b, r \in N(R)$. [7, Proposition 2.1] implies that nil-semicommutative rings are GQN, but the converse is not true because Abel rings need not be nil-semicommutative. The following proposition generalizes [7, Corollary 2.3].

Proposition 2.8 *Let R be a GQN ring and $e \in E(R)$ and $x \in R$. Then:*

(1) If M is a maximal left ideal of R and $e \notin M$, then $(1-e)R \subseteq M$;

(2) $Rx + R(xe-1) = R$;

(3) $Re + R(ex-1) = R$;

(4) If M is a maximal left ideal of R and $1-xe \in M$, then $1-ex \in M$;

(5) If M is a maximal left ideal of R and $1-ex \in M$, then $1-xe \in M$;

(6) If $x, z \in R$ satisfy $x + z \in zxE(R)$, then $xR = zR$.

Proof (1) Clearly, $Re + M = R$. Let $1 = ae + m$ for some $a \in R$ and $m \in M$. Then for any $z \in R$, one has

$$(1-e)z = (1-e)zae + (1-e)zm. \tag{2.5}$$

For any $y \in R$, by Theorem 2.1(4), $(1-e)zaey(1-e) \in N(R)$, so there exists $m \geq 1$ such that $((1-e)zaey(1-e))^m = 0$, and this gives $((1-e)zaey)^{m+1} = ((1-e)zaey(1-e))^m zaey = 0$. Thus, for each $y \in R$, one has

$$(1-e)zaey \in N(R). \tag{2.6}$$

Hence, $(1-e)zae \in J(R) \subseteq M$; this implies $(1-e)z \in M$ by equation (2.5), so $(1-e)R \subseteq M$.

(2) If $Rx + R(xe-1) \neq R$, then there exists a maximal left ideal M of R containing $Rx + R(xe-1)$. Since $xe - 1 \in M$, $e \notin M$, by (1), $1 - e \in M$, so $x - xe = x(1-e) \in M$. Since $x \in M$, $xe \in M$, this implies $1 = xe - (xe-1) \in M$, which is a contradiction. Hence, $Rx + R(xe-1) = R$.

(3) If $Re + R(ex-1) \neq R$, then there exists a maximal left ideal M of R containing $Re + R(ex-1)$. Since $e \in M$, $1 - e \notin M$, by (1), $eR \subseteq M$, so $ex \in M$. Since $ex - 1 \in M$, $1 \in M$, which is a contradiction. Hence, $Re + R(ex-1) = R$.

(4) Since $1 - xe \in M$, $e \notin M$. By (1), $(1-e)R \subseteq M$. Since $1 - xe = (1-x) + (x - xe)$, $1 - x \in M$. Since $1 - ex = (1-x) + ((1-e)x)$, $1 - ex \in M$.

(5) Assume that $1 - ex \in M$. If $e \in M$, then $eR \subseteq M$ by (1), and it follows that $1 = (1-ex) + ex \in M$, a contradiction. Hence, $e \notin M$, also by (1), and $(1-e)R \subseteq M$. Since $1 - ex = (1-x) + (1-e)x$, $1 - x \in M$, this gives $1 - xe = (1-x) + (x(1-e)) \in M$.

(6) Let $x + z = zxg$ for some $g \in E(R)$. Then $x = z(xg-1)$, by (2), and one has $R = Rx + R(xg-1)$. Hence, $R = R(xg-1)$, by Corollary 2.6(1), $xg - 1$ is invertible, and this gives $xR = z(xg-1)R = zR$. □

The following theorem addresses how to construct more examples of GQN rings from a given GQN ring.

Theorem 2.9 *The following conditions are equivalent for a ring R:*

(1) R is a GQN ring;

(2) The $n \times n$ upper triangular matrix ring $UTM_n(R)$ is a GQN ring for some $n \geq 2$;

(3) The $n \times n$ upper triangular matrix ring $UTM_n(R)$ is a GQN ring for each $n \geq 2$.

Proof (3) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (1) Let $E = e_{11} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in S = UTM_n(R)$. Then $ESE \cong R$. Since

ESE is a subring of S and every subring of GQN ring S is GQN, R is GQN.

(1) $\Longrightarrow$ (3) Let $n \geq 2$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in N(S)$ and

$E = \begin{pmatrix} e_{11} & e_{12} & e_{13} & \cdots & e_{1n} \\ 0 & e_{22} & e_{23} & \cdots & e_{2n} \\ 0 & 0 & e_{33} & \cdots & e_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e_{nn} \end{pmatrix} \in E(S)$, where $S = UTM_n(R)$. Then $a_{ii} \in N(R)$ and $e_{ii} \in E(R)$, $i = 1, 2, \cdots, n$. Since R is GQN, $e_{ii}a_{ii} \in N(R)$, $i = 1, 2, \cdots, n$. Hence,

$EA \in \begin{pmatrix} N(R) & R & R & \cdots & R \\ 0 & N(R) & R & \cdots & R \\ 0 & 0 & N(R) & \cdots & R \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & N(R) \end{pmatrix} \subseteq N(S)$, and this shows that $UTM_n(R)$ is GQN. □

Theorem 2.10 *If R is a finite subdirect product of a family of GQN rings $\{R_i : i = 1, 2, \cdots, n\}$, then R is GQN.*

Proof Let $R_i = R/A_i$ where A_i is ideals of R with $\cap_{i=1}^{n} A_i = 0$. Let $e \in E(R)$ and $a \in N(R)$. Then $e_i = e + A_i \in E(R_i)$ and $a_i = a + A_i \in N(R_i)$ for any i. Since each R_i is GQN, $e_i a_i \in N(R_i)$; this implies that for each i there exists $m_i \geq 1$ such that $(ea)^{m_i} \in A_i$. Choose $m = max\{m_1, m_2, \cdots, m_n\}$. Then $(ea)^m \in A_i$ for each i, so $(ea)^m \in \cap_{i=1}^{n} A_i = 0$, which implies $ea \in N(R)$. Therefore, R is GQN. □

A left ideal I of a ring R is called *regular* if $a \in aIa$ for each $a \in I$. A ring R is called left $MVNR$ if R contains a regular maximal left ideal of R. Clearly, strongly regular rings are left $MVNR$.

Lemma 2.11 *Let R be a GQN ring. If R is left $MVNR$, then R is reduced.*

Proof Suppose that M is a regular maximal left ideal of R and suppose that $a \in R$ with $a^2 = 0$. If $a \notin M$, then $Ra + M = R$. Write $1 = sa + m$ for some $s \in R$ and $m \in M$. Clearly, $a = ma$. Since $am \in M$, $am = ambam$ for some $b \in M$. Set $e = amb$. Then $am = eam$ and $e^2 = e$. Let $h = am - ame$. Then $h = e(am)(1 - e)$. If $e \in M$, then by Proposition 2.8(1), $eR \subseteq M$, so $h \in M$. If $e \notin M$, then by Proposition 2.8(1), $1 - e \in M$, and we also have $h \in M$. Hence, in any case, we have $h \in M$, so $h = hdh$ for some $d \in M$. Choose $f = dh + dhd(1 - dh)$. Then $f^2 = f$. Since R is a GQN ring and $h^2 = 0$, $fh \in N(R)$. Clearly, $fh = dh$ is an idempotent element, so $dh = 0$, and it follows that $h = hdh = 0$. Hence, $am - ame = h = 0$, which implies that $am = am(amb) = a(ma)mb = a^2mb = 0$. Therefore, $a = a1 = a(sa + m) = asa$. If $a \in M$, then, certainly, $a = asa$ for some $s \in M$. Hence, in any case, one has $a = aca$ for some $c \in R$. Write $g = ca + cac(1 - ca)$. Then $g^2 = g$. Since $a^2 = 0$, $ga \in N(R)$. Since $ga = ca$ is an idempotent element of R, $ca = 0$ and $a = aca = 0$. Therefore, R is reduced. □

Theorem 2.12 *Let R be a GQN ring. If R is $MVNR$, then R is strongly regular.*

Proof First, by Lemma 2.11, R is a reduced ring. Since reduced regular rings are strongly regular, we only need to show that R is a regular ring. Assume that $a \in R$. If $a \in M$, we are done. If $a \notin M$, then

$R = Ra + M$. Write $1 = sa + m$ for some $s \in R$ and $m \in M$. Since $am \in M$, $am = amdam$ for some $d \in M$. Set $e = amd$ and $g = dam$. Then $e^2 = e$, $g^2 = g$ and $am = eam = amg$. Since R is a reduced ring, R is *abelian*, and it follows that $eg = ge$; this gives $amd^2am = eg = ge = gamd = agmd = adam^2d$. Since R is a reduced ring and $a(md^2am - dam^2d) = 0$, $aR(md^2am - dam^2d) = 0$, and this implies that $(md^2am - dam^2d)^2 = 0$, so $md^2am = dam^2d = dammd = gmd = mgd$. Similar to the proof mentioned above, one obtains that $d^2am = gd = damd$. Further, one has $dam = amd$; that is, $e = g$. Since $a(m - mdam) = 0$ and R is reduced, $ma = mdama = mga = mea = mamda$, and this gives $ma(1 - mda) = 0$. Since R is symmetric, $m(1 - mda)a = 0$, and it follows that $ma = m^2da^2$, so $a = 1a = sa^2 + ma = (s + m^2d)a^2$. Hence, R is a strongly regular ring. □

3. Generalized GQN rings

An idempotent element e of a ring R is called *left minimal idempotent* if Re is a minimal left ideal of R. Write $ME_l(R) = \{e \in E(R) | e \text{ is a left minimal idempotent of } R\}$. A ring R is called a *generalized* GQN ring if $ea \in N(R)$ for all $e \in ME_l(R)$ and $a \in N(R)$. Clearly, GQN rings are generalized GQN, but the converse is not true. In fact, for any ring R, $R[x]$ is a generalized GQN ring because $ME_l(R[x]) = \varnothing$, while $R[x]$ need not be GQN. Clearly, a ring R is a generalized GQN ring if and only if $ae \in N(R)$ for all $e \in ME_l(R)$ and $a \in N(R)$.

Proposition 3.1 *Let R be a ring. Then:*

(1) If $R/J(R)$ is a generalized GQN ring, then so is R;

(2) If $R/Z_l(R)$ is a generalized GQN ring, then so is R.

Proof (1) Let $e \in ME_l(R)$. Then $e \notin J(R)$. Let $\bar{R} = R/J(R)$ and $\bar{e} = e + J(R)$. Then we claim that $\bar{e} \in ME_l(R)$. In fact, assume that $a \in R$ such that $\bar{a}\bar{e} \neq \bar{0}$. Then $ae \neq 0$, and it follows that $Rae = Re$, so $e = bae$ for some $b \in R$; this implies $\bar{e} = \bar{b}\bar{a}\bar{e}$. Hence, $\bar{e} \in ME_l(\bar{R})$. Since $\bar{R}$ is a generalized GQN ring, $\bar{x}\bar{e} \in N(\bar{R})$ for all $x \in N(R)$. Let $n \geq 1$ satisfy $(\bar{x}\bar{e})^n = \bar{0}$. Then we have $(xe)^n \in J(R)$. If $(xe)^n \neq 0$, then $R(xe)^n = Re$ because Re is a minimal left ideal of R, and this implies $e \in J(R)$, which is a contradiction. Hence, $(xe)^n = 0$, and it follows that $xe \in N(R)$. Thus, R is a generalized GQN ring.

Similarly, we can show (2). □

Proposition 3.2 *Let R be a generalized GQN ring and $f \in E(R)$. If $RfR = R$, then fRf is generalized GQN.*

Proof Let $e \in ME_l(fRf)$ and choose $a \in R$ such that $ae \neq 0$. Since $RfR = R$, $1 = \sum_{i=1}^n s_i f t_i$ for $s_i, t_i \in R$, it follows that $ae = \sum_{i=1}^n s_i f t_i a f e$, and this implies that there exists $i_0 \in \{1, 2, \cdots, n\}$ such that $ft_{i_0}afe \neq 0$. Since $e \in ME_l(fRf)$, there exists $x \in fRf$ such that $e = xft_{i_0}afe = (xft_{i_0})ae$, and this shows that $e \in ME_l(R)$. Since R is a generalized GQN ring, $ey \in N(R)$ for all $y \in N(fRf)$, so $ey \in N(fRf)$ for all $y \in N(fRf)$. Hence, fRf is generalized GQN. □

Proposition 3.3 *R is a generalized GQN ring if and only if the 2×2 upper triangular matrix ring $T_2(R)$ is a generalized GQN ring.*

Proof Let $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in E(T_2(R))$. Then $T_2(R)e_{11}T_2(R) = T_2(R)$ and $e_{11}T_2(R)e_{11} \cong R$. Hence, the sufficiency is an immediate result of Proposition 3.2.

Now we assume that R is a generalized GQN ring and $E = \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix} \in ME_l(T_2(R))$. Then $e_1^2 = e_1$, $e_3^2 = e_3$ and $e_2 = e_1e_2 + e_2e_3$.

If $e_1 \neq 0$, then $e_1 \in ME_l(R)$ and $e_3 = 0$. In fact, assume that $a \in R$ satisfies $ae_1 \neq 0$. Then $AE \neq 0$ where $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in T_2(R)$. Since $E \in ME_l(T_2(R))$, there exists $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in T_2(R)$ such that $BAE = E$; that is, $\begin{pmatrix} b_1ae_1 & b_1ae_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix}$. Hence, $b_1ae_1 = e_1$ and $e_3 = 0$, and it follows that $e_1 \in ME_l(R)$. Now let $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} \in N(T_2(R))$. Then $c_1, c_3 \in N(R)$. Since R is a generalized GQN ring, $e_1c_1 \in N(R)$, this gives $EC = \begin{pmatrix} e_1c_1 & e_1c_2 + e_2c_3 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} N(R) & R \\ 0 & N(R) \end{pmatrix} = N(T_2(R))$.

If $e_1 = 0$, then $0 \neq e_3 \in ME_l(R)$. In fact, assume that $x \in R$ such that $xe_3 \neq 0$, and then $DE \neq 0$ where $D = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \in T_2(R)$; it follows that $E = GDE$ for some $G = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} \in T_2(R)$, and this implies $e_3 = y_3xe_3$, so $e_3 \in ME_l(R)$. For any $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} \in N(T_2(R))$, one has $EC = \begin{pmatrix} 0 & e_2c_3 \\ 0 & e_3c_3 \end{pmatrix} \in N(T_2(R))$ because $e_3c_3 \in N(R)$. Hence, $T_2(R)$ is a generalized GQN ring. □

Recall that a ring R is left quasi-duo [15] if every maximal left ideal of R is an ideal, and R is said to be $MELT$ if every essential maximal left ideal of R is an ideal. Clearly, left quasi-duo rings are $MELT$, but the converse is not true.

Proposition 3.4 *R is a left quasi-duo ring if and only if R is a generalized GQN $MELT$ ring.*

Proof We first assume that R is a left quasi-duo ring. Then R is $MELT$. Now let $a \in N(R)$ and $e \in ME_l(R)$. If $ea = 0$, then we are done. If $ea \neq 0$, then $l(ea) = l(e)$ is a maximal left ideal of R. Since R is a left quasi-duo ring, $l(ea)$ is an ideal of R. Since $1 - e \in l(ea)$, $(1 - e)a \in l(ea)$, this gives $aea = eaea$. Since $a \in N(R)$, there exists $n \geq 1$ such that $a^n = 0$, so $(ea)^{n+1} = a^nea = 0$. Hence, R is generalized GQN.

Conversely, assume that R is a generalized GQN $MELT$ ring. Let M be a maximal left ideal of R. If M is an essential left ideal of R, then M is an ideal because R is $MELT$. If M is not essential, then $M = l(e)$ for some $e \in ME_l(R)$. Choose $m \in M$ and $b \in R$. If $mb \notin M$, then $mbe \neq 0$. Write $h = (1 - e)be$. If $h = 0$, then $be = ebe$ and $mbe = mebe = 0$, a contradiction. Hence, $h \neq 0$ and $Rh = Re$. Let $e = ch$ for some $c \in R$. Then $ec(1 - e)h = e$. Write $g = e + ec(1 - e)$. Then $g \in ME_l(R)$ and $gh = e$. Since R is a generalized GQN ring, $gh \in N(R)$, which is a contradiction. Hence, $mbe = 0$, and this gives $mb \in M$. Thus, in any case, we have that M is an ideal, so R is a left quasi-duo ring. □

The following corollary is an immediate result of Proposition 3.4.

Corollary 3.5 *GQN $MELT$ rings are left quasi-duo.*

Proposition 3.6 *The following conditions are equivalent for a ring R:*

(1) R is a generalized GQN ring;

(2) $Ra + R(ae - 1) = R$ for all $a \in R$ and $e \in ME_l(R)$;

(3) $Ra + R(ae - 1) = R$ for all $a \in R$ and $e \in ME_l(R)$.

(1) $\Longrightarrow$ (2) If $Ra+R(ae-1) \neq R$, then there exists a maximal left ideal M of R containing $Ra+R(ae-1)$. If M is essential, then $Re \subseteq M$, and it follows that $ae \in M$, so $1 = ae+(1-ae) \in M$, a contradiction. Hence, M is not essential, and this gives $M = l(g)$ for some $g \in ME_l(R)$. Hence, $ag = 0$ and $g = aeg$. Since R is a generalized GQN ring, similar to the sufficiency proof of Proposition 3.4, one can show that $eg = geg$, and it follows that $g = aeg = ageg = 0$, a contradiction. Thus, $Ra + R(ae - 1) = R$.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) Let $a \in N(R)$ and $e \in ME_l(R)$. If $ae \notin N(R)$, then $Re = Rae$. Let $e = cae$ for some $c \in R$ and $h = ae - eae$. If $h \neq 0$, then $Re = Rh$. Write $e = dh$ for some $d \in R$ and $g = e + ed - ede$. Then $gh = e$ and $g \in ME_l(R)$. By (3), we have $Rh+R(hg-1) = R$, so $Rh = Rh^2+R(hg-1)h = 0$, a contradiction. Hence, $ae = eae$, and it follows that $e = cae = ceae = c^2aeae = c^2a^2e = c^2ea^2e = c^3aea^2e = c^3a^3e = \cdots = c^na^ne$ for all $n \geq 1$. Since $a \in N(R)$, $e = 0$, a contradiction. Hence, $ae \in N(R)$; this implies R is a generalized GQN ring. □

4. GQN exchange rings

An element $x \in R$ is said to be exchange if there exists $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$. The ring R is said to be exchange if all of its elements are exchange. An element $x \in R$ is said to be clean if $x = u + f$ for some $u \in U(R)$ and $f \in E(R)$. The ring R is said to be clean if all of its elements are clean. In [5, Proposition 1.8] it is shown that clean rings are exchange, but the converse is not true by [4, Example 1]. In [5] it is shown that abelian exchange rings are clean; [15] showed that left quasi-duo exchange rings are clean; [14, Proposition 4.1] showed that quasinormal exchange rings are clean. Clearly, the integral ring $\mathbb{Z}$ is GQN but not exchange. The full matrix ring over a field $\mathbb{F}$ is exchange but not GQN. In the following, we will study the exchange property of GQN rings.

Theorem 4.1 *Let R be a GQN ring and $x \in R$. If x is exchange, then x is clean.*

Proof Let $e \in E(R)$ satisfy $e = xa$ and $1 - e = (1 - x)b$ for some $a, b \in R$. Let $y = ae$ and $z = b(1 - e)$. Then $e = xy$ and $1-e = (1-x)z$. By simple calculation we obtain that $(x-(1-e))(y-z) = 1-ez-(1-e)y$. Since $(ez)^2 = 0 = ((1-e)y)^2$, $(x-(1-e))(y-z) = (1-ez)(1-(1+ez)(1-e)y)$. Clearly, $((1+ez)(1-e)y)^2 = (1+ez)(1-e)yz(1-e)y$. Write $g = 1-e+ez$. Then $g^2 = g$. Since R is a GQN ring and $((1-e)yz(1-e)y)^2 = 0$, $g(1 - e)yz(1 - e)y \in N(R)$; that is, $(1 + ez)(1 - e)yz(1 - e)y \in N(R)$, so $(1 + ez)(1 - e)y \in N(R)$, and this implies $(x - (1 - e))(y - z) \in U(R)$. By Corollary 2.6(1), $x - (1 - e) \in U(R)$, so x is clean. □

Theorem 4.1 implies the following corollary.

Corollary 4.2 *(1) Let R be a GQN ring. If R is exchange, then R is clean.*

(2) Let R be a quasinormal ring. If R is exchange, then R is clean.

(3) Let R be an NI ring. If R is exchange, then R is clean.

Recall that a ring R is said to have stable range 1 (cf. [9]) if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. It is well known that an exchange ring has stable range 1 if and only if every regular element is unit-regular.

It is well known that an exchange ring R has stable range 1 if and only if for any $a, x \in R$ and $e \in E(R)$, $ax + e = 1$ implies $a + ey \in U(R)$ for some $y \in R$.

Proposition 4.3 *An exchange ring R has stable range 1 if and only if for every regular element a of R, there exists $u \in U(R)$ such that $a - aua \in Z_l(R)$.*

Proof The necessity is clear.

Now assume $ax + e = 1$, where $a, x \in R$ and $e \in E(R)$. Then $a = axa + ea$. If $ea = 0$, then $a = axa$. By hypothesis, there exists $u \in U(R)$ such that $a - aua \in Z_l(R)$. Let $a = aua + z$ for some $z \in Z_l(R)$. Then $1 - e = ax = auax + zx = au(1 - e) + zx$ and $(au - e)^2 = auau - aue - eau + e = au - zu - aue + e = au(1 - e) + e - zu = 1 - e - zx - zu + e = 1 - (zx + zu)$. Clearly, $zx + zu \in Z_l(R)$. Since R is an exchange ring, there exists $g \in E(R)$ such that $g \in (zx + zu)R \subseteq Z_l(R)$ and $1 - g \in (1 - zx - zu)R$; it follows that $g \in Z_l(R)$, so $g = 0$, and this gives $1 \in (1 - zx - zu)R$. Write $1 = (1 - zx - zu)t$ for some $t \in R$. Then $1 - zx - zu = (1 - zx - zu)t(1 - zx - zu)$ and $1 - (1 - zx - zu)t \in l(1 - zx - zu)$. Since $zx + zu \in Z_l(R)$ and $l(zx + zu) \cap l(1 - zx - zu) = 0$, $l(1 - zx - zu) = 0$. Hence, $(1 - zx - zu)t = 1$, and it follows that $1 - zx - zu \in U(R)$; that is, $au - e \in U(R)$. Let $au - e = v$ for some $v \in U(R)$. Then $a - eu^{-1} = vu^{-1} \in U(R)$. If $ea \neq 0$, then $a \neq axa$. Let $f = ax = 1 - e$ and $r = fa - a$. Then $rx = (fa - a)x = (axa - a)x = (ax - 1)ax = -e(1 - e) = 0$ and $fr = f^2a - fa = 0$. Let $a' = a + r$. Then $a'x = ax + rx = ax = f$, $a'xa' = fa' = fa + fr = fa = r + a = a'$, and $a'x + e = f + e = ax + e = 1$. Since $ea' = ea + er = efa = eaxa = e(1 - e)a = 0$, by a similar proof as above, there exists $w \in U(R)$ such that $a' - ew = s \in U(R)$. Since $fr = 0$, $r = (1 - f)r = er$, and this leads to $s = a' - ew = a + r - ew = a + e(r - w)$. Therefore, R has stable range 1. □

Theorem 2.4 and Proposition 4.3 imply the following corollary, which is a generalization of [14, Theorem 4.8].

Corollary 4.4 *Exchange GQN rings have stable range 1.*

A ring R is called *left topologically boolean*, or a *left tb-ring* [2] for short, if for every pair of distinct maximal left ideals of R there is an idempotent in exactly one of them.

Theorem 4.5 *Let R be a GQN exchange ring. Then R is a left tb-ring.*

Proof Suppose that M and N are distinct maximal left ideals of R. Let $a \in M \backslash N$. Then $Ra + N = R$ and $1 - xa \in N$ for some $x \in R$. Clearly, $xa \in M \backslash N$. Since R is a GQN exchange ring, R is clean by Corollary 4.2, so there exist an idempotent $e \in E(R)$ and a unit u in R such that $xa = e + u$. If $e \in M$, then $u = xa - e \in M$, from which it follows that $R = M$, a contradiction. Thus, $e \notin M$. If $e \notin N$, then $1 - e \in N$ by Proposition 2.8(1) and hence $u = (1 - e) + (xa - 1) \in N$. It follows that $N = R$, which is also not possible. We thus have that e belongs to N only. □

Theorem 4.6 *Let R be a GQN exchange ring. Then R/P is a division ring for every left primitive ideal P of R.*

Proof According to [10, Theorem 1], an exchange ring with only two idempotents is a local ring. Now let $a \in R$ satisfy $a - a^2 \in P$. Since R is an exchange ring, idempotents can be lifted modulo P, and there exists $e \in E(R)$ such that $e - a \in P$. If $eR(1-e) \subsetneq P$, then there exists a maximal left ideal M of R such that $P = (0 : R/M) = \{x \in R | xR \subseteq M\}$ and $eR(1-e)R \subsetneq M$. Since R is a GQN ring, by Proposition 2.8(1), $e \in M$, so $1 - e \notin M$, and again by Proposition 2.8(1), $eR \subseteq M$; this implies $e \in P$. If $eR(1-e) \subseteq P$, then either $e \in P$ or $1 - e \in P$. Hence, in any case, we have either $a \in P$ or $1 - a \in P$, and it follows that R/P has only two idempotents. Since R/P is an exchange ring, R/P is a local ring. Since R/P is a left primitive ring, R/P is a division ring. □

It is well known that abelian rings need not be left quasi-duo and GQN rings need not be left quasi-duo. The following corollary shows that exchange GQN rings are left quasi-duo, which is also a corollary of [14, Theorem 3.12].

Corollary 4.7 *Let R be an exchange GQN ring. Then R is a left and right quasi-duo ring.*

Proof Assume that M is a maximal left ideal of R. Write $P = (0 : R/M)$. Then P is a left primitive ideal of R, and this gives that R/P is a division ring by Theorem 4.6. If M is not an ideal, then there exist $m \in M$ and $a \in R$ such that $ma \notin M$; it follows that $ma \notin P$, so there exists $b \in R$ such that $1 - bam \in P \subseteq M$, and this gives $1 = (1 - bam) + bam \in M$, which is a contradiction. Hence, M is an ideal of R and R is a left quasi-duo ring. By the proof of Proposition 2.1 of [15], one has that $R/J(R)$ is a left quasi-duo ring. By [15, Corollary 2.4], $R/J(R)$ is reduced. By [14, Lemma 3.5], R is right quasi-duo. □

Corollary 4.8 *Let R be an exchange GQN ring. If every prime ideal of R is left primitive, then R is strongly π-regular and $R/J(R)$ is strongly regular.*

Proof It follows from [15, Theorem 2.5] and Corollary 4.7. □

Corollary 4.9 *Let R be an exchange GQN ring. Then the following conditions are equivalent:*

(1) Every prime ideal of R is maximal and $J(R) = 0$;

(2) Every prime ideal of R is left primitive and $J(R) = 0$;

(3) R is strongly regular.

Proof It is an immediate result of Corollary 4.8. □

Recall that R is left (right) weakly regular if $a \in RaRa$ $(a \in aRaR)$ for all $a \in R$, and R is said to be a left (right) V-ring if every simple left (right) R-module is injective. Clearly, strongly regular rings are left and right V-rings and left (right) V-rings are left (right) weakly regular. Since left (right) quasi-duo left (right) weakly regular rings are strongly regular, Corollary 4.7 implies the following corollary.

Corollary 4.10 *Let R be an exchange GQN ring. Then the following conditions are equivalent:*

(1) R is a strongly regular ring;

(2) R is a left V-ring;

(3) R is a right V-ring;

(4) R is a left weakly regular ring;

(5) R is a right weakly regular ring.

5. GQN semiperiodic rings

Following [1], a ring R is said to be *semiperiodic* if for each $x \in R\backslash(J(R) \cup Z(R))$, there exist $m, n \in \mathbb{Z}$, of opposite parity, such that $x^n - x^m \in N(R)$. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings.

Lemma 5.1 *Let R be a GQN ring. If R is a semiperiodic ring, then $N(R) \subseteq J(R)$.*

Proof Let $a \in N(R)$ with $a^k = 0$, and let $x \in R$. If $ax \in J(R)$, then ax is right quasiregular, and if $ax \in Z(R)$, then ax is nilpotent and again ax is right quasiregular. Suppose, then, that $ax \notin J(R) \cup Z(R)$, in which case [1, Lemma 2.3(iii)] gives $q \in \mathbb{Z}^+$ and an idempotent e of form axy such that $(ax)^q = (ax)^q e$. Since $e = axy = eaxy = ea(1-e)xy + eaexy = ea(1-e)xy + ea^2(xy)^2 = ea(1-e)xy + ea^2(1-e)(xy)^2 + ea^2e(xy)^2 = ea(1-e)xy + ea^2(1-e)(xy)^2 + ea^3(xy)^3 = \cdots = \Sigma_{i=1}^{k-1} ea^i(1-e)(xy)^i + ea^k(xy)^k = \Sigma_{i=1}^{k-1} ea^i(1-e)(xy)^i$. For any $r \in R$, let $g = 1-e+(1-e)re$. Then $g^2 = g$. Since R is a GQN ring, $(1-e)rea^i(1-e) = g(ea^i(1-e)) \in N(R)$, there exists $n_i \geq 1$ such that $((1-e)rea^i(1-e))^{n_i} = 0$. Hence, $(rea^i(1-e))^{n+1} = 0$, and it follows that $ea^i(1-e) \in J(R)$ for all i, so $\Sigma_{i=1}^{k-1} ea^i(1-e) \in J(R)$. Therefore, $e = \Sigma_{i=1}^{k-1} ea^i(1-e)(xy)^i \in J(R)$, and this leads to $e = 0$ and $(ax)^q = 0$, which shows that ax is right quasiregular. Thus, $a \in J(R)$. □

Theorem 5.2 *If R is a GQN semiperiodic ring, then $R/J(R)$ is commutative.*

Proof By [1, Theorem 4.3], R is either commutative or periodic, so we may assume that R is periodic. Since $J(R)$ contains no nonzero idempotents, $J(R)$ is contained in $N(R)$ and hence $J(R) = N(R)$ by Lemma 5.1 and one has that $R/J(R) = R/N(R)$ is reduced; since $R/N(R)$ is also semiperiodic, it is commutative by [1, Theorem 4.4]. □

Theorem 5.3 *Let R be a GQN semiperiodic ring. Then:*

(1) $N(R)$ is an ideal of R.

(2) If $J(R) \neq N(R)$, then R is commutative.

Proof In the proof of Theorem 5.2, we obtain that if R is not commutative, then $J(R) = N(R)$. Hence, (2) holds and (1) also holds for noncommutative ring R. But also if R is commutative, $N(R)$ is an ideal; hence, (1) holds in any case. □

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